On Directed Trades

Nasrin Soltankhah

Department of Mathematical Sciences
Sharif University of Technology
P.O. Box 11365–9415
Tehran, I.R. Iran
soltan@irearn.bitnet

Abstract. A (v,k,t) directed trade (or simply a (v,k,t)DT) of volume s consists of two disjoint collections T_1 and T_2, each containing s ordered k-tuples of distinct elements of a v-set V, called blocks, such that the number of blocks containing any t-tuple of V is the same in T_1 as in T_2. Our study shows that the volume of a (v,k,t)DT is at least $2^{\lfloor t/2 \rfloor}$ and that directed trades with minimum volume and minimum foundation exist. Also it is shown that for each $s \geq 2$, there exists a $(v,k,2)$DT and a $(v,k,3)$DT each of volume s, with one exception, that is, no $(v,4,3)$DT of volume three exists.

1 Introduction

Let $0 < t \leq k \leq v$ and $\lambda > 0$ be integers, and V be a set of v elements. In this note by an n-tuple of V, we mean an ordered n-subset of V. Each k-tuple of distinct elements of V is called a *block*. A t-(v,k,λ) *directed design* (or a simply t-(v,k,λ)DD) is a pair $(V,\mathcal{B})$, where V is a v-set, and $\mathcal{B}$ is a collection of blocks, such that each t-tuple of V appears in precisely λ blocks. Note that a t-tuple is said to appear in a k-tuple if its components are contained in that block as a set, and they appear in the same order. For example the 4-tuple *abcd* contains the ordered pairs *ab*, *ac*, *ad*, *bc*, *bd* and *cd*.

Definition. A (v,k,t) *directed trade* (or simply a (v,k,t)DT) of volume s consists of two disjoint collections T_1 and T_2, each of s blocks, such that the number of blocks containing any t-tuple of V is the same in T_1 as in T_2. When $s = 0$, the directed trade is said to be void.

Example 1. Some directed trades:

a $(7,4,2)$DT of volume 5:

T_1	T_2
1367	1357
1457	1467
2357	2367
2647	2457
3467	3647

a $(4,3,2)$DT of volume 2:

T_1	T_2
213	231
431	413

a $(4,4,3)$DT of volume 2:

T_1	T_2
1234	1243
2143	2134

Note that the definition of (v,k,t) directed trades allows repeated blocks in T_1 or in T_2. It is an easy exercise to prove that a (v,k,t) directed trade is also a (v,k,t') directed trade, for all t' with $0 < t' < t$.

Clearly, when a t-(v,k,λ) directed design D contains the collection of blocks of T_2 in a (v,k,t) directed trade, then by substituting the blocks of T_1 for the blocks of T_2 in the design, the resulting design is still a t-(v,k,λ) directed design. Thus by applying a proper directed trade to a given directed design, we may obtain a new directed design. This method is called the *method of trade off.* Therefore, it is important to understand the structure of directed trades, and conditions for their existence and nonexistence .

Directed designs were introduced in 1973 by Hung and Mendelsohn [4] and there are a few papers which deal with the existence of directed designs; for example, [1] and [7] and the references therein.

Trades have been used in the discussion of t-(v,k,λ) designs. Graver and Jurkat [2] called them null designs. There has been extensive research on (v,k,t) trades. For a survey on this, see Hedayat [3]. Papers by Hwang [5] and Mahmoodian and Soltankhah [6] deal with the existence and nonexistence of (v,k,t) trades.

In this paper we investigate the necessary and sufficient conditions for the existence of (v,k,t) directed trades. We have the following results: (for the definitions see Section 2)

(*i*) the minimum foundation size and minimum volume of a non-void (v,k,t)DT are k and $2^{\lfloor t/2 \rfloor}$ respectively;

(*ii*) (v,k,t)DTs with both the minimum foundation size k, and the minimum volume $2^{\lfloor t/2 \rfloor}$ exist;

(*iii*) for each $s \geq 2$, there exist at least a $(v,k,2)$DT and a $(v,k,3)$DT each of volume s, with one exception, that is, no $(v,4,3)$DT of volume three exists.

