Blocking set preserving embeddings of partial $K_4 - e$ designs

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Abstract

In this paper we show that a partial $K_4 - e$ design of order n and index λ that has a blocking set S can be embedded in a $K_4 - e$ design of order $v \leq 10n + 20\sqrt{n} + 56$ and index λ that has a blocking set S^* such that $S \subseteq S^*$. This also improves upon the smallest known embedding for partial $K_4 - e$ designs.

1 Introduction

A (partial) H-design of a graph G is an ordered pair (V, B), where V is the vertex set of G and where B is a collection of edge-disjoint copies of H with the property that each edge of G is in (at most one) exactly one copy of H in B. If G is (a subgraph of) λK_n then we say that (V, B) is a (partial) H-design of *order* n and *index* λ.

An H-design (V, B) of G is said to be *embedded* in an H-design (V', B') of λK_n if $V \subseteq V'$ and $B \subseteq B'$. There have been many papers written on the embedding of H-designs, especially in the case where $H = K_3$ [1, 2], but also for example when H is a cycle [7, 8] and when $H = K_4 - e$ [6]. The most common embedding question asked seems to be: What is the smallest integer v such that any partial H-design of order n and index λ can be embedded in an H-design of λK_v? Of course, v is a function of n, and conceivably also of λ. The most famous outstanding problem in this area is to show that if $H = K_3$ and $\lambda = 1$ then $v = 2n + 1$ (it has been shown that if $H = K_3$ and 4 divides λ then $v = 2n + 1$, and this is best possible).

To date, the smallest known embedding for any partial $K_4 - e$ design of order n and index λ is in a $K_4 - e$ design of order $v = 15n + 46$ [6], but this is certainly

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not the smallest possible value of v. However, even obtaining this embedding was a breakthrough, produced by using a generalization of Cruse's Theorem [5] for embedding partial idempotent commutative quasigroups to the embedding of partial groupoids (see Section 2). This construction is quite flexible, a fact that we demonstrate in this paper by showing that not only can a small embedding be produced, but also that any blocking set (see below) of the original partial $K_4 - e$ design can be extended to a blocking set of the containing $K_4 - e$ design of λK_v.

A *blocking set* of an H-design (V, B) is a set $S \subseteq V$ such that each copy $h \in B$ of H satisfies $V(h) \cap S \neq \emptyset$ and $V(h) \cap S \neq V(h)$. Again, there have been many papers written in this area. For example, a long series of papers finally culminated in the settling of the existence of K_4-designs of λK_v that have a blocking set, with a couple of possible exceptions [3], and the existence of H-designs of K_v with blocking sets has also been settled for all connected graphs H with at most 5 edges [4, 9] (and in particular for $K_4 - e$ designs).

In this paper we show that any partial $K_4 - e$ design of order n and index λ that has a blocking set S, can be embedded in a $K_4 - e$ design of λK_v that has a blocking set S^* such that $S \subseteq S^*$ and $v \leq 10n + 20\sqrt{n} + 56$; so in addition to extending the blocking set, we also improve upon the best known embedding for partial $K_4 - e$ designs for $n \geq 16$ (see the remark following Theorem 2.2).

Let (a, b, c, d) denote the copy of $K_4 - e$ with edge set $\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}\}$.

2 Embedding Groupoids

A partial groupoid $(P, \circ)$ is said to be idempotent if $x \circ x = x$ for all $x \in P$. A partial groupoid $(P, \circ)$ is called an *embedding groupoid* if (1) $(P, \circ)$ is idempotent, (2) if $x \neq y$ then either both or neither of the products $x \circ y$ and $y \circ x$ is defined, (3) $(P, \circ)$ is row latin, and (4) each $x \in P$ occurs as a product an odd number of times.

Theorem 2.1 ([5]) *Any partial embedding groupoid of order n can be embedded in an idempotent groupoid of order $2n + 1$ which is (1) row latin, and (2) the main diagonal together with all products not defined in the given partial embedding groupoid form a partial symmetric idempotent quasigroup.*

Remark The fact that in (2) we form a partial quasigroup and not just a partial groupoid is important in what follows.

Certainly a stronger result than the following can be proved, but this will suffice for our purposes. Let $(a; a_1, \ldots, a_m)$ denote the m-star $K_{1,m}$ on the vertex set $\{a, a_1, \ldots, a_m\}$ in which a has degree m.

