A STABILITY THEOREM FOR THE AUTOMORPHISM GROUPS OF POWERS OF THE n-CUBE

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ABSTRACT

The kth power of a graph is, by definition, the graph obtained by joining all vertices at distance less than or equal to k in the original graph. The size of the automorphism groups of the kth powers of the n-dimensional cube, $1 \le k \le n$, arises naturally in an application of the second moment method for determining a probability threshold for bandwidth in hypercubes. In this paper we give an elementary method for determining these groups. The groups of the nth and $(n-1)$th powers are easily obtained. The group of the second power is calculated and shown to be strictly larger than that of the n-cube itself. However, to our surprise, the groups stabilize with the second power (until $n-1$), and we show that all the even powers (less than $n-1$) have groups equal to that of the second power, while all the odd powers (less than $n-1$) have groups equal to that of the original cube.

1. INTRODUCTION

The *k-th power* of a graph Γ is by definition the graph $\Gamma^{(k)}$ whose vertices are the vertices of Γ, two being adjacent if and only if they are distinct and at distance $\le k$ in Γ (see, for example, [**BCL**] or [**H**]). In this paper we are interested in determining by elementary methods the automorphism groups of powers of the n-cube $\Gamma(n)$. Denote by $G^{(k)}(n)$ the automorphism group of the k-th power of the n-cube, $\Gamma^{(k)}(n)$, for $1 \le k \le n$. In a standard application of the second moment method to finding a probability threshold for bandwidth in hypercubes [**BGM**], one needs the expected number of subgraphs isomorphic to $\Gamma^{(k)}(n)$ contained in a which provided our original motivation for investigating the behavior of $|G^{(k)}(n)|$ as k increases. Surprisingly we find that the value of $|G^{(k)}(n)|$ stabilizes very quickly, alternating between only two possible values depending on whether k is even or odd, until k reaches the special cases $n-1$ and n, where the answer is obvious. In view of the accelerating interest in the study of hypercubes, both in mathematics and computer science, we feel that this result may also be of some use in other investigations.

In the paper, [**M**], the author gives a characterization of the subschemes of the Hamming association scheme. The method there proceeds by identifying the Hamming association scheme as a cellular ring and then characterizing the subrings of this ring. From this characterization it is possible to deduce the main result of this paper (the Theorem below). However, this approach to our Theorem requires a substantial amount of algebraic machinery. Our purpose here is to present an elementary proof of this elegant and potentially useful result, so that it may be accessible to a wider audience. (In fact, we obtained the proof of our Theorem before learning of the work of Muzichuk. A less complete form was also presented in [**MP1**] and [**MP2**].) It is hoped that our results and the ideas in this paper will stimulate further investigations of this type.

In Section 2, we outline a proof that $G^{(1)}(n)$, the automorphism group of the cube, is isomorphic to a semi-direct product of $\mathbf{Z}_2^n$ with S_n, of order $2^n \cdot n!$, and $G^{(n)}(n)$ is isomorphic to S_{2^n}, of order $(2^n)!$. What is the situation for $2 \le k \le n-1$? We prove the following.

THEOREM. (*i*) For $n \ge 2$, $G^{(n-1)}(n)$ is isomorphic to a semidirect product of $\mathbf{Z}_2^{2^{n-1}}$ with $S_{2^{n-1}}$, of order $2^{2^{n-1}}(2^{n-1})!$.
(*ii*) For $n \ge 4$, $G^{(2)}(n)$ is isomorphic to a semi-direct product of $\mathbf{Z}_2^n$ with S_{n+1}, of order $2^n(n+1)!$.
(*iii*) For $2 \le k < n-1$, $G^{(k)}(n) = \begin{cases} G^{(2)}(n), & \text{if } k \text{ is ever} \\ G^{(1)}(n), & \text{if } k \text{ is odd} \end{cases}$.

(We note that S. Burr, [**B**], has independently proven (*i*) and the following part of (*iii*): for $2 \le k < n-1$ we have $G^{(k)}(n) = G^{(1)}(n)$ when k is odd, and $G^{(1)}(n) \le G^{(k)}(n) \le G^{(2)}(n)$ when k is even.)