2 Definitions and preliminary results

Unless stated otherwise, all the directed trades in this paper are non-void.

(i) A (v,k,t)DT of volume s will be represented by

$$T = T_1 - T_2 = \sum_{i=1}^{s} B_{1i} - \sum_{i=1}^{s} B_{2i},$$

where B_{1i}'s and B_{2i}'s are the blocks contained in T_1 and T_2, respectively.

(ii) In a (v,k,t)DT, both collections of blocks must cover the same set of elements. This set of elements is called the *foundation* of the directed trade. The foundation of a directed trade T will be denoted by found(T). Thus by definition $|\text{found}(T)| \leq v$.

(iii) (v,k,t) trades and (v,k,t) directed trades may be obtained from each other. By arranging the elements of each block of a given (v,k,t) trade of volume s in, say, increasing (or decreasing) order, we obtain a (v,k,t)DT of volume s.
Also, if we consider the blocks of a given (v,k,t)DT of volume s to be unordered, we obtain a (v,k,t) trade of volume s', where $0 \leq s' \leq s$ (it should be noted that the foundation size may also decrease). If we consider the directed trades of example 1 to be unordered, we obtain a $(7,4,2)$ trade of volume 4 and two void trades respectively. This leads us to the following definition.

Definition. A directed trade is called *strictly directed* if when we consider its blocks without order then we obtain a void trade.

By definition, each strictly directed trade T has a structure such as the following:

$$T = T_1 - T_2 = \sum_{i=1}^{s} B_i - \sum_{i=1}^{s} B_i\alpha_i,$$

where each α_i is a permutation on the elements of B_i, for $i = 1,\ldots,s$.

Hwang [5] showed that, when $v < k+t+1$, there is no non-void (v,k,t) trade, while in the case of $v \geq k+t+1$, the volume of a (v,k,t) trade is at least 2^t. It follows from this result that each (v,k,t)DT with $|\text{found}(T)| < k+t+1$ or volume $s < 2^t$ must be a strictly directed trade. The second and the third cases of example 1 are strictly directed trades.

(iv) Let D be a collection of blocks and $x_1 \ldots x_i$ be an i-tuple of V, $0 < i < k$. We define $r_{D(x_1 \ldots x_i)}$ to be the number of blocks in D which contain $x_1 \ldots x_i$. To avoid messy notation, we shall use $r_{x_1 \ldots x_i}$ for $r_{D(x_1 \ldots x_i)}$.

(v) Let $T = T_1 - T_2$ and $T^* = T_1^* - T_2^*$ be two (v,k,t)DTs. Then it can be easily seen that $T + T^* = T_1T_1^* - T_2T_2^*$ and $T \setminus T^* = T_1T_2^* - T_2T_1^*$ are also (v,k,t)DTs, where for two collections A and B, AB denotes the union of A and B.

(vi) Let T be a (v,k,t)DT of volume s and the set of elements $\{x_1,\ldots,x_c\}$ be disjoint from found(T). Then by adding the "tail" $x_1 \ldots x_c$ to the end of each block of T, we obtain a $(v,k+c,t)$DT of volume s. Conversely, let T be a (v,k,t)DT of volume s, and suppose that $x_1,\ldots,x_c \in \text{found}(T)$ with $r_{x_i} = s$ for $1 \leq i \leq c$. Then by omitting these elements from all the blocks of T, we obtain a $(v,k-c,t)$DT of volume s', where $0 \leq s' \leq s$.

3 Necessary conditions

First we state the following lemma. Although this lemma is trivial, it provides the basis for some useful results which will be derived later.

Lemma 1. Let T be a (v,k,t)DT of volume s, and $x \in \text{found}(T)$ such that $r_x < s$. Let

$$T_{1x} = \sum_{i:B_{1i}\ni x} B_{1i} \quad , \quad T_{2x} = \sum_{i:B_{2i}\ni x} B_{2i}$$

and

$$T'_{1x} = \sum_{i:B_{1i}\not\ni x} B_{1i} \quad , \quad T'_{2x} = \sum_{i:B_{2i}\not\ni x} B_{2i}.$$

Then:

(i) $T_x = T_{1x} - T_{2x}$ is a $(v,k,t-1)$DT of volume r_x;

(ii) $T'_x = T'_{1x} - T'_{2x}$ is a $(v-1,k,t-1)$DT of volume $s - r_x$.

