A new upper bound for the Bondage Number of graphs with small domination number

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Abstract

Let γ be the domination number of a graph. The bondage number $b(G)$ of a nonempty graph G was first introduced by Fink, Jacobson, Kinch and Roberts [3] to be the minimum cardinality among all sets of edges X for which $\gamma(G-X) > \gamma(G)$ holds.
In this paper we show that $b(G) \leq \frac{3}{2}\Delta(G)$ for any graph G satisfying $\gamma(G) \leq 3$, which is best possible.

1 Introduction

Let $G = (V, E)$ be a finite, undirected graph with neither loops nor multiple edges. For $u \in V(G)$ we denote by $N(u)$ the neighborhood of u. More generally we define $N(U) = \bigcup_{u \in U} N(u)$ for a set $U \subseteq V$ and $N[U] = N(U) \cup U$.
A set D of vertices in G is a dominating set if $N[D] = V$. A dominating set of minimum cardinality in G is called a minimum dominating set (MDS), and its cardinality is termed the domination number of G and denoted by $\gamma(G)$.
Fink, Jacobson, Kinch, and Roberts [3] defined the bondage number $b(G)$ of a nonempty graph to be the minimum cardinality among all sets of edges X for which $\gamma(G-X) > \gamma(G)$ holds. Brigham, Chinn, and Dutton [2] defined a vertex v to be critical iff $\gamma(G-v) < \gamma(G)$ and G to be a vertex domination-critical graph (from now on called 'vc-graph') iff each vertex of G is critical. For graph theory terminology not presented here we follow [4].
From [1] we know that $b(G) \leq deg\ v \leq \Delta(G)$, when G has a non-critical vertex v. For vc-graphs it is more difficult to find good upper bounds. On one hand, we have trivial-type upper bounds like $b(G) \leq \Delta(G) + \delta(G) - 1$ [3]. On the other hand it was conjectured in [3], that $b(G) \leq \Delta(G) + 1$. This conjecture was disproved in [5] and further it was pointed out in [7], that there is no fixed natural number c such that $b(G) \leq \Delta(G) + c$ is valid for any graph G. This result was obtained by indicating an infinite class of graphs $G_i \quad (i \in \mathbb{N})$ for which $\Delta(G_i) = 2(i-1)$ and

$b(G_i) = 3(i-1)$. Since the class G_i looked 'somehow extremal', a new hypothesis arose, namely $b(G) \leq \frac{3}{2}\Delta(G)$. In this paper we prove $b(G) \leq \frac{3}{2}\Delta(G)$ for graphs G satisfying $\gamma(G) \leq 3$.

2 Preliminary results

Proposition 2.1 *[6] Let G be a graph. Suppose q edges can be removed from G to give a graph H with $b(H) = 1$. Then $b(G) \leq q + 1$.*

Proposition 2.2 *[6] Let G be a nonempty graph with a unique MDS. Then $b(G) = 1$.*

Proposition 2.3 *[6] Let G be a nonempty graph with $K_t \subseteq G$. Then*

$$b(G) \leq min\{deg\ u + deg\ v - t + 1\ ;\ u \text{ and } v \text{ belong to the same } K_t\}$$

To arrange the proof of our main result as smoothly as possible we will prove some preliminary results. Furthermore we need a few helpful definitions:

Definitions:

1. Let $u \in V(G)$. Then $E_u := \{x \in E(G); x \text{ incident to } u\}$
2. Let $DOM(G) := \langle\{v \in V(G); v \text{ belongs to at least one } MDS(G)\}\rangle$, where $\langle V' \rangle$ is the subgraph induced by $V' \subseteq V(G)$.
   Observation: If G is a vc-graph then $DOM(G) = G$.
3. Let $b_k(G) := min\{|X|; X \subseteq E(G), \gamma(G - X) \geq \gamma(G) + k\}$ for $1 \leq k \leq |V(G)| - \gamma(G), k \in \mathbb{N}$ be the k-bondage number of G.
   Observation: $b(G) = b_1(G)$.