In Section 2 of this paper, we collect together the definitions and notation to be used in the paper as well as give a description of the automorphism group of the n-cube in the form we shall need it later in the paper. The rest of the paper is devoted to proving the main theorem above. In Section 3, we prove parts (*i*) and (*ii*) of the theorem. In Section 4 we analyze the triangle structure in the powers of the n-cube, finally providing a count of the number of such triangles. This is crucial for the proof of part (*iii*) of the theorem.

2. DEFINITIONS, NOTATION AND PRELIMINARY RESULTS

We denote by $\Gamma(n)$ the graph of vertices and edges of the n-dimensional cube. The vertex set of $\Gamma(n)$ can be regarded as the set of all 2^n n-tuples of zeros and ones, i.e. as the vector space $\Omega = \mathbf{Z}_2^n$, where $\mathbf{Z}_2$ denotes the field of two elements, and where two vertices are adjacent if and only if the corresponding n-tuples differ in exactly one coordinate position.

Throughout this paper, if x and y are in V, then we will denote by $d(x, y)$ the *distance* from x to y in $\Gamma(n)$, i.e. $d(x, y)$ is the *Hamming distance* function, being the number of coordinates in which x and y differ. For convenience, denote by $\mathbf{0}$ the vertex $(0, 0, \dots, 0)$ and by $\mathbf{1}$ the vertex $(1, 1, \dots, 1)$, and define the *weight* of x to be $wt(x) = d(x, \mathbf{0})$ = the number of non-zero coordinates of x. So $d(x, y) = wt(x - y)$ and x is adjacent to y in $\Gamma(n)$ if and only if $d(x, y) = 1$, or, if and only if $wt(x - y) = 1$.

For $k = 1, \dots, n$, denote by e_k the vector in Ω consisting entirely of zeros except for a single one in the k-th coordinate position. Then the set $\Delta = \{e_k : k = 1, \dots, n\}$ forms a basis for Ω as a vector space over $\mathbf{Z}_2$, and, in $\Gamma(n)$, Δ is the set of vertices adjacent to $\mathbf{0}$.

Let $G(n) = \mathrm{Aut}(\Gamma(n))$, the automorphism group of the graph $\Gamma(n)$. We will view $G(n)$ as a semidirect product and, as consequences of this viewpoint will be used later in the paper, we will discuss them in some detail now. If we let ϕ_k be the function $\Omega \rightarrow \Omega$ defined by $\phi_k(x) = x + e_k$, for all $x \in \Omega$, then ϕ_k is in $G(n)$, and the subgroup N of $G(n)$ generated by ϕ_k, for $1 \le k \le n$, is a regular normal subgroup of $G(n)$. If $\phi \in N$, then there is a unique v_ϕ in Ω such that $\phi(x) = x + v_\phi$, for all $x \in \Omega$, so N is isomorphic to $\mathbf{Z}_2^n$. Since N is transitive on Ω, we have that $G(n) = N \cdot G_0$, where G_0 denotes the subgroup of $G(n)$ fixing the vertex $\mathbf{0}$.

We now examine in more detail the action of $G(n)$ on $\Gamma(n)$. We first remark that G_0 is isomorphic to the symmetric group S_n of degree n, as can be seen as follows. Any permutation of the n coordinate positions clearly preserves Hamming distance, fixes $\mathbf{0}$, and so is in G_0. On the other hand there is a natural map $\mu: G_0 \rightarrow S_n$ under which $\mu(\alpha)$, for α in G_0, is the permutation in S_n induced by the action of α on the points Δ of $\Gamma(n)$ adjacent to $\mathbf{0}$. Since G_0 acts faithfully (as a set of permutations) on Δ, this map is easily seen to be an injection. Now each permutation of the basis Δ may be extended by linearity to a permutation of Ω which is easily checked to be, in fact, an automorphism of the graph $\Gamma(n)$. So μ is surjective and $G_0 \cong S_n$. Thus we may view each $\alpha \in G_0$ as acting on an arbitrary vertex x of $\Gamma(n)$ in the manner specified above, so that, in particular, α permutes the coordinates of x according to the action of $\mu(\alpha)$ as an element of S_n.

