A STUDY OF HENDECADS

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In honour of Albert Leon Whiteman's eightieth birthday

1. Introduction.

In earlier papers, we have illustrated how the packings of pairs into k-sets depend critically upon whether the total number of varieties v does or does not exceed the value $k^2 - k + 1$. The behaviour is critically affected by the existence or non-existence of a projective geometry on $k^2 - k + 1$ points.

In this paper, we shall study the packings of v elements into hendecads for those values of v less than 111. The values k = 7 and k = 11 are the first values for which a geometry does not exist, and the situation for k = 11 displays considerably more variety than that which occurred for k = 7 (cf. **[15]**). First, we recapitulate terminology; we start with v elements, and wish to determine the packing number D(2,k,v) for v small. D(2,k,v) is the cardinality of the largest family of k-sets chosen from the v elements in such a way that no pair occurs more than once. Saying that v is "small" means that $v \le k^2 - k + 1$ (we shall thus be considering the case that $k = 11, v \le 111$).

Earlier papers have given various results for packings and coverings with k = 6 to 10 (cf. **[1]** - **[14]**). The numerical results have suggested and complemented general results, and the same is true of the results now presented for k = 11.

2. The Cases $v \le 66$.

We refer to **[10]** for the concept of a *peeling array*; the peeling array for hendecads is shown below. It is the dual of the set of all pairs on 12 elements, and from it we make the table of packing numbers that immediately follows.

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Proof. The existence of the design with parameters (v,b,r,k,1) guarantees [3] that the weight of the design is

$$v(v-1) - bk(k-1) = 0.$$

Now suppose that $D(2,r,b+x) \geq v+1$. Then we may write

$$r(v+1) = rv + r = bk + r = (b+x)k + r - xk.$$

Now $k(k+1)[r/k - 2(r-1)/k(k+1)] = r(k+1) - 2(r-1) = r(k-1) + 2 = v+1$. It follows from the given condition that $x < r/k$, that is, r-xk is positive. We can then calculate the maximum weight of an array in v+1 blocks as

$$\begin{aligned}
&(v+1)v - (r-xk)(k+1)k - (b+x-r+xk)k(k-1) \\
&= (v+1)v - k(r-xk)2 - (b+x)k(k-1) \\
&= v(v-1) + 2v - (r-xk)2k - bk(k-1) - xk(k-1) \\
&= 2v - 2rk + 2xk^2 - xk^2 + xk \\
&= xk(k+1) - 2(r-1).
\end{aligned}$$

But the given condition guarantees that this weight is negative; so a packing array in v+1 blocks is impossible. It follows that $D(2,r,b+x) = v$.

In our case, r = 11, b = 66, k = 2; hence, for $x < 20/6$, we have the packing number still equal to 12.

For v = 70, the weight bound is 13, and the weight is 4. This corresponds to a weight distribution of $3^3,2^{67}$. Such a packing would be the dual of a design with replication number 11 that comprises 3 triples and 67 pairs. To achieve such a design, take 9 elements 1,2,3,4,5,6,7,8,9, and 4 starred elements 1*,2*,3*,4*. Take blocks 123, 456, 789, as the triples. Take the other 27 pairs that do not involve starred elements, and the 36 pairs that involve one unstarred and one starred element. Take also the pairs 1*3*,1*4*,2*3*,2*4*. It is easily checked that each element occurs 11 times in this design (which is a PBD missing 1*2* and 3*4*). So D(2,11,70) = 13. The weight bound immediately establishes that D(2,11,71) and D(2,11,72) are also equal to 13.

The weight bound for v = 74 is 14; again the weight is 4, corresponding to a weight distribution of 8 triples and 65 pairs. We achieve this by dualizing a design on symbols 1,2,3,4,5,6,7,8,9,0, and 1*,2*,3*,4*. The design comprises the triples 1*12,2*34, 3*56, 4*78, 913, 057, 924, 068, and all pairs not in these triples, saving only 1*2*, 3*4*. Again, we can check that each element occurs 11 times in this system. The weight bound then shows that D(2,11,74) is also 14.