Lemma 2.4 *Let G be a vc-graph, $u \in V(G)$ and $\gamma(DOM(G-u)) = \gamma(G) - k$, where $k \in \{1,2\}$. Then $b(G) \leq deg\ u + b_k(DOM(G-u))$ if $b_k(DOM(G-u))$ exists, and $b(G) \leq deg\ u + 1$ if $b_k(DOM(G-u))$ doesn't exist.*

Proof: Since u is critical, $\gamma(G-u) = \gamma(G) - 1$ and $N(u) \cap V(DOM(G-u)) = \emptyset$. Hence any dominating set of $G-u$ including a vertex $w \notin V(DOM(G-u))$ has at least $\gamma(G)$ elements because it can't be a MDS.
Now let $\gamma(DOM(G-u)) = \gamma(G) - k$ where $k \in \{1,2\}$. We consider two cases.
<u>*Case 1:*</u> $G-u$ has a unique MDS. Then $b_k(DOM(G-u))$ may not exist, but from Proposition 2.2 we have $b(G-u) = 1$ and thus by Proposition 2.1 $b(G) \leq deg\ u + 1$.
<u>*Case 2:*</u> $G-u$ has more than only one MDS. Then $|DOM(G-u)| \geq \gamma(G) = \gamma(DOM(G-u)) + k$. Thus $b_k(DOM(G-u))$ is well defined. Now let X be a minimum set of edges such that $\gamma(DOM(G-u) - X) = \gamma(DOM(G-u)) + k$.
Let D be a $MDS(G-u-X)$.

Case 2.1: There is a $w \in D$ with $w \notin V(DOM(G-u))$. Then $\gamma(G-u-X) = |D| \geq \gamma(G)$ by the argument at the beginning of the proof.
Case 2.2: $D \cap V(DOM(G-u)) = D$. Then
$\gamma(G-u-X) \geq \gamma(DOM(G-u)-X) = \gamma(DOM(G-u)) + k = \gamma(G)$.
In both cases we have $\gamma(G-u-X) \geq \gamma(G)$ and therefore $\gamma(G-E_u-X) \geq \gamma(G)+1$ which proves the lemma. •

The next two lemmata are almost trivial but useful. We will omit the proofs.

Lemma 2.5 *Let G be a graph with $\gamma(G) = 1$ and $t := |\{v \in V(G); deg\ v = |V(G)|-1\}|$. Then $b(G) = \lceil \frac{t}{2} \rceil$.*

Lemma 2.6 *Let G be a vc-graph, $u \in V(G)$ and $G' := DOM(G-u)$. Let $R := V(G-u)-V(G')$ be the vertices not belonging to any $MDS(G-u)$. And let $m[w,G']$ be the number of edges leading from w to G' for any vertex $w \in R$. Then $b(G) \leq deg\ u + \min_{w \in R}\{m[w,G']\}$.*

Remark: Since $N(u) \subseteq R$ for any vc-graph, the set R can not be empty and hence is well defined in the above lemma.

Lemma 2.5 especially shows that $b(G) \leq \frac{1}{2}\Delta(G) + 1$ if $\gamma(G) = 1$. Before finally concentrating on the case $\gamma = 3$ we also treat the much easier case $\gamma = 2$.