Finally, given any σ in $G(n)$, with $\sigma = \phi\alpha$, for $\phi \in N$ and $\alpha \in G_0$, the action of σ on an arbitrary vertex x of $\Gamma(n)$ is given by $\sigma(x) = (\phi\alpha)(x) = \phi(\alpha(x)) = \alpha(x) + v_\phi$. We also observe that $G(n)$ is isomorphic to the semi-direct product of $\mathbf{Z}_2^n$ with S_n, of order $2^n \cdot n!$, where the action is coordinate-wise.

We have defined the k-th power of a graph Γ in section 1. We will let $\Gamma^{(1)} = \Gamma$. If we let $G^{(k)} = \mathrm{Aut}(\Gamma^{(k)})$, then it is not difficult to verify that $G^{(k)} \leq G^{(km)}$, for all $k, m \geq 1$.

Since the diameter (i.e. the maximum distance between any two vertices) of $\Gamma(n)$ is n, it is clear that for $k \geq n, \Gamma^{(k)}(n)$ is isomorphic to K_{2^n}, the complete graph on 2^n vertices, so that the corresponding group $G^{(k)}(n)$ is isomorphic to S_{2^n} of order $(2^n)!$. Thus, for the remainder of the paper we will assume when dealing with $\Gamma^{(k)}(n)$ and $G^{(k)}(n)$, that $2 \leq k \leq n-1$.

3. PROOF OF THE THEOREM, PARTS (*i*) AND (*ii*)

We now determine the structure of $G^{(k)}(n)$, for $2 \leq k \leq n-1$.

Part (*i*) $\qquad k = n - 1$.

In $\Gamma(n)$ every vertex x is at distance exactly n from the unique vertex $x + \mathbf{1}$ (where the addition is modulo 2), and so the complement of $\Gamma^{(n-1)}(n)$ is a matching consisting of 2^{n-1} edges. Since the group of $\Gamma^{(n-1)}(n)$ is isomorphic to the group of its complement, we have that $G^{(n-1)}(n)$ is isomorphic to the semidirect product of $\mathbf{Z}_2^{2^{n-1}}$ with $S_{2^{n-1}}$, of order $2^{2^{n-1}} \cdot (2^{n-1})!$, where the action is coordinate-wise. This completes the proof of case (*i*).

REMARK. For the remainder of the proof of the theorem, it is helpful to observe the following. Since $G(n) \leq G^{(k)}(n)$ for all k, the transitive group N defined in Section 2 above is a subgroup of $G^{(k)}(n)$ for all k. Thus $G^{(k)}(n) = N \cdot G_0^{(k)}$ for all k, where $G_0^{(k)}$ is the subgroup of $G^{(k)}(n)$ fixing the origin $\mathbf{0}$. Hence we will concentrate on the structure of $G_0^{(k)}$. Note also that $G_0^{(k)}$ is a subgroup of $G_0^{(km)}$ for all k and m.

Part (ii) $k = 2$.

For all m, $1 \leq m \leq n$, define the following permutation of the vertices of the n-cube.

$$\text{For } x \in \mathbf{Z}_2^n,\ \tau_m : \begin{cases} x \to x + e_m \text{, if } wt(x) \text{ is even and } x \text{ has 1 in position } m \\ x \to x + e_m \text{, if } wt(x) \text{ is odd and } x \text{ has 0 in position } m \\ x \to x \text{, otherwise}. \end{cases}$$

It is clear from this definition that τ_m fixes $\mathbf{0}$ for all m. We will show that in fact, since τ_m fixes $\mathbf{0}$ and changes the weight of some vertices, it follows that $\tau_m \in G_0^{(2)} - G_0$.

LEMMA 3.1 For all m, $\tau_m \in G_0^{(2)} - G_0$.

Proof. We will prove that $\tau_1 \in G_0^{(2)} - G_0$. Then the fact that $\tau_m \in G_0^{(2)} - G_0$, for all m, follows by observing that $G_0 \leq G_0^{(2)}$, and if α in G_0 maps e_1 to e_m, then $\alpha\tau_1\alpha^{-1} = \tau_m$ (since by the permutation action of α on the coordinate positions of elements of Ω, α maps the first coordinate to the m-th coordinate).