Lemma 2.1 *For all $\ell \geq 1$ there exists a simple graph G on 2ℓ vertices with at least $\binom{2\ell}{2} - 3\ell$ edges for which there exists a $K_{1,4}$-design with the additional property that each $K_{1,4}$ can be split into two copies of $K_{1,2}$ so that the resulting $K_{1,2}$-design of G has a blocking set of size ℓ.*

Proof: We construct such a graph on the vertex set $\mathbb{Z}_\ell \times \mathbb{Z}_2$, with blocking set $\mathbb{Z}_\ell \times \{0\}$. Define the set B of copies of $K_{1,4}$ as follows.

If $\ell = 4x+1$ then $B = \{((i,k);(i+2j+1,k),(i+2j+2,k),\ (i+2j+1,k+1),(i+2j+2,k+1)) \mid i \in \mathbb{Z}_\ell,\ j \in \mathbb{Z}_x, k \in \mathbb{Z}_2\}$; there are ℓ edges in no copy of $K_{1,4}$ in B.

If $\ell = 4x+3$ then $B = \{((i,k);(i+2j,k),(i+2j+1,k),(i+2j,k+1),(i+2j+1,k+1)) \mid i \in \mathbb{Z}_\ell,\ 1 \le j \le x,\ k \in \mathbb{Z}_2\} \cup \{(i,0);(i+1,0),\ (i+1,1),(i,1),(i-1,1) \mid i \in \mathbb{Z}_\ell\}$; there are ℓ edges in no copy of $K_{1,4}$ in B.

If $\ell = 4x$ then $B = \{(i,k);(i+2j,k),(i+2j+1,k),(i+2j,k+1),(i+2j+1,k+1)) \mid i \in \mathbb{Z}_\ell,\ 1 \le j \le x-1,\ k \in \mathbb{Z}_2\} \cup \{(i,0);(i+1,0),\ (i+1,1),(i,1),(i-1,1) \mid i \in \mathbb{Z}_\ell\}$; there are 3ℓ edges in no copy of $K_{1,4}$ in B.

If $\ell = 4x+2$ then $B = \{((i,k);(i+2j+1,k),(i+2j+2,k),\ (i+2j+1,k+1),\ (i+2j+2,k+1)) \mid i \in \mathbb{Z}_\ell,\ j \in \mathbb{Z}_x,\ k \in \mathbb{Z}_2\}$; there are 3ℓ edges in no copy of $K_{1,4}$ in B.

It is trivial to see that each copy of $K_{1,4}$ can be split into two copies of $K_{1,2}$ so that each copy of $K_{1,2}$ has a vertex in $\mathbb{Z}_\ell \times \{0\}$ and a vertex in $\mathbb{Z}_\ell \times \{1\}$, so indeed $\mathbb{Z}_\ell \times \{0\}$ is a blocking set.

Lemma 2.2 *There exists a $K_4 - e$ design of K_6 with a blocking set of size 3, and one of size 4.*

Proof: $\{1,2,3\}$ and $\{1,2,3,4\}$ are each blocking sets for the $K_4 - e$ design $(\mathbb{Z}_6, \{(i, i+3, i+1, i+4) \mid i \in \mathbb{Z}_3\})$ of K_6.

We are now ready for the main result.

Theorem 2.2 *A partial $K_4 - e$ design of order n and index λ that has a blocking set S can be embedded in a $K_4 - e$ design of λK_v that has a blocking set S^* such that $S \subseteq S^*$ and $v \le 10n + 20\sqrt{n} + 56$.*

Remark It may be worth noting that the theorem proves a slightly stronger result, namely that $v \le 10n + 10\alpha + 6$, where α is at most the smallest even integer with $\alpha \ge 2\sqrt{n} + 3$. Also, the size of the blocking set produced is $|S| + 4n + 9\alpha/2 + 3$.

Proof: Let (P, B) be a partial $K_4 - e$ design of order n and index λ with a blocking set S.

For $1 \le i \le \lambda$, let G_i be the simple graph with vertex set P and with $\{u,v\} \in E(G_i)$ iff $\{u,v\}$ occurs in at least i copies of $K_4 - e$ in B.