Lemma 2.7 *Let G be a graph with $\gamma(G) = 2$. Then $b(G) \leq \frac{3}{2}\Delta(G)$.*

Proof: If G is not a vc-graph, then $b(G) \leq \Delta(G)$ is immediate (see Section 1). Now let G be a vc-graph. From [2] we know that the vc-graphs with domination number 2 are exactly the graphs K_{2k} with a 1-factor removed ($k \geq 2$). From [6] we know that for these graphs $b(G) = \Delta(G) + 1$. Since $\Delta(G) \geq 2$ we have $\Delta(G) + 1 \leq \frac{3}{2}\Delta(G)$. •

Lemma 2.8 *[2] If G is a vc-graph, then $|V(G)| \leq (\gamma(G)-1)(\Delta(G)+1)+1$.*

Proof: Since G is a vc-graph, $\gamma(G-u) \leq \gamma(G) - 1$ for any $u \in V(G)$. Hence $|V(G)| - 1 \leq (\gamma(G)-1)(\Delta(G)+1)$. •

Corollary 2.9 *If G is a vc-graph with $\gamma(G) = 3$, then $|V(G)| \leq 2\Delta(G) + 3$.*

Theorem 2.10 *Let G be a vc-graph with $\gamma(G) = 3$ and $|V(G)| = 2\Delta(G) + 3$. Then $b(G) \leq \Delta(G) + 1$.*

Proof: For $\Delta(G) \leq 2$ the conclusion is trivially true by the result $b(G) \leq \Delta(G) + \delta(G) - 1$ in [3].
Now let $\Delta(G) \geq 3$, $u \in V(G)$ and $G' := G - u$. Since G is a vc-graph $\gamma(G') = 2$. Let $\{v,w\} \in MDS(G')$. If v and w had a common neighbor we would have $|V(G)| \leq 2\Delta(G)+2$, which is a contradiction. Hence $d(v,w) > 2$ and $|N(v)| = |N(w)| = \Delta(G)$. ($v, w \notin N_G(u)$, otherwise $\gamma(G) = 2$, contradiction as well)
Case 1: $\{v,w\}$ is the unique MDS of G'. Then we get $b(G') = 1$ by Proposition 2.2 and thus $b(G) \leq deg\ u + 1 \leq \Delta(G) + 1$, as required.
Case 2: There is a set $D \in MDS(G')$, $D \neq \{v,w\}$.

Case 2.1 : $D \cap \{v, w\} \neq \emptyset$, say $v \in D$. Then the second element of D must be an element of $N(w) := \{t_1, \dots, t_{\Delta(G)}\}$, say $D = \{v, t_1\}$. Hence $N(t_1) = \{w, t_2, \dots, t_{\Delta(G)}\}$ because v and t_1 have no common neighbor (see above). Now consider $\tilde{G} := G - t_1$. To dominate w in $\tilde{G}$ we conclude that $N(t_1) \cap MDS(\tilde{G}) \neq \emptyset$. But then $\gamma(\tilde{G}) = \gamma(G)$ and t_1 is not critical in G, which is a contradiction since G was a vc-graph.
Case 2.2 : $D \cap \{v, w\} = \emptyset$, say $D = \{s_1, t_1\}$ where $N(v) := \{s_1, \dots, s_{\Delta(G)}\}$.
Without loss of generality let t_1 be adjacent to $t_2, \dots, t_i, s_{i+1}, \dots, s_{\Delta(G)}$ and let s_1 be adjacent to $s_2, \dots, s_i, t_{i+1}, \dots, t_{\Delta(G)}$ where $1 \leq i \leq \Delta(G) - 1$. (t_1 must have a s_j-neighbor, otherwise we have the same contradiction as in Case 2.1.).
Consider $\tilde{G} := G - s_1$. What does a $MDS(\tilde{G})$ look like?
One element of $V_1 := \{s_{i+1}, \dots, s_{\Delta(G)}\}$ must belong to any $MDS(\tilde{G})$ because v has to be dominated in $\tilde{G}$. (v itself can't belong to a $MDS(\tilde{G})$ otherwise $\gamma(G) = \gamma(\tilde{G})$).
Also w can't belong to a $MDS(\tilde{G})$ because w and any vertex of V_1 have the common neighbor t_1 (which still is not possible).
Thus one element of $V_2 := \{t_1, \dots, t_i\}$ must be the second element of a $MDS(\tilde{G})$, since w has to be dominated in $\tilde{G}$. But $d(V_1, V_2) \leq 2$, hence we again have the contradiction $|V(G)| \leq 2\Delta(G) + 2$, even in the extreme case $i = 1$.
Thus case 2 can not arise, which completes the proof. •

3 The main result

3.1 The 'finding' algorithm

Before we prove our main result it is useful to present the algorithm that will be used to find a 'bondage-edge-set', i.e. a minimum set X of edges such that $\gamma(G - X) > \gamma(G)$:
Let G be a vc-graph with $\gamma(G) = 3$.