Now we can prove the following theorem.

Theorem 1. If T is a (v,k,t)DT, then:

(i) $|\text{found}(T)| \geq k$;

(ii) the volume of T is at least $2^{\lfloor t/2 \rfloor}$.

Proof. (i) is evident.
(ii) Proof is by induction on t. For t=1 there is nothing to prove. For t= 2,3, it can be easily seen that there exists no $(v,k,2)$DT and $(v,k,3)$DT of volume 1. Assume that $t > 3$ and the theorem holds for all values less than t. We show that it holds for t also. Hence we may assume the volume of a $(v,k,t-1)$DT is at least $2^{\lfloor (t-1)/2 \rfloor}$. Let T be a (v,k,t)DT. If there exists $x \in \text{found}(T)$ such that $r_x < s$, then by Lemma 1, T_x and T'_x are $(v,k,t-1)$DTs and by assumption each has volume at least $2^{\lfloor (t-1)/2 \rfloor}$, which in turn implies that the volume of T is at least $2^{\lfloor t/2 \rfloor}$. If for each $x \in \text{found}(T)$ $r_x = s$, then there exist $x, y \in \text{found}(T)$ such that $r_{xy} < s$. (Note that ordered 2-tuples xy and yx cannot appear in the same block). Thus T_{xy} (the blocks in T which contain the 2-tuple xy) and T'_{xy} (the blocks in T which do not contain the 2-tuple xy) are $(v,k,t-2)$DTs and by assumption each of them has volume at least $2^{\lfloor (t-2)/2 \rfloor}$, which it implies that the volume of T is at least $2^{\lfloor t/2 \rfloor}$. ∎

Definition. A (v,k,t)DT of foundation size k and volume $2^{\lfloor t/2 \rfloor}$ is called a *minimal directed trade.*

From Lemma 1 and Theorem 1(ii), we obtain the following fact about minimal directed trades.

Lemma 2. If T is a minimal directed trade, then for any i-tuple of V, $0 < i \leq t$,

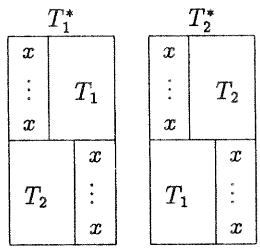
$$r_{T_{x_1 \ldots x_i}} = r_{T_{1(x_1 \ldots x_i)}} = r_{T_{2(x_1 \ldots x_i)}} = 2^{\lfloor t/2 \rfloor}, 2^{\lfloor t-1/2 \rfloor}, \ldots, 2^{\lfloor t-i/2 \rfloor}, \quad \text{or} \quad 0.$$

4 (v, k, t)DTs of minimum volume

In this section we show that (v, k, t)DTs with volume $2^{\lfloor t/2 \rfloor}$ exist for all $v \geq k$. First we state and prove two lemmas, from which we may obtain some new directed trades from a given directed trade.

Lemma 3. If there exists a (v, k, t)DT, T of volume s, then there exists a $(v+1, k+1, t+1)$DT, T^*, of volume $2s$.

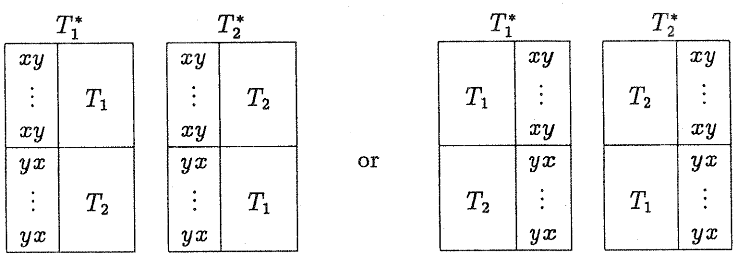
Proof. Let x be a new element. Then we can construct blocks of T^* as follows:



Clearly each T^* constructed in this way is a $(v+1, k+1, t+1)$DT of volume $2s$. □

Lemma 4. If there exists a (v, k, t)DT, T of volume s, then there exists a $(v+2, k+2, t+2)$DT, T^*, of volume $2s$.