Let $2x_i$ $(\le n)$ be the number of vertices of odd degree in G_i. Let α be the smallest even integer satisfying $\binom{\alpha}{2} - 3\alpha/2 \ge 2n$ (clearly $(2\sqrt{n}+3)(2\sqrt{n}-1)/2 \ge 2n$, so certainly $\alpha \le 2\sqrt{n}+5$). Let A be a set of $\alpha = 2\ell$ vertices with $P \cap A = \emptyset$. For $1 \le i \le \lambda$ let H_i be a graph with vertex set A containing $4x_i$ $(\le 2n)$ edges as described in Lemma 2.1. In each case, let $A' \subseteq A$ be a blocking set for the $K_{1,2}$-design of H_i (A' is independent of i).

For $1 \le i \le \lambda$, arbitrarily gather the $2x_i$ vertices of odd degree into x_i pairs, and to each pair arbitrarily assign one of the x_i copies of $K_{1,4}$ in the $K_{1,4}$ design of H_i.

For each such pair u_1 and u_2 of odd degree vertices and their corresponding $K_{1,4}$, say $(a; b, c, d, e)$, if $K_{1,4}$ is split into the two copies $(a; b, c)$ and $(a; d, e)$ of $K_{1,2}$ in Lemma 2.1 then let B_i contain the two copies (u_1, a, b, c) and (u_2, a, d, e) of $K_4 - e$. Let G_i' be the simple graph with vertex set $P' = P \cup A$ formed from G_i by adding the edges in the copies of $K_4 - e$ in B_i. If u has odd degree in G_i, then $d_{G_i'}(u) = d_{G_i}(u) + 3$, so u has even degree in G_i'. Also, since the copies of $K_{1,2}$ can be paired to form copies of $K_{1,4}$, each vertex in A has even degree in G_i'. So for $1 \le i \le \lambda$, each vertex in G_i' has even degree. Clearly A' is a blocking set for B_i.

For $1 \le i \le \lambda$ form a partial idempotent groupoid $(P', \circ_i)$ where

(i) $x \circ_i x = x$ for all $x \in P'$, and

(ii) if $x \neq y$, then $x \circ_i y = y$ and $y \circ_i x = x$ if $\{x, y\} \in E(G_i')$ and otherwise $x \circ_i y$ and $y \circ_i x$ are undefined.

Then clearly $(P', \circ_i)$ is an embedding groupoid, satisfying property (4) of embedding groupoids because each vertex in G_i' has even degree. Therefore we can apply Theorem 2.1 to embed $(P', \circ_i)$ in a groupoid $(Q, \circ_i)$ of order $2|P'| + 1$ which satisfies properties (1) and (2) of Theorem 2.1.

We can now define a $K_4 - e$ design $(\{\infty\} \cup (Q \times \mathbb{Z}_5), B^*)$ as follows.

(i) For each $a \in Q$ let $(\{\infty\} \cup (\{a\} \times \mathbb{Z}_5), B_a)$ be a $K_4 - e$ design of λK_6 in which $\{\infty, (a,0), (a,1), (a,2)\}$ is a blocking set if $a \in S \cup A'$, and in which $\{\infty, (a,1), (a,2)\}$ is a blocking set if $a \notin S \cup A'$, and let $B_a \subseteq B^*$.

(ii) For each $(a, b, c, d) \in B \cup (\bigcup_{i=1}^{\lambda} B_i)$ and for each $x, y \in \mathbb{Z}_5$ (including $x = y$) let $((a, x), (b, y), (c, x \otimes_1 y), (d, x \otimes_2 y)) \in B^*$, where $(\mathbb{Z}_5, \otimes_1)$ and $(\mathbb{Z}_5, \otimes_2)$ are defined by the following quasigroups.

$\otimes_1$	0	1	2	3	4
0	0	2	3	4	1
1	4	3	1	2	0
2	2	1	4	0	3
3	3	0	2	1	4
4	1	4	0	3	2

$\otimes_2$	0	1	2	3	4
0	0	4	2	1	3
1	1	2	3	4	0
2	4	3	1	0	2
3	2	0	4	3	1
4	3	1	0	2	4

(iii) For $1 \le i \le \lambda$, if $\{a, b\} \notin E(G_i')$ then $((a, j), (b, j), (a \circ_i b, j+1), (a \circ_i b, j+3)) \in B^*$ for all $j \in \mathbb{Z}_5$ (reducing sums modulo 5).