1. Isolate a vertex v_0 with $deg\ v_0 = \Delta(G)$. Let $H_1 := DOM(G - v_0)$.

2. Consider $\gamma(H_1)$. Since G is a vc-graph $\gamma(H_1) \leq 2$.

3. Let $v_1 \in V(H_1)$ be a vertex with $deg_{H_1} v_1 = \delta(H_1)$. Isolate v_1 *in H_1*. Let $H_2 := DOM(H_1 - v_1)$ and let $t := |\{w \in (V(H_1) - v_1);\ deg_{(H_1 - v_1)} w = |V(H_1)| - 2\}|$.

The following cases have to be discussed:
Case 1 : $\gamma(H_1) = 2$. Then $b(G) \leq \Delta(G) + b(H_1)$
(or $b(G) \leq \Delta(G) + 1$ if $b(H_1)$ doesn't exist) by Lemma 2.4.
Case 1.1 : v_1 is not critical in H_1. Then $b(H_1) \leq \deg_{H_1} v_1 = \delta(H_1)$.
Case 1.2 : v_1 is critical in H_1. Then $b(H_1) \leq \deg_{H_1} v_1 + \lceil \frac{t}{2} \rceil$ by Lemma 2.5.
Case 2 : $\gamma(H_1) = 1$. Then $b(G) \leq \Delta(G) + b_2(H_1)$
(or $b(G) \leq \Delta(G) + 1$ if $b_2(H_1)$ doesn't exist) by Lemma 2.4.
Case 2.1 : $\gamma(H_2) > 1$. Then $b_2(H_1) \leq \deg_{H_1} v_1 = \delta(H_1)$.

Case 2.2 : $\gamma(H_2) = 1$. Then $b_2(H_1) \leq \deg_{H_1} v_1 + \lceil \frac{t}{2} \rceil$.
If one of the bondage numbers doesn't exist we have $b(G) \leq \Delta(G) + 1$.
Otherwise we have $b(G) \leq \Delta(G) + \delta(H_1) + \lceil \frac{t}{2} \rceil$ in the worst case. So we now have to calculate $\delta(H_1)$ and $\lceil \frac{t}{2} \rceil$.

3.2 The main theorem

Theorem 3.1 *Let G be a vc-graph with $\gamma(G) = 3$, then $b(G) \leq \frac{3}{2}\Delta(G)$.*

Proof: Without loss of generality let $\Delta := \Delta(G) \geq 3$ (for $\Delta \leq 2$ the conclusion is trivially true as in Theorem 2.10) and likewise without loss of generality let $|V(G)| \leq 2\Delta + 2$ (for the case $|V(G)| = 2\Delta + 3$ we have Theorem 2.10 to prove the conclusion).
Following the 'finding algorithm' we firstly isolate the vertex v_0 ($deg\ v_0 = \Delta$) in G and define $H_1 := DOM(G - v_0)$.
Since $N(v_0) \cap V(H_1) = \emptyset$ (otherwise v_0 was not critical) we conclude that $n_1 := |V(H_1)| \leq |V(G)| - (\Delta + 1) \leq \Delta + 1$.
Assumption: $b(G) > \frac{3}{2}\Delta$.