We have already observed that τ_1 fixes $\mathbf{0}$ and $\tau_1 \notin G_0$, so it remains to be shown that $\tau_1 \in G_0^{(2)}$. The proof of this fact is facilitated by the observation that, in fact, τ_1 is a *linear transformation* of the vector space $\mathbf{Z}_2^n$. To see this, let T be the n by n matrix with 1's in the first row, 0's in the remainder of the first column, and the rest an $n-1$ by $n-1$ identity matrix;

$$T = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then it is easily verified that for all vectors $x \in \mathbf{Z}_2^n$, we have $\tau_1(x) = Tx$ (with reference to the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbf{Z}_2^n$).

Now for all i, j, $1 \leq i \neq j \leq n$, let e_{ij} be the vector in Ω consisting entirely of zeros except for 1's in both the i and j coordinate positions. Then, for $x \neq y$, $d(x, y) \leq 2 \Leftrightarrow wt(x - y) \leq 2 \Leftrightarrow x - y \in \Delta'$, where $\Delta' = \{e_k, e_{ij} : 1 \leq k \leq n,\ 1 \leq i \neq j \leq n\}$. Thus, for all $x \neq y$, with $x - y \in \Delta'$, we have $\tau_1 \in G_0^{(2)} \Leftrightarrow d(\tau_1(x),$

$\tau_1(y)) \le 2 \Leftrightarrow \tau_1(x) - \tau_1(y) \in \Delta' \Leftrightarrow T(x-y) \in \Delta'$, since T is linear, $\Leftrightarrow T$ fixes Δ' as a set. This last condition is easily verified so $\tau_1 \in G_{\mathbf{0}}^{(2)}$.

This completes the proof of the lemma. ∎

LEMMA 3.2 The subgroup of $G_{\mathbf{0}}^{(2)}$ generated by $G_{\mathbf{0}}$ and τ_1 is isomorphic to the symmetric group S_{n+1} acting (linearly) on $\mathbf{Z}_2^n$.

Proof. We can define an action of S_{n+1} on $\mathbf{Z}_2^n$, which is linear, as follows. Let $e_{n+1} = e_1 + \dots + e_n$. Then the action of S_{n+1} on $\{e_1, \dots, e_n, e_{n+1}\}$, defined by the natural permutation of the subscripts, can be shown to extend linearly to a group H of linear transformations of $\mathbf{Z}_2^n$. As a subgroup of the group GL$(n, 2)$ of linear transformations of $\mathbf{Z}_2^n$, $H = \langle G_{\mathbf{0}}, \sigma\rangle$, where σ is the involution interchanging e_1 with e_{n+1} and fixing all other e_k. Now let S be the n by n matrix with 1's in the first column, 0's in the remainder of the first row, and the rest an $n-1$ by $n-1$ identity matrix;

$$S = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then it is easily verified that for all vectors $x \in \mathbf{Z}_2^n$, we have $\sigma(x) = Sx$ (with reference to the standard basis $\{e_1, \dots, e_n\}$ of $\mathbf{Z}_2^n$).

Now, as a group of matrices in GL$(n, 2)$, $G_{\mathbf{0}}$ can be generated by symmetric matrices. Since $S^* = T$, where $*$ denotes transpose, we thus see that $\langle G_{\mathbf{0}}, \tau_1\rangle \cong H \cong S_{n+1}$. ∎

LEMMA 3.3 For $n \ge 4$ we have $G_{\mathbf{0}}^{(2)} \cong S_{n+1}$.

Proof. Let $\Delta' = \{e_k, e_{ij} : 1 \le k \le n, 1 \le i \ne j \le n\}$ be as in the proof of lemma 3.1. Then by definition of the second power of a graph, $G_{\mathbf{0}}^{(2)}$ acts on Δ', and since its subgroup $G_{\mathbf{0}}$ is transitive on the vectors of weight 1 and 2 in Ω, by lemma 3.1 $G_{\mathbf{0}}^{(2)}$ is transitive on Δ'. Further $|\Delta'| = n + \binom{n}{2} = \binom{n+1}{2}$. Hence $|G_{\mathbf{0}}^{(2)} : G_{\mathbf{0}, e_1}^{(2)}| = \binom{n+1}{2}$, where $G_{\mathbf{0}, e_1}^{(2)}$ is the subgroup of $G^{(2)}(n)$ fixing both $\mathbf{0}$ and e_1. We investigate the structure of $G_{\mathbf{0}, e_1}^{(2)}$.

