On shifted intersecting families with respect to posets

Morimasa Tsuchiya

Department of Mathematical Sciences

Tokai University

Hiratsuka, Kanagawa 259-12, JAPAN

Abstract. In this paper, we show that for a shifted complex $\mathcal{I} \subseteq 2^P$ with respect to a poset $P$ with minimum element $0$ and an intersecting subfamily $\mathcal{G} \subseteq \mathcal{I}$, $\#\mathcal{G} \leq \#\{ F \in \mathcal{I} ; 0 \in F \}$.

We denote the set $\{ 1,2,\ldots,n \}$ by $[n]$, the family of all subsets of a set $X$ by $2^X$. $\# F$ denotes the number of elements of a set $F$. Let $\mathcal{I}$ be a family of subsets of $[n]$, i.e., $\mathcal{I} = \{ F_1,\ldots,F_m \}$ where $F_1,\ldots,F_m$ are distinct subsets of $[n]$. A family $\mathcal{I}$ is intersecting if for every $F_i, F_j \in \mathcal{I}$, $F_i \cap F_j \neq \emptyset$. For families $\mathcal{G}, \mathcal{I} \subseteq 2^{[n]}$, $\mathcal{G}$ and $\mathcal{I}$ are cross-intersecting if $G \cap F \neq \emptyset$ for $\forall G \in \mathcal{G}$ and $\forall F \in \mathcal{I}$. A family $\mathcal{I} \subseteq 2^{[n]}$ is called a complex if $G \subseteq F \in \mathcal{I}$ implies $G \in \mathcal{I}$. We already know the following results. For an intersecting family $\mathcal{I} \subseteq 2^{[n]}$, $\# \mathcal{I} \leq 2^{n-1}$ ([1]) and for a complex $\mathcal{I} \subseteq 2^{[n]}$ and cross-intersecting subfamilies $\mathcal{G}, \mathcal{H} \subseteq \mathcal{I}$, $\# \mathcal{G} + \# \mathcal{H} \leq \# \mathcal{I}$ ([4]).

For $F, G \subseteq [n]$, if there exists a one-to-one mapping $f : F \rightarrow G$ with $x \leq f(x)$ for each $x \in F$, then we write $F \leq G$. $\mathcal{I} \subseteq 2^{[n]}$

Australasian Journal of Combinatorics 5(1992), pp.53-58
is \( V \)-hereditary if \( G \trianglelefteq F \in \mathcal{F} \) implies \( G \in \mathcal{F} \). V.Chvátal introduced this notion and proved the next result.

**Theorem A (1974 [2]).** Let \( \mathcal{F} \subseteq 2^{[n]} \) be a \( V \)-hereditary family and \( \mathcal{G} \) be an intersecting subfamily of \( \mathcal{F} \). Then \( \# \mathcal{G} \leq \# \{ F \in \mathcal{F} ; 1 \in F \} \). ■

H.Era extended the notion of V.Chvátal and also showed the following result. Let \( P \) be a finite ranked poset with the minimum element 0. For \( F, G \subseteq P \), if there exists a one-to-one mapping \( f : F \rightarrow G \) with \( x \leq f(x) \) in \( P \) or \( x \) and \( f(x) \) are incomparable for each \( x \in F \), then we write \( F \leq P G \). \( \mathcal{F} \subseteq 2^{P} \) is \( P \)-hereditary if \( G \leq P F \in \mathcal{F} \) implies \( G \in \mathcal{F} \).

**Theorem B ([3]).** Let \( P \) be a finite ranked poset with the minimum element 0 and \( \mathcal{F} \subseteq 2^{P} \) be a \( P \)-hereditary family. For an intersecting subfamily \( \mathcal{G} \) of \( \mathcal{F} \), \( \# \mathcal{G} \leq \# \{ F \in \mathcal{F} ; 0 \in F \} \). ■

Let \( P \) be a finite poset with the minimum element 0. For a family \( \mathcal{F} \subseteq 2^{P} \) and \( \alpha \leq \beta \) in \( P \), we define
\[
S_{\alpha, \beta}(F) = \begin{cases} 
(F - \{ \beta \}) \cup \{ \alpha \} & \text{if } \alpha \notin F, \beta \in F, (F - \{ \beta \}) \cup \{ \alpha \} \notin \mathcal{F} \\
F & \text{otherwise}
\end{cases}
\]
for each \( F \in \mathcal{F} \) and \( S_{\alpha, \beta}(\mathcal{F}) = \{ S_{\alpha, \beta}(F) ; F \in \mathcal{F} \} \). Then \( \#S_{\alpha, \beta}(\mathcal{F}) = \#\mathcal{F} \) and if \( \mathcal{F} \) is complex and intersecting, then \( S_{\alpha, \beta}(\mathcal{F}) \) is also complex and intersecting.