Then the relevant bondage numbers of H_1 must exist and we can start calculating $\delta(H_1)$ and $\lceil \frac{t}{2} \rceil$.
Let $N(v_0) := \{w_1, \ldots, w_\Delta\}$. We want to apply Lemma 2.6:
If we had a $w_i \in N(v_0)$ with $m[w_i, H_1] \leq \frac{1}{2}\Delta$ we would obtain $b(G) \leq deg\ v_0 + \frac{1}{2}\Delta = \frac{3}{2}\Delta$ by Lemma 2.6, a contradiction.
Thus we know that $m[w_i, H_1] > \frac{1}{2}\Delta$ for each i.
Now we can estimate the number of edges in H_1:

$$m_1 := |E(H_1)| \leq \frac{1}{2}\left(\sum_{u \in V(H_1)} deg\ u - \sum_{1 \leq i \leq \Delta} m[w_i, H_1]\right)$$
$$\leq \frac{1}{2}\Delta\left(n_1 - \left\lceil \frac{1}{2}\Delta + \frac{1}{2} \right\rceil\right) \leq \frac{1}{2}\Delta \cdot \left\lceil \frac{1}{2}\Delta \right\rceil.$$

Case 1 : Δ even

We already know $n_1 \leq \Delta + 1$, $m_1 \leq \frac{1}{4}\Delta^2$. Then

$$\delta(H_1) \leq \left\lfloor \frac{2m_1}{n_1} \right\rfloor \leq \left\lfloor \frac{\Delta^2}{2(\Delta+1)} \right\rfloor = \frac{1}{2}\Delta - 1$$

(For the second inequality, notice that since m_1 also depends on n_1 we can easily check the derivation to see that the maximal value of $\delta(H_1)$ is obtained by taking the maximal value of n_1.)

Following the algorithm we now isolate v_1 in H_1 (remember that v_1 is a vertex of

degree $\delta(H_1)$ in H_1). The remaining object is to estimate t (as defined in section 3.1), i.e. to estimate the number of vertices in $H_1 - v_1$ with degree $n_1 - 2$.
The following formula, counting the edges of H_1, is obvious:

$$t(n_1-2)+(n_1-t)\cdot\delta(H_1)\leq 2m_1 \quad if\ and\ only\ if \quad t\leq\frac{2m_1-n_1\cdot\delta(H_1)}{n_1-2-\delta(H_1)},$$

$$which\ implies\ that \quad t\leq\frac{\Delta(n_1-\lceil\frac{\Delta+1}{2}\rceil)-n_1\cdot\delta(H_1)}{n_1-2-\delta(H_1)}. \tag{1}$$

Again the derivation shows that t is maximal for the greatest possible n_1. Hence we can evaluate (1) by putting the term $\frac{1}{2}\Delta-1-s$ instead of $\delta(H_1)$ (where $s\geq 0,\ s\in\mathbf{N}$) and taking the maximal n_1:

$$t \quad \leq \quad \frac{\frac{1}{2}\Delta^2-(\Delta+1)(\frac{1}{2}\Delta-1-s)}{\Delta-1-(\frac{1}{2}\Delta-1-s)} \quad = \quad 2s+1+\frac{2-4s^2}{\Delta+2s}.$$

Discussion of t:

$s=0$: $\quad t\leq 1+\left\lfloor\frac{2}{\Delta}\right\rfloor=1$

$s\geq 1$: $\quad t\leq 2s \quad$ (since the fraction is negative)

Now we can estimate $b(G)$:

$$b(G)\leq deg\ v_0+deg_{H_1}v_1+\left\lceil\frac{t}{2}\right\rceil \quad\leq\quad \Delta+\delta(H_1)+\left\lceil\frac{t}{2}\right\rceil$$

$$=\begin{cases}\Delta+\frac{1}{2}\Delta-1+\left\lceil\frac{1}{2}\right\rceil\leq\ \frac{3}{2}\Delta\,, & for\ s=0\\ \Delta+\frac{1}{2}\Delta-1-s+\left\lceil\frac{2s}{2}\right\rceil\leq\ \frac{3}{2}\Delta-1\,, & for\ s\geq 1.\end{cases}$$

In both cases we have a contradiction to our assumption. Hence the first case is complete.