Let $\alpha \in G_{\mathbf{0}, e_1}$, the subgroup of $G(n)$ fixing both $\mathbf{0}$ and e_1. From the first paragraph of the proof of lemma 3.1, we have $\alpha\tau_1\alpha^{-1} = \tau_1$, so $\tau_1\alpha = \alpha\tau_1$. Hence, $G^{(2)}_{\mathbf{0},\,e_1} \geq \langle G_{\mathbf{0},\,e_1}, \tau_1\rangle = G_{\mathbf{0},\,e_1} \times \langle\tau_1\rangle \cong S_{n-1} \times \mathbf{Z}_2$.

Let $\Sigma = \{e_k, e_{1j} : 2 \leq k \leq n,\ 2 \leq j \leq n\} \subset \Delta'$. Then Σ is an orbit for the action of $G^{(2)}_{\mathbf{0},\,e_1}$ on Δ'. Since $d(e_k, e_{1j}) = 3$ for $1 \neq k \neq j$, $d(e_k, e_{1k}) = 1$ for $1 \neq k$, and $d(e_k, e_j) = d(e_{1k}, e_{1j}) = 2$ for $j \neq k$, the subgraph of $\Gamma^{(2)}(n)$ induced by Σ is isomorphic to $K_{n-1} \times K_2$, the product of the complete graphs K_{n-1} and K_2. Now, the automorphism group of this product is isomorphic to the direct product $S_{n-1} \times \mathbf{Z}_2$, if $n > 3$, and to D_4, the dihedral group of degree 4 and order 8, if $n = 3$. Since it is easily verified that $G^{(2)}_{\mathbf{0},\,e_1}$ acts faithfully on Σ, we thus have $G^{(2)}_{\mathbf{0},\,e_1} = G_{\mathbf{0},\,e_1} \times \langle\tau_1\rangle \cong S_{n-1} \times \mathbf{Z}_2$.

So now we have $|G^{(2)}_{\mathbf{0}}| = \binom{n+1}{2}\cdot 2(n-1)! = (n+1)!$. Hence by lemma 3.2, $G^{(2)}_{\mathbf{0}} = \langle G_{\mathbf{0}}, \tau_1\rangle \cong S_{n+1}$, completing the proof of the lemma. ∎

This completes the proof of part *(ii)* of the theorem.

4. PROOF OF THE THEOREM, PART *(iii)*

The idea in the proof of part *(iii)* of the theorem is to analyze the triangle structure in the powers of the n-cube by obtaining a count of the number of triangles having an edge whose vertices were at a specified distance in the original cube. Then, by using the fact that the automorphism group of the kth power of the n-cube fixes (as a set) unions of certain of these triangles, we are able to reach the conclusion. We first have the following definitions and results in $\Gamma^{(k)}(n)$.

Suppose that for x and y in Ω, $d(x, y) = m \leq k$. Then x is adjacent to y in $\Gamma^{(k)}(n)$, and we call the edge from x to y in $\Gamma^{(k)}(n)$ an *m-edge*. For x in Ω, with $d(\mathbf{0}, x) = m$, let $T^{(k)}(m)$ be the number of vertices (in $\Gamma^{(k)}(n)$) adjacent to both $\mathbf{0}$ and x. Then it can be seen that $T^{(k)}(m)$ is independent of x (with $d(\mathbf{0}, x) = m$). Also, if $m \leq k$, then $T^{(k)}(m)$ is the number of triangles in $\Gamma^{(k)}(n)$ on the m-edge from $\mathbf{0}$ to x.

For all $i \leq n$ and $w \in \Omega$, let $\Delta_i(w) = \{y \in \Omega : d(y, w) = i\}$. For $x \in \Delta_p(\mathbf{0})$, let $\mu_{pqr} = |\Delta_q(\mathbf{0}) \cap \Delta_r(x)|$.

LEMMA 4.1 *(a)* We have $\mu_{pqr} = \binom{p}{s}\binom{n-p}{q-s}$, where s is such that $2s = p + q - r$ (and so μ_{pqr} is independent of the choice of $x \in \Delta_p(\mathbf{0})$).

(*b*) $\mu_{pqr} = \mu_{prq}$; $\mu_{pqr} = 0$ if $p + q + r$ is odd; and $\mu_{pqr} = 0$ if any of $|p - q| \le r$, $|q - r| \le p$, or $|r - p| \le q$ fail to hold.
(*c*) $T^{(k)}(m) = \sum_{q,\,r} \mu_{mqr}$, where the sum is taken over all $1 \le q,\ r \le k$.