Proof. Let x and y be two new elements. We can construct blocks of T^* as follows:



Clearly T^* is a $(v+2, k+2, t+2)$DT of volume $2s$. □

Theorem 2. Minimal (v, k, t) directed trades exist.

Proof. The theorem is established by applying Lemma 3 and Lemma 4 to a $(2, 2, 1)$DT of volume 1, namely $T_1 = 12$; $T_2 = 21$. ■

In a directed trade with minimum volume the foundation size can be greater than k. This is shown in the following theorem.

Theorem 3. The possible foundation sizes of a (v, k, t)DT of minimum volume are:

(i) $|\text{found}(T)| = k$, if t is odd;

(ii) $k \leq |\text{found}(T)| \leq 2k - t$, if t is even.

Proof. Let T be a (v, k, t)DT of minimum volume.
(i) If t is odd then by, Lemma 2, for each $x \in \text{found}(T)$, $r_x = 2^{\lfloor t/2 \rfloor}$. Thus the foundation size of T must equal k.
(ii) If t is even then by, Lemma 2, for each $x \in \text{found}(T)$, $r_x = 2^{\lfloor t/2 \rfloor}$ or $2^{\lfloor t-1/2 \rfloor}$. If the foundation size is greater than k, then there exists $x \in \text{found}(T)$ with $r_x = 2^{\lfloor t-1/2 \rfloor}$. By Lemma 1, each of T_x and T'_x is a $(v, k, t-1)$DT of minimum volume. Thus by (i) of this theorem $|\text{found}(T_x)| = |\text{found}(T'_x)| = k$ and each element in $\text{found}(T_x)$ or in $\text{found}(T'_x)$ appears in each block of T_x or in each block of T'_x respectively. Now there exists at least one t-tuple in T_{1x}, say $x_1 \cdots x_t$, which does not appear in T_{2x}, for otherwise T_x will be a (v, k, t)DT of volume $2^{\lfloor t-1/2 \rfloor}$, which is impossible. Then $x_1 \cdots x_t$ must appear in T'_{2x}. Thus $x_1, \cdots, x_t \in \text{found}(T_x)$ and $x_1, \cdots, x_t \in \text{found}(T'_x)$. Therefore these elements appear in each block of T. Now let a be the number of elements which appear in all blocks of T, and b be the number of elements which appear in exactly $2^{\lfloor t-1/2 \rfloor}$ blocks of T, so $a \geq t$. We have that $a + b = |\text{found}(T)|$ and $a.2^{\lfloor t/2 \rfloor} + b.2^{\lfloor t-1/2 \rfloor} = k.2^{\lfloor t/2 \rfloor}$. Since t is even, it follows that $2a + b = 2k$, and hence $2k - a = |\text{found}(T)|$, which implies that $|\text{found}(T)| = 2k - a \leq 2k - t$. ∎

5 Existence of some more (v, k, t)DTs

We first introduce the following lemma for the general case when $t \geq 1$.

Lemma 5. If T is a (v, k, t)DT of volume s, then for any $x \in \text{found}(T)$, either $r_x = s$ or $2^{\lfloor t-1/2 \rfloor} \leq r_x \leq s - 2^{\lfloor t-1/2 \rfloor}$.

Proof. This follows from Lemma 1 and Theorem 1(ii). □

In the case of ordinary trades, the minimum possible volume for a (v, k, t) trade is 2^t, and there does not exist a (v, k, t) trade of volume s, when $2^t + 1 \leq s \leq 2^t + 2^{t-1} - 1$, see [5] and [6]. The following theorems dealing with cases $t = 2$ and $t = 3$ show that no such general result holds for directed trades.

Theorem 4. For each $s \geq 2$, there exist directed trades of volume s for some v in the following cases:

(i) a $(v, k, 2)$DT for each k;

(ii) a $(v, k, 3)$DT for each k $(k \neq 4)$.