<u>*Case 2:*</u> Δ odd
We already know $n_1\leq\Delta+1$, $m_1\leq\frac{1}{4}\Delta(\Delta+1)$. Then

$$\delta(H_1)\leq\left\lfloor\frac{2m_1}{n_1}\right\rfloor\leq\left\lfloor\frac{\Delta(\Delta+1)}{2(\Delta+1)}\right\rfloor=\frac{1}{2}(\Delta-1)$$

(again the derivation tells us that $\delta(H_1)$ is maximal by taking the maximal n_1).

We carry out the same steps as in Case 1, i.e. we isolate v_1 in H_1, we use formula (1) to estimate t and evaluate (1) this time by putting the term $\frac{1}{2}(\Delta-1)-s$ instead of $\delta(H_1)$ (where $s\geq 0$, $s\in\mathbf{N}$ again):

$$t \leq \frac{\frac{1}{2}\Delta(\Delta+1)-(\Delta+1)(\frac{1}{2}(\Delta-1)-s)}{\Delta-1-(\frac{1}{2}(\Delta-1)-s)} = 2s+1+\frac{2+2s-4s^2}{\Delta+2s-1}$$

Discussion of t:

$s=0$: $t \leq 1+\left\lfloor\frac{2}{\Delta-1}\right\rfloor$

$s=1$: $t \leq 3$

$s \geq 2$: $t \leq 2s$

Hence, for $s \geq 2$ we conclude

$$b(G) \leq \Delta+\delta(H_1)+\left\lceil\frac{t}{2}\right\rceil \leq \Delta+\frac{1}{2}(\Delta-1)-s+\left\lceil\frac{2s}{2}\right\rceil \leq \frac{3}{2}\Delta$$

which is a contradiction to our assumption.
But we still have to inspect the cases $s=0$ and $s=1$.

Case 2.1: $s=0$, so that $\delta(H_1)=\frac{1}{2}(\Delta-1)$.
Let $\{w,w'\} \subseteq V(H_1) := DOM(G-v_0)$ such that $\{w,w'\} \in MDS(H_1)$ and $deg_{H_1}w = \Delta(H_1)$. Since $\delta(H_1)=\frac{1}{2}(\Delta-1)$, w' can dominate at most $\Delta-\delta(H_1)=\frac{1}{2}(\Delta+1)$ vertices of $R := V(G-v_0)-V(H_1)$. Remember $N(v_0) \subseteq R$ and therefore $|R| \geq |N(v_0)| = \Delta$. Hence w has to dominate at least $\frac{1}{2}(\Delta-1)$ vertices of R, and consequently

$$\Delta(H_1) \leq \frac{1}{2}(\Delta+1). \tag{2}$$

Remember the definition of t. For the existence of t we necessarily need $\Delta(H_1) \geq n_1-2$. Hence by (2) we have $n_1 \leq \frac{1}{2}(\Delta+1)+2$ and by recalculation

$$m_1 \leq \frac{1}{2}[\Delta \cdot n_1 - \Delta \cdot \frac{1}{2}(\Delta+1)] \leq \Delta. \tag{3}$$

Obviously we get a new bound for $\delta(H_1)$:

$$\delta(H_1) \leq \left\lfloor\frac{2m_1}{n_1}\right\rfloor = \left\lfloor\frac{2\Delta}{\frac{1}{2}(\Delta+5)}\right\rfloor = \left\lfloor 4-\frac{20}{\Delta+5}\right\rfloor$$

For $\Delta \geq 7$ we immediately get a contradiction to the condition of Case 2.1. Thus the cases $\Delta=3$ and $\Delta=5$ remain to be investigated.