Proof. (*a*) Let y be in $\Delta_q(\mathbf{0}) \cap \Delta_r(x)$. Suppose that y has exactly s coordinates equal to 1 in the positions where x has its coordinates equal to 1. Then, since $wt(y) = q$, y has exactly $q - s$ coordinates equal to 1 in the positions where x has its coordinates equal to 0. The number of such y's is thus $\binom{p}{s}\cdot\binom{n-p}{q-s}$. Now since x and y differ in exactly r coordinates, we have that $(p - s) + (q - s) = r$, whence $2s = p + q - r$.

The proofs of (*b*) and (*c*) are left to the reader. ∎

LEMMA 4.2 (*a*) If $k = n$, then $T^{(k)}(m) = 2^n - 2$ for all m.
(*b*) If $k = n-1$, then $T^{(k)}(m) = 2^n - 4$ for all $m \le n - 1$, and $T^{(k)}(n) = 2^n - 2$.
(*c*) For all $k \ge 2$, $T^{(k)}(1) = T^{(k)}(2) = 2\left[\binom{n-1}{1} + \binom{n-1}{2} + \ldots + \binom{n-1}{k-1}\right]$.

Proof. Part (*a*) follows from the fact that $\Gamma^{(n)}(n)$ is isomorphic to K_{2^n}, and part (*b*) follows from the description of $\Gamma^{(n-1)}(n)$ given in the proof of part (*i*) in section 3.

(*c*) The proof is by induction on $k \ge 2$. When $k = 2$,

$$T^{(2)}(1) = \mu_{112} + \mu_{121} = 2\mu_{112} = 2\binom{1}{0}\binom{n-1}{1} = 2\binom{n-1}{1}, \text{ and}$$
$$T^{(2)}(2) = \mu_{211} + \mu_{222} = \binom{2}{1}\binom{n-2}{0} + \binom{2}{1}\binom{n-2}{1} = 2\binom{n-1}{1}.$$

For $k > 2$,
$$\begin{aligned} T^{(k)}(1) &= T^{(k-1)}(1) + \mu_{1(k-1)k} + \mu_{1k(k-1)}, \quad \text{since } \mu_{1kk} = 0 \text{ by Lemma 4.1}(b) \\ &= T^{(k-1)}(1) + 2\mu_{1(k-1)k} \\ &= T^{(k-1)}(1) + 2\binom{n-1}{k-1}. \end{aligned}$$

Also,
$$\begin{aligned} T^{(k)}(2) &= T^{(k-1)}(2) + \mu_{2(k-2)k} + \mu_{2kk} + \mu_{2k(k-2)} \\ &= T^{(k-1)}(2) + 2\mu_{2(k-2)k} + \mu_{2kk} \\ &= T^{(k-1)}(2) + 2\binom{2}{0}\binom{n-2}{k-2} + \binom{2}{1}\binom{n-2}{k-1} \\ &= T^{(k-1)}(2) + 2\binom{n-1}{k-1}, \text{ and we are done.} \end{aligned}$$
∎

LEMMA 4.3 (*a*) For all m with $2 \le 2m \le k$, $T^{(k)}(2m-1) = T^{(k)}(2m)$.
(*b*) For all m with $3 \le 2m+1 \le k$, $T^{(k)}(2m) = T^{(k)}(2m+1) + D^{(k)}(m)$, where

$$D^{(k)}(m) = \binom{2m}{m}\cdot\binom{n-2m-1}{k-m} .$$

Proof. We first observe the following. From Lemma 4.1(*c*), for all $m \le k$,

$$T^{(k)}(m) = \sum_{q,\,r} \mu_{mqr} = \sum_{q,\,r} \binom{m}{s}\cdot\binom{n-m}{q-s},$$

where the sum is taken over all q and r with $1 \le q, r \le k$, and s is such that $2s = m+q-r$. By the restriction $|q-r| \le m$, we have $-m \le r-q \le m$, so that s takes on all values from 0 to m. Observe that in this sum, $q-s = s+r-m$, so $q-s \le s+k-m$. Further, there is no term with both $s = m$ and $q-s = 0$, since such a term corresponds to $r = 0$.