Then we claim that $(\{\infty\} \cup (Q \times \mathbb{Z}_5), B^*)$ is a $K_4 - e$ design of λK_v with $v = 5(2(n + \alpha) + 1) + 1 \le 10n + 20\sqrt{n} + 56$ in which $S^* = \{\infty\} \cup ((S \cup A') \times \{0\}) \cup (Q \times \{1, 2\})$ is a blocking set. From these observations the result follows, because since $0 \otimes_1 0 = 0 = 0 \otimes_2 0$, from (ii) we have that for each $(a, b, c, d) \in B$, $((a, 0), (b, 0), (c, 0), (d, 0)) \in B^*$ and $S \times \{0\} \subseteq S^*$, so the embedding that preserves the blocking set has been produced.

To see that S^* is a blocking set, consider (ii) and (iii) in the construction. Notice that for all x, y other than $x = 0 = y$, at least one of $x, y, x \otimes_1 y$ and $x \otimes_2 y$ is in $\{1, 2\}$ and at least one is in $\{3, 4\}$, so S^* is a blocking set for the copies of $K_4 - e$ arising from these values of x and y in (ii). If $x = 0 = y$ then also $x \otimes_1 y = 0 = x \otimes_2 y$, so $(S \cup A') \times \{0\} = S^* \cap (Q \times \{0\})$ ensures that S^* is a blocking set for the copies of $K_4 - e$ arising from these values of x and y in (ii). Finally, in (iii), for each $j \in \mathbb{Z}_5$, at least one of j, $j + 1$ and $j + 3$ is in $\{1, 2\}$ and at least one is in $\{3, 4\}$, so S^* is a blocking set for the copies of $K_4 - e$ defined in (iii) since $Q \times \{1, 2\} \subseteq S^*$ and $Q \times \{3, 4\} \cap S^* = \emptyset$. Therefore S^* is a blocking set as claimed.

To see that B^* defines a $K_4 - e$ design of λK_v, consider the edge $e = \{(u, s), (w, t)\}$. If $u = w$ then e is in λ copies of $K_4 - e$ defined in (i), so suppose that $u \neq w$. For each graph G_i' containing the edge $\{u, w\}$ there is a copy of $K_4 - e$ in $B \cup (\cup_{i=1}^{\lambda} B_i)$ containing $\{u, w\}$, and corresponding to this copy of $K_4 - e$, say (a, b, c, d) there are copies $((a, x), (b, y), (c, x \otimes_1 y), (d, x \otimes_2 y))$ in B^*. Since $(\mathbb{Z}_5, \otimes_1)$ and $(\mathbb{Z}_5, \otimes_2)$ are quasigroups, it is easy to check that regardless of which of a, b, c and d u and w happen to be, x and y are uniquely determined by s and t. So if $\{u, w\}$ occurs in ℓ of the λ graphs $G_1, \ldots, G_\lambda$ then we have just found ℓ copies of $K_4 - e$ in B^* that contain $\{u, w\}$. Now $\lambda - \ell$ of the graphs $G_1, \ldots, G_\lambda$ do not contain $\{u, w\}$, so for each such graph G_i the product $u \circ_i w$ in $(P', \circ_i)$ is undefined, and so there is no $z \in P'$ such that $u \circ_i z = w$ (since from (ii) if such a z existed it would be $u \circ_i w = w$). So by (1) of Theorem 2.1 there is a unique $z \in Q$ such that $u \circ_i z = w$ in $(Q, \circ_i)$, and by (2) of Theorem 2.1 we have that $u \circ_i z = w = z \circ_i w$. So, if $s = t$ then $\{u, w\}$ is in the copy $((u, s), (w, s), (u \circ_i w, s + 1), (u \circ_i w, s + 3))$ defined in (iii), and if $s \neq t$ then we can assume that $t - s \pmod 5 \in \{1, 3\}$ and so $\{u, w\}$ is in the copy $((u, s), (z, s), (u \circ_i z = w, s + 1), (u \circ_i z = w, s + 3))$ defined in (iii).

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