**Proposition 1.** For a finite poset \( P \) and \( \alpha \leq \beta \) in \( P \), if \( \mathcal{F} \subseteq 2^{P} \) is complex, then \( S_{\alpha, \beta}(\mathcal{F}) \) is also complex.

**Proof.** We suppose that there exist \( G, F \subseteq P \) such that \( G \subseteq F \in S_{\alpha, \beta}(\mathcal{F}) \) and \( G \notin S_{\alpha, \beta}(\mathcal{F}) \).

Case 1. \( F \in \mathcal{F} \).
Since \( \mathcal{I} \) is complex, \( G \in \mathcal{I} \). So \( \alpha \notin G, \beta \in G, (G-\{\beta\}) \cup \{\alpha\} \notin \mathcal{I} \) and \( \beta \in F \). If \( \alpha \in F \), then \( (G-\{\beta\}) \cup \{\alpha\} \subseteq F \), which contradicts the property that \( \mathcal{I} \) is complex. If \( \alpha \notin F \), then \( (F-\{\beta\}) \cup \{\alpha\} \in \mathcal{I} \) and \( (G-\{\beta\}) \cup \{\alpha\} \subseteq (F-\{\beta\}) \cup \{\alpha\} \), which contradicts the property that \( \mathcal{I} \) is complex.

Case 2. \( F \notin \mathcal{I} \).

Then \( \alpha \in F, \beta \notin F \) and \( (F-\{\alpha\}) \cup \{\beta\} \in \mathcal{I} \). If \( G \in \mathcal{I} \), then \( \alpha \notin G, \beta \in G \) and \( (G-\{\beta\}) \cup \{\alpha\} \notin \mathcal{I} \). So \( G \notin F \), which is a contradiction. If \( G \notin \mathcal{I} \), then \( (G-\{\alpha\}) \cup \{\beta\} \subseteq (F-\{\alpha\}) \cup \{\beta\} \) and \( G' = (G-\{\alpha\}) \cup \{\beta\} \in \mathcal{I} \). Since \( G' \cap \{\alpha, \beta\} = \{\beta\}, (G'-\{\beta\}) \cup \{\alpha\} = G \in S_{\alpha, \beta}(\mathcal{I}) \), which is a contradiction. ■

**Proposition 2.** For a finite poset \( P \) and \( \alpha \leq \beta \) in \( P \), if \( \mathcal{I} \subseteq 2^P \) is intersecting, then \( S_{\alpha, \beta}(\mathcal{I}) \) is also intersecting.

**Proof.** We suppose that there exist \( G, F \in S_{\alpha, \beta}(\mathcal{I}) \) such that \( G \cap F = \emptyset \). Since \( \mathcal{I} \) is intersecting, both of \( G \) and \( F \) do not belong to \( \mathcal{I} \). We assume that \( F \notin \mathcal{I} \). Thus there exists \( H \in \mathcal{I} \) such that \( S_{\alpha, \beta}(H) = F \) and \( H \notin F \). By the definition of \((\alpha, \beta)\)-shifting, \( H = (F-\{\alpha\}) \cup \{\beta\} \in \mathcal{I} \), \( \alpha \in F \) and \( \beta \notin F \). If \( G \notin \mathcal{I} \), then \( \alpha \in G \) and \( F \cap G \neq \emptyset \), which is a contradiction. Thus \( G \in \mathcal{I}, \beta \in G \) and \( \alpha \notin G \).