$\Delta = 3$:
We already know $\delta(H_1)=1$, $\Delta(H_1) \leq 2$, $\gamma(H_1) \leq 2$ and $n_1 \leq 4$, $m_1 \leq 3$. To determine which existing graphs satisfy these conditions we consider the following possibilities:

$H_1 \cong K_2$: Then $n_1=2$ and therefore by (3) $m_1 \leq 0$, a contradiction.

$H_1 \cong K_{1,2}$: Then $n_1=3$ and therefore by (3) $m_1 \leq 1$, a contradiction.

$H_1 \cong 2K_2$: Then $b(H_1) = 1$ and by Lemma 2.4 $b(G) \le 3 + 1 \le \frac{3}{2}\Delta$, a contradiction.

$H_1 \cong P_4$: Any possible 'mother graph' G is cubic and has 8 vertices. There is only one graph satisfying all these conditions, but it has bondage number 4, a contradiction to our assumption.

$\Delta = 5$:
We already know $\delta(H_1) = 2$, $\Delta(H_1) \le 3$, $\gamma(H_1) \le 2$ and $n_1 \le 5$, $m_1 \le 5$. There are the following graphs satisfying these conditions:

$H_1 \cong C_3$: Then $n_1 = 3$ and therefore by (3) $m_1 \le 0$, a contradiction.

$H_1 \cong C_4$ or $H_1 \cong (K_4 - edge)$: Then $n_1 = 4$ and therefore by (3) $m_1 \le 2$, a contradiction.

$H_1 \cong C_5$: Then $b(H_1) = 2$ and by Lemma 2.4 $b(G) \le 5 + 2 \le \frac{3}{2}\Delta$, a contradiction.

This completes the Case 2.1.

Case 2.2 : $s = 1$, so that $\delta(H_1) = \frac{1}{2}(\Delta - 3)$.
Analogously to Case 2.1 we conclude that $\Delta(H_1) \le \frac{1}{2}(\Delta + 3)$. Continuing analogously to Case 2.1 we compute $n_1 \le \frac{1}{2}(\Delta + 3) + 2$ and with (3) $m_1 \le \frac{3}{2}\Delta$. Hence

$$\delta(H_1) \le \left\lfloor \frac{2m_1}{n_1} \right\rfloor = \left\lfloor \frac{3\Delta}{\frac{1}{2}(\Delta + 7)} \right\rfloor = \left\lfloor 6 - \frac{42}{\Delta + 7} \right\rfloor .$$

For $\Delta \ge 11$ we immediately get a contradiction to the condition of Case 2.2. Thus the cases $3 \le \Delta \le 9$ remain to be investigated.
If $t \le 2$ we immediately have a contradiction to our assumption. Hence the only remaining case is $t = 3$ and therefore $n_1 \ge 4$. We evaluate (1):

$$3(n_1 - 2) + (n_1 - 3)\delta(H_1) \le 2m_1 \qquad (4)$$

$\Delta = 3$:
We know $\delta(H_1) = 0$, $n_1 \le 5$ and $m_1 \le 4$. We investigate any possible value of n_1 and obtain the following results:

$n_1 = 5$: This contradicts (4).

$n_1 = 4$: $H_1 \cong C_3 \cup K_1$ is the only possibility for H_1. But then H_1 and thus G contains a triangle, and by Proposition 2.3 we get $b(G) \le 2\Delta - 2 = 4 \le \frac{3}{2}\Delta$, a contradiction.

$\Delta = 5$:
We know $\delta(H_1) = 1$, $n_1 \le 6$ and by (3) $m_1 \le \frac{1}{2}(5n_1 - 15)$. Then all possible values of n_1 ($4 \le n_1 \le 6$) contradict (4).

The cases $\Delta = 7$ and $\Delta = 9$ can be worked out in the same manner.

This completes the proof of the main theorem. ●

Finally we want to state the following

Conjecture: $b(G) \leq \frac{3}{2}\Delta(G)$ is valid for any graph G (even for vc-graphs with $\gamma \geq 4$).

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