Now in the above sum, let $t = q-s \le k-s$ so that there are no terms with $s+k-m+1 \le t \le k-s$, or, equivalently, there are no terms corresponding to both $s \le (m-1)/2$ and $s+k-m+1 \le t$.

In the expansion of $(1+x)^n$ in powers of x, denote by $S(k)$ the sum of the coefficients of x^q, for $1 \le q \le k$. Then from the preceeding paragraphs, by writing $(1+x)^n$ as the product $(1+x)^m\cdot(1+x)^{n-m}$, we see that $T^{(k)}(m) = S(k) - 1 - E^{(k)}(m)$, where the *excess*, $E^{(k)}(m)$, is given by

$$E^{(k)}(m) = \sum_{s=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor} \sum_{t=s+k-m+1}^{k-s} \binom{m}{s}\binom{n-m}{t} .$$

(*a*) Now suppose that $2 \le 2m \le k$. Then

$$E^{(k)}(2m-1) = \sum_{s=0}^{m-1} \sum_{t=s+k-2m+2}^{k-s} \binom{2m-1}{s}\cdot\binom{n-2m+1}{t} \text{, and}$$

$$E^{(k)}(2m) = \sum_{s=0}^{m-1} \sum_{t=s+k-2m+1}^{k-s} \binom{2m}{s}\cdot\binom{n-2m}{t}$$

$$= \sum_{s=0}^{m-1} \sum_{t=s+k-2m+1}^{k-s} \left[\binom{2m-1}{s-1} + \binom{2m-1}{s}\right]\binom{n-2m}{t} \text{ , where } \binom{2m-1}{-1} = 0$$

$$= \sum_{s=0}^{m-2} \sum_{t=s+k-2m+2}^{k-s-1} \binom{2m-1}{s}\binom{n-2m}{t} + \sum_{s=0}^{m-1} \sum_{t=s+k-2m+2}^{k-s+1} \binom{2m-1}{s}\binom{n-2m}{t-1}$$

$$= \sum_{s=0}^{m-2} \sum_{t=s+k-2m+2}^{k-s-1} \binom{2m-1}{s}\left[\binom{n-2m}{t} + \binom{n-2m}{t-1}\right]$$

$$+ \sum_{s=0}^{m-1} \binom{2m-1}{s}\left[\binom{n-2m}{k-s-1} + \binom{n-2m}{k-s}\right]$$

$$= \sum_{s=0}^{m-2} \sum_{t=s+k-2m+2}^{k-s-1} \binom{2m-1}{s}\binom{n-2m+1}{t} + \sum_{s=0}^{m-1} \binom{2m-1}{s}\binom{n-2m+1}{k-s}$$

$$= \sum_{s=0}^{m-1} \sum_{t=s+k-2m+2}^{k-s} \binom{2m-1}{s}\binom{n-2m+1}{t} = E^{(k)}(2m-1).$$

Thus $T^{(k)}(2m-1) = T^{(k)}(2m)$.

(b) Now suppose that $3 \le 2m+1 \le k$. Then

$$E^{(k)}(2m) = \sum_{s=0}^{m-1} \sum_{t=s+k-2m+1}^{k-s} \binom{2m}{s}\binom{n-2m}{t} \text{ , and}$$

$$E^{(k)}(2m+1) = \sum_{s=0}^{m} \sum_{t=s+k-2m}^{k-s} \binom{2m+1}{s}\binom{n-2m-1}{t}$$

$$= \sum_{s=0}^{m} \sum_{t=s+k-2m}^{k-s} \left[\binom{2m}{s-1} + \binom{2m}{s}\right]\binom{n-2m-1}{t} \text{ , where } \binom{2m}{-1} = 0$$

$$= \sum_{s=0}^{m-1} \sum_{t=s+k-2m+1}^{k-s-1} \binom{2m}{s}\binom{n-2m-1}{t} + \sum_{s=0}^{m} \sum_{t=s+k-2m+1}^{k-s+1} \binom{2m}{s}\binom{n-2m-1}{t-1}$$

$$= \sum_{s=0}^{m-1} \sum_{t=s+k-2m+1}^{k-s-1} \binom{2m}{s}\left[\binom{n-2m-1}{t} + \binom{n-2m-1}{t-1}\right]$$