Since \( S_{\alpha, \beta}(G) = G, (G-\{\beta\}) \cup \{\alpha\} \in \mathcal{I} \) by the definition of \((\alpha, \beta)\)-shifting. Then \((F-\{\alpha\}) \cup \{\beta\}) \cap ((G-\{\beta\}) \cup \{\alpha\}) = ((F-\{\alpha\}) \cap (G-\{\beta\})) \cup (\{\beta\} \cap (G-\{\beta\})) \cup ((F-\{\alpha\}) \cap \{\alpha\}) \cup (\{\alpha\} \cap \{\beta\}) = (F-\{\alpha\}) \cap (G-\{\beta\}) = \emptyset \), contradicting the fact that \( \mathcal{I} \) is an intersecting family. ■

A family \( \mathcal{I} \) is *shifted* if \( S_{\alpha, \beta}(\mathcal{I}) = \mathcal{I} \) for all \( \alpha, \beta \) such that \( \alpha < \beta \) in \( P \). We obtain the following result which is concerned with shifted complexes and intersecting families.
Theorem 3. Let $P$ be a finite poset with the minimum element 0 and $\mathcal{F} \subseteq 2^P$ be a shifted complex. For an intersecting subfamily $\mathcal{G}$ of $\mathcal{F}$, $\# \mathcal{G} \leq \# \{ F \in \mathcal{F} ; 0 \in F \}$.

Proof. Let $\mathcal{F}(0) = \{ F - \{0\} ; 0 \in F \in \mathcal{F} \}$ and $\mathcal{F}_0 = \{ F \in \mathcal{F} ; 0 \notin F \}$. By Proposition 2, we can assume that $\mathcal{G}$ is shifted. Then we define the family $\mathcal{G}_* = \{ H ; H \subseteq \exists G \in \mathcal{G} \}$, that is, if $G \in \mathcal{G}$ and $H \subseteq G$, then $H \in \mathcal{G}_*$. In the following we show that $\mathcal{G}_* = \{ H ; H \subseteq \exists G \in \mathcal{G} \}$ is a shifted complex.

Suppose that $\mathcal{G}_*$ is not a shifted complex. Then there exist $\alpha, \beta \in P$ and $H \in \mathcal{G}_*$ such that $\alpha \subseteq \beta$, $H \cap \{\alpha, \beta\} = \{\beta\}$ and $(H - \{\beta\}) \cup \{\alpha\} \notin \mathcal{G}_*$. By definition of $\mathcal{G}_*$, there exists $G \in \mathcal{G}$ such that $H \subseteq G$. If $G \cap \{\alpha, \beta\} = \{\beta\}$, then $(G - \{\beta\}) \cup \{\alpha\} \in \mathcal{G}$ because $\mathcal{G}$ is shifted. Since $(H - \{\beta\}) \cup \{\alpha\} \subseteq (G - \{\beta\}) \cup \{\alpha\} \in \mathcal{G}$, $(H - \{\beta\}) \cup \{\alpha\} \in \mathcal{G}_*$, which is a contradiction. If $G \cap \{\alpha, \beta\} \neq \{\beta\}$, then $\alpha, \beta \in G$. Since $(H - \{\beta\}) \cup \{\alpha\} \subseteq (G - \{\beta\}) \cup \{\alpha\} \subseteq G \in \mathcal{G}$, $(H - \{\beta\}) \cup \{\alpha\} \in \mathcal{G}_*$, which is a contradiction.

Thus $\mathcal{G}_* = \{ H ; H \subseteq \exists G \in \mathcal{G} \}$ is a shifted complex and $\mathcal{G} \subseteq \mathcal{G}_* \subseteq \mathcal{F}$. So for $\mathcal{G}_*(0) = \{ G - \{0\} ; 0 \in G \in \mathcal{G}_* \}$, $\# \mathcal{G}_*(0) \leq \# \mathcal{F}(0)$. Therefore without loss of generality we can assume that $\mathcal{G}_* = \mathcal{F}$. For $\forall H \in \mathcal{G}_* - \mathcal{G}$, $H \subseteq \exists G \in \mathcal{G}$. Since $\mathcal{G}_*$ is shifted, $0 \notin H$ implies $H \cup \{0\} \in \mathcal{G}_*$. Let $\mathcal{G}_0 = \{ G \in \mathcal{G} ; 0 \notin G \}$ and $\mathcal{C} = \{ C \in \mathcal{F}_0 ; \exists G \in \mathcal{G}_0, C \cap G = \emptyset \}$. Since $\mathcal{G}_0$ and $\mathcal{F}_0 - \mathcal{C}$ are cross-intersecting, $\# \mathcal{G}_0 + \# \mathcal{F}_0 - \mathcal{C} \leq \# \mathcal{F}_0$ and therefore $\# \mathcal{C} \geq \# \mathcal{G}_0$. For $\mathcal{C}^+ = \{ C \cup \{0\} ; C \in \mathcal{C} \}$, $\# \mathcal{C}^+ = \# \mathcal{C}$.