01	02	03	04	05	06	07	08	09	10	11
01	12	13	14	15	16	17	18	19	20	21
02	12	22	23	24	25	26	27	28	29	30
03	13	22	31	32	33	34	35	36	37	38
04	14	23	31	39	40	41	42	43	44	45
05	15	24	32	39	46	47	48	49	50	51
06	16	25	33	40	46	52	53	54	55	56
07	17	26	34	41	47	52	57	58	59	60
08	18	27	35	42	48	53	57	61	62	63
09	19	28	36	43	49	54	58	61	64	65
10	20	29	37	44	50	55	59	62	64	66
11	21	30	38	45	51	56	60	63	65	66

Table 1. Peeling array for 66 elements

v	D(v)	v	D(v)
11-20	1	56-59	7
21-29	2	60-62	8
30-37	3	63-64	9
38-44	4	65	10
45-50	5	66	12
51-55	6		

Table 2. Packing numbers D(2,11,v) for v < 67

3. Packings Related to PBDs with Triples.

There is no triple system with replication number 11; otherwise, we could proceed as in **[6]** or **[8]**. However, the procedure is very analogous.

The weight bound for v = 67, 68, and 69 is 12. Hence, we can use the result for D(2,11,66) to give D(2,11,67) = D(2,11,68) = D(2,11,69) = 12.

This result is also an application of the following simple lemma.

Lemma 3.1. Suppose that there is a BIBD with parameters (v,b,r,k,1). Then D(2,r,b+x) = v, provided that $x < 2(r-1)/k(k+1)$.

The weight bound for 75 is zero, corresponding to a distribution of $3^{15},2^{60}$. This requires a PBD on 15 elements that comprises 15 triples, 60 pairs. This is easily obtained from a triple system on 15 elements; keep three resolution classes and split the other 4 resolution classes into pairs. Application of the weight bound gives D(2,11,76) = D(2,11,77) = 15.

The weight bound for 78 is 16, with a weight of 4 corresponding to a distribution of $3^{20},2^{58}$. Take elements 1,2,3,4,5,6,7,8, along with 1*,2*,3*,4*, and a,b,c,d. Use the following triples: 1*12, 2*34, 3*56, 4*78; 1*ab, 2*cd, 3*13, 4*24; 1*57, 2*68, 3*ac, 4*bd; a15, b26, c37, d48; a45, b16, c27, d38. Then use all the unused pairs except for 1*2*, 3*4*. Each element occurs 11 times, and so dualization gives the required packing. The weight bound then shows that D(2,11,79) = 16.

The weight bound for 80 is 17, with a weight of 4 corresponding to a distribution of $3^{27}, 2^{53}$. Again, we omit the pairs 1*2*, 3*4*. Our triples are

1*12	2*90	3*45	4*bc	7a1	a83	8c5
1*34	2*ab	3*67	4*13	8b1	b93	9a6
1*56	2*c1	3*89	4*25	9c2	c06	045
1*78	2*23	3*0a	4*46	072	7b4	

The weight bound for 81 is 18, with a weight of 0 corresponding to the distribution $3^{27}, 2^{53}$. We use the PBD generated by blocks 125 and 138, modulo 18, together with the remaining 45 pairs.

Since the weight bound for 82 is 18, we have D(2,11,82) = 18. Now consider v = 84; the weight bound is 21, with a weight of 0 corresponding to the distribution $3^{63}, 2^{21}$. We merely dualize a PBD obtained by cycling the blocks (1,2,5), (1,3,10), (1,6,12), (1,9), modulo 21. By deletion, we meet the weight bound for 83; so D(2,11, 83) = 19.

The weight bound for 85 is 22, with a weight of 4 corresponding to a distribution of $3^{72}, 2^{13}$. Again, we omit the pairs 1*2*, 3*4*. Our blocks are on elements 1,2, ... , 18, and 1*,2*,3*,4*. Take 36 triples generated by (1,5,11) and (1,3,8), modulo 18. Take triples generated by 1* with (1,2), (3,4), ... , (17,18); 2* with (2,3), (4,5), ... , (18,1); 3* with (1,4), (3,6), ... , (17,2); 4* with (2,5), (4,7), ... , (18,3). The pairs are (1,10), (2,11), ... , (9,18), together with 4 pairs on 1*, 2*, 3*, 4*.

We summarize the results of this section in Table 3.

v	D(v)	v	D(v)
67-69	12	80	17
70-72	13	81-82	18
74	14	83	19
75-77	15	84	21
78-79	16	85	22

Table 3. Packing numbers D(2,11,v) for 67 $\leq$ v $\leq$ 85

4. Packings Related to PBDs with Quadruples.

The weight bound for 86 is 24, with a weight of 0 corresponding to a frequency distribution of 4^6, 3^{80}. We generate a PBD by cycling blocks (1,3,12), (1,4,11), (1,2,6), modulo 24. We also use 8 blocks (1,9,17), (2,10,18), ... , (8,16,24), and 8 blocks (1,7,13,19), (2,8,14,20), ... , (6,12,18,24). The dual provides our packing.