Proof. It is sufficient to show that there exists a $(v, 3, 2)$DT of volume s.
(i) If $s = 2l$, take l copies of a $(v, 3, 2)$DT of volume 2. If $s = 2l + 1$, take $l - 1$ copies

of a $(v,3,2)$DT of volume 2 and a $(v,3,2)$DT of volume 3 with a distinct foundation, namely

A $(4,3,2)$DT of volume 3:

T_1	T_2
123	213
231	312
324	234.

(ii) This case may be argued similar to the case (i) by applying a $(v,5,3)$DT of volume 3, namely

A $(5,5,3)$DT of volume 3:

T_1	T_2
21345	12354
13254	13245
12453	21453. ■

Theorem 5. A $(v,4,3)$DT of volume s exists if and only if $s=2$ or $s \geq 4$.

Proof. For the existence of a $(v,4,3)$DT of each volume s ($s \geq 2$, $s \neq 3$), if $s=2l$, take l copies of a $(v,4,3)$DT of volume 2. If $s=2l+1$ take $l-2$ copies of a $(v,4,3)$DT of volume 2 and a $(v,4,3)$DT of volume 5 with a distinct foundation, namely

A $(4,4,3)$DT of volume 5:

$$T_1 = \{1234, 1432, 2413, 3412, 3214\}; \; T_2 = \{3241, 3142, 2134, 4132, 1243\}.$$

Now we show that there is no $(v,4,3)$DT of volume 3. Let T be a $(v,4,3)$DT of volume 3. By Lemma 5, $r_x = 3$ for all $x \in \text{found}(T)$. Then there exist $x, y \in \text{found}(T)$ such that $r_{xy} < 3$, and $r_{xy} = 1$ or 2. Without loss of generality assume that $r_{xy} = 2$. Then by Lemma 1, T'_{xy}(or T_{yx}) is a $(v,4,1)$DT of volume 1. Also T_{yx} must contain all of the 3-tuples which contain yx. The only possibility for T_{1yx} is one of the 4-tuples $yabx$, $abyx$ or $yxab$. If T_{1yx} is $yabx$, then T_{2yx} must be $ybax$. Then yba and bax must appear in two disjoint blocks of T_{1xy} and yab and abx must appear in two disjoint blocks of T_{2xy}. It means that the 2-tuple ba appears twice in T_1 and once in T_2. This is a contradiction. If T_{1yx} is $abyx$, then T_{2yx} must be $bayx$. Thus bax must appear in T_{1xy}, and abx must appear in T_{2xy}. Therefore the block $baxy$ appears in T_1 and the block $abxy$ appears in T_2. But these two blocks in T_1 and in T_2 form a $(v,4,3)$DT, T_*, of volume 2, implying $T \setminus T_*$ is a $(v,4,3)$DT of volume 1, which is impossible. The last case may be argued similarly. ■

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References

[1] F.E. Bennett, A. Mahmoodi, R. Wei, and J. Yin. Existence of DBIBDs with

block size six. *Utilitas Math.*, 43:205–217, 1993.

[2] J.E. Graver and W.B. Jurkat. The module structure of integral designs. *J. Combin. Theory Ser. A*, 15:75–90, 1973.

[3] A. Hedayat. The theory of trade-off for t-designs. *Coding Theory and Design Theory, Part II, Design Theory, edited by D. Ray-Chaudhuri, IMA Volumes in Mathematics and its Applications, Springer-Verlag*, 21:101–126, 1990.

[4] S.H.Y. Hung and N.S. Mendelsohn. Directed triple systems. *J. Combin. Theory Ser. A*, 14:310–318, 1973.

[5] H.L. Hwang. On the structure of (v, k, t) trades. *J. Statist. Plann. and Inference*, 13:179–191, 1986.

[6] E.S. Mahmoodian and N. Soltankhah. On the existence of (v, k, t) trades. *Australas. J. Combin.*, 6:279–291, 1992.

[7] N. Soltankhah. Directed quadruple designs. *(submitted).*

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