For $C \in \mathcal{C}$ and $G \in \mathcal{G}_0$, since $0 \notin G$ and $C \cap G = \emptyset$, $\{0\} \cup C \notin \mathcal{G}_*$, $(\{0\} \cup C) \cap G = \emptyset$. By the fact that $\mathcal{G}$ is intersecting, $\{0\} \cup C \notin \mathcal{G}$. So $\mathcal{C}^+ \cap \mathcal{G}_*$ = $\emptyset$. Since every element of $(\mathcal{G} - \mathcal{G}_0) \cup \mathcal{C}^+$ contains 0 and $\mathcal{C}^+ \subseteq \mathcal{F}$, $\# \mathcal{G} \leq \# \mathcal{G} - \# \mathcal{G}_0 + \# \mathcal{C} = \# ((\mathcal{G} - \mathcal{G}_0) \cup \mathcal{C}^+) \leq \# \mathcal{F}(0)$. $lacksquare$

Proposition 4. Let $P$ be a finite poset with the minimum element 0. If
$\mathcal{F} \subseteq 2^P$ is a $P$-hereditary family, then $\mathcal{F}$ is a shifted complex.

**Proof.** We assume that $G \subseteq 2^P$ and $G \subseteq \exists F \in \mathcal{F}$. Since the mapping $f$ from $G$ to $F$ such that $f(x) = x$ is a one-to-one mapping, $G \trianglelefteq_P F$. By the property that $\mathcal{F}$ is a $P$-hereditary family, $G \in \mathcal{F}$. Thus $\mathcal{F}$ is complex.

We assume that $\mathcal{F}$ is not shifted. Then there exist $\alpha$ and $\beta$ such that $\alpha, \beta \in P$ and $\alpha \trianglelefteq \beta$ and $F \in \mathcal{F}$ such that $F \cap \{\alpha, \beta\} = \{\beta\}$ and $(F - \{\beta\}) \cup \{\alpha\} \notin \mathcal{F}$. We define the mapping $f$ from $(F - \{\beta\}) \cup \{\alpha\}$ to $F$ as follows:

$$f(x) = \begin{cases} x & \text{if } x \neq \alpha \\ \beta & \text{if } x = \alpha. \end{cases}$$

Since $\alpha \trianglelefteq \beta$ in $P$, $x \trianglelefteq f(x)$ for $\forall x \in (F - \{\beta\}) \cup \{\alpha\}$. Thus $f$ is a one-to-one mapping and $(F - \{\beta\}) \cup \{\alpha\} \trianglelefteq_P F$. By the property that $\mathcal{F}$ is a $P$-hereditary family, $(F - \{\beta\}) \cup \{\alpha\} \in \mathcal{F}$, which is a contradiction.

By Proposition 4 and Theorem 3, we also obtain Theorem B. However the converse of Proposition 4 does not hold. For example, for the poset of Figure 1, $\mathcal{F} = \{\{0,1,2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0\}, \{1\}, \{2\}\}$ is a shifted complex. Since $\{3\} \trianglelefteq_P \{2\}$ and $\{3\} \notin \mathcal{F}$, $\mathcal{F}$ is not $P$-hereditary. So we do not obtain Theorem 3 from Theorem B.

We can easily see that $\mathcal{F}$ is a $V$-hereditary family if and only if $\mathcal{F}$ is a shifted family with respect to a linear order set. Let $P$ be a poset with the minimum element and $l(P)$ be a linear extension of $P$. If $\mathcal{F}$ is a shifted family with respect to $l(P)$, then $\mathcal{F}$ is a shifted family with respect to $P$. So we also obtain Theorem A by Theorem 3. But the converse does not hold. For example, $\mathcal{F} = \{\{0,1,2\}, \{0,3,4\}\}$ is a shifted family with respect to the poset of Figure 1 and is not a shifted family with respect to the linear extension $0 \trianglelefteq 1 \trianglelefteq 2 \trianglelefteq 3 \trianglelefteq 4$. 

57
References.