Since the weight bound for 87 is 24, we also have D(2,11,87) = 24.

The weight bound for 88 is 25, with a weight of 6 corresponding to a distribution of 4^{11}, 3^{77}. We take 22 elements 1, 2, 3, ... , 22, and 3 elements 1*,2*,3*. Our quadruples are (2,7,15,20), (4,8,17,22), ... , (22,5,13,18). We cycle (1,4,11), modulo 22, for 22 blocks. Then we get 55 more blocks by taking (a) (1,3,7), (3,5,9), ... , (21,1,5); (b) (2,4,10), (4,6,12), ... , (22,2,8); (c) 1* with (1,12), (2,13), ... , (11,22); (d) 2* with (1,2), (3,4), (21,22); (e) 3* with (2,3), (4,5), ... , (22,1). All starred pairs are missing. The dual of this array is our required packing; consequently, D(2,11,88) = 25.

The weight bound for 89 is 26, with a weight of 2; this corresponds to a distribution of 4^{19}, 3^{70}.

The weight bound for 90 is 27, with a weight of 0 corresponding to a distribution of 4^{27}, 3^{63}. Form the required PBD by cycling (1,3,8,16), (1,2,5), and (1,7,17), modulo 27. We might note that there are many such designs; another is generated by (1,2,5,11), (1,3,14), (1,6,13). Hence D(2,11,90) = 27.

The weight bound for 91 is 28; the design weight is 0 corresponding to a frequency distribution of 4^{35}, 3^{56}. Form a PBD by taking 7 blocks (1,8,15,22) , ... , (7,14,21,28), and then cycling (1,5,6,18), (1,3,11), and (1,4,10). Another suitable design is given by cycling (1,2,5,14), (1,3,11), (1,6,12). Hence D(2,11,91) = 28.

The weight bound for 92 is 29, with a weight of 2 corresponding to a distribution of 4^{43}, 3^{49}.

The weight bound for 93 is 31, with a weight of 0 corresponding to a distribution of 4^{62}, 3^{31}. A PBD on 31 elements in 93 blocks is obtained by cycling (mod 31) the initial blocks (1,2,6,14), (1,3,12,18), and (1,4,11). Dualizing this PBD shows that D(2,11,93) = 31.

The weight bound for 94 is 34, with a weight of 6 corresponding to a distribution of 4^{92}, 3^{2}.

5. Packings Related to PBDs with Quintuples.

From the BIBD (45,99,11,5,1), we have D(2,11,99) = 45. Also, deletion (cf. [3]) shows that D(2,11,98) = 40.

The weight bound for 97 is 38, with a weight of 2 corresponding to a distribution of 5^{30}, 4^{67}.

The weight bound for 96 is 36, with a weight of 12 corresponding to a distribution of 5^{12}, 4^{84}. A PBD would have distribution 5^{18}, 4^{72},3^{6}.

The weight bound for 95 is 35, with a weight of 10 corresponding to a distribution of 5^{5}, 4^{90}. Since D(2,4,34) = 92, we can construct 92 quadruples on 34 elements; 28 of these elements occur 11 times, and the other 6 occur 3 times each. We get an (11,1) PBD in 95 blocks by adding 3 pairs. Dualizing proves that D(2,11,95) $\geq$ 34.

We summarize the results of the last two sections in Table 4.

v	D(v)	v	D(v)
86-87	24	94	32-34
88	25	95	34-35
89	25-26	96	34-36
90	27	97	36-38
91	28	98	40
92	28-29	99	45
93	31		

Table 4. Packing numbers D(2,11,v) for $86 \le v \le 99$

6. Packings in the region of the Conic Bound.

Since there is no geometry on 111 points, we can not use the lower bounds provided by a conic in the geometry (cf. **[13]**). So we can only use the upper bounds W given by the weight bound, and (as the example of $k = 7$ shows) these are not close. We record them in Table 5.

v	W	wt	v	W	wt
111	111	0	105	63	0
110	100	0	104	59	2
109	90	0	103	56	10
108	81	0	102	51	0
107	73	6	101	48	6
106	69	2	100	46	10

Table 5. Weight bounds on D(2,11,v) for $100 \le v \le 111$

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