$$+\sum_{s=0}^{m-1}\binom{2m}{s}\left[\binom{n-2m-1}{k-s-1}+\binom{n-2m-1}{k-s}\right]+\binom{2m}{m}\binom{n-2m-1}{k-m}$$

$$=\sum_{s=0}^{m-1}\sum_{t=s+k-2m+1}^{k-s-1}\binom{2m}{s}\binom{n-2m}{t}+\sum_{s=0}^{m-1}\binom{2m}{s}\binom{n-2m}{k-s}+\binom{2m}{m}\binom{n-2m-1}{k-m}$$

$$=\sum_{s=0}^{m-1}\sum_{t=s+k-2m+1}^{k-s}\binom{2m}{s}\binom{n-2m}{t}+D^{(k)}(m)$$

$$=E^{(k)}(2m)+D^{(k)}(m).$$

Thus $T^{(k)}(2m) = T^{(k)}(2m+1) + D^{(k)}(m)$. ∎

COROLLARY 4.4 For $2m + 1 \le k$, we have

$$T^{(k)}(2m+1) = 2\sum_{s=1}^{k-1}\binom{n-1}{s} - \sum_{s=1}^{m}\binom{2s}{s}\binom{n-2s-1}{k-s},$$

and $T^{(k)}(2i-1) \neq T^{(k)}(2j-1)$ for any $1 \le i \neq j \le (k+1)/2$.

Proof. The corollary follows by lemma 4.2 (c) and by lemma 4.3 (a) and (b) (from which it is also clear that $T^{(k)}(2i-1) > T^{(k)}(2j-1)$ for any $1 \le i < j \le (k+1)/2$). ∎

We now finish the proof of the theorem.

Part (iii) $2 \le k < n-1$ (so $n \ge 4$).

Suppose k is even. Then certainly $G^{(2)}(n) \le G^{(k)}(n)$. Let σ be an element of $G^{(k)}(n)$. As pointed out earlier, $G^{(k)}(n) = N \cdot G_0^{(k)}$ where the subgroup N is transitive on the vertices of the graph and preserves the weight of vectors. Thus to show that σ is in $G^{(2)}(n)$, we may assume that σ is in $G_0^{(k)}$ and it then suffices to show that σ fixes the set $\Delta_1(\mathbf{0}) \cup \Delta_2(\mathbf{0})$ of vertices of weight 1 or 2. The latter assertion now follows by corollary 4.4 and lemma 4.3 (a), since σ has to map the set of triangles on any 1-edge (or 2-edge) of $\Gamma^{(k)}(n)$ onto the set of triangles on the image of that 1-edge (or 2-edge).

Suppose now that k is odd. Again we get, as in the preceding paragraph, that any automorphism σ in $G^{(k)}(n)$ must be in $G^{(2)}(n)$. We now show, in fact, that σ is in $G^{(1)}(n)$.

Again we may assume σ fixes $\mathbf{0}$ and suppose to the contrary that σ maps the 1-edge $(\mathbf{0}, \boldsymbol{x})$ to the 2-edge $(\mathbf{0}, \boldsymbol{y})$. Now since k is odd, by corollary 4.4 and lemma 4.3 (a), σ must fix the set $\Delta_k(\mathbf{0})$ of vectors of weight k. It is easy to show that the number of $(k-1)$-edges $(\boldsymbol{x}, z)$, with z in $\Delta_k(\mathbf{0})$, is $\binom{n-1}{k-1}$, and the number of $(k-2)$-edges $(\boldsymbol{y}, z)$, with z in $\Delta_k(\mathbf{0})$, is $\binom{n-2}{k-2}$. Since σ maps the 1-edge $(\mathbf{0}, \boldsymbol{x})$ to the 2-edge $(\mathbf{0}, \boldsymbol{y})$, by lemma 4.3 (a), σ must map the former set of $(k-1)$-edges onto the latter set of $(k-2)$-edges, (since $k-2$ is odd, and $k-1$ is even). Hence we must have $\binom{n-1}{k-1} = \binom{n-2}{k-2}$, from which we get $k = n$, a contradiction. Thus σ must fix the set $\Delta_1(\mathbf{0})$, whence σ is in $G^{(1)}(n)$.

This concludes the proof of the theorem. ∎

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