Cycles and Paths in Multigraphs

by

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Abstract
We consider cycles and paths in multigraphic realizations of a degree sequence \( \mathbf{d} \). In particular we show that there exists a realization of \( \mathbf{d} \) in which no cycle has order greater than three and no path has length greater than four.

In addition we show which orders of cycles and which lengths of paths exist in some realization of \( \mathbf{d} \).

1. Introduction
Throughout we consider \( \mathbf{d} = (d_1, d_2, \ldots, d_n) \) to be a sequence of \( n \) non-negative integers. The sequence \( \mathbf{d} \) will be called a degree sequence if there is a multigraph \( G \) (without loops) with vertex set \( \{ v_i : i = 1, 2, \ldots, n \} \) such that \( \text{deg } v_i = d_i \) for \( i = 1, 2, \ldots, n \). We say that \( \mathbf{d} \) is an ordered sequence if \( d_i \geq d_j \) whenever \( i \leq j \). It is well known (see [2]) that any ordered sequence \( \mathbf{d} \) is a degree sequence if and only if

\[
\begin{align*}
(i) & \quad \sum_{i=1}^{n} d_i \equiv 0 \pmod{2}, \quad \text{and} \\
(ii) & \quad d_1 \leq \sum_{i=2}^{n} d_i.
\end{align*}
\]

Furthermore \( \mathbf{d} \) is a positive degree sequence if \( d_i > 0 \) for all \( i \).

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For convenience we will use $\sum d$ for $\sum d_i$ and we abbreviate such sequences as $(2, 2, \ldots, 2)$ (a twos) and $(2, 2, \ldots, 2, 1, \ldots, 1)$ (a twos and b ones) as $(2^a)$ and $(2^a, 1^b)$, respectively. By $\max d$ we mean the largest term in the sequence $d$. Thus if $\mathbf{d}$ is ordered, $\max \mathbf{d} = d_1$.

The **skeleton**, skel $(G)$, of a multigraph $G$ is the simple graph with vertex set $VG$ such that two vertices are adjacent if and only if they are adjacent in $G$. We say that the multigraph $G$ is **acyclic** if and only if skel$(G)$ is a forest, even if $G$ contains a cycle isomorphic to $C_2$.

In [1] Erdős and Gallai showed that if $\sum d \geq k(n - 1)$, then there is some simple realization of $\mathbf{d}$ which contains a cycle of order at least $k$. Here we consider the same problem for multigraphs and also determine what paths can be found in multigraphic realizations of $d$.

2. **Cycles and Paths which are Forced.**

In this section we show that given any multigraphic sequence $\mathbf{d}$, there exists a realization which contains at most one triangle and no longer cycle and there exists a $P_5$ but no longer path.

**Theorem 2.1:**

The sequence $\mathbf{d}$ has an acyclic realization if and only if it has a bipartite realization.

**Proof:**

Clearly an acyclic realization is bipartite.

If $G$ is a bipartite realization of $\mathbf{d}$, then it is already acyclic, or it contains a cycle $C_{2m}$ for some $m \geq 2$. Let $a$ be the smallest multiplicity of any edge on this cycle. Let $v_0, v_1, \ldots, v_{2m-1}$ be the vertices of $C_{2m}$, where $v_i$ is joined to $v_j$ if and only if $i \equiv j \pm 1$ (mod $2m$). Suppose the multiplicity of $v_0v_1$ is $a$. If we decrease the multiplicities of $v_0v_1, v_2v_3, \ldots, v_{2m-2}v_{2m-1}$ by $a$ and increase the multiplicities of the remaining edges by the same amount then the cycle $C_{2m}$ is destroyed while the degree sequence of the original multigraph is preserved. Every even cycle can be destroyed in this way to give an acyclic realization of $\mathbf{d}$. 


Having destroyed the even cycles, we may then operate on odd cycles in multigraphs by similar methods and reduce them to triangles. These techniques can therefore be used to show that there is some multigraphic realization of \( d \) which contains no cycle of order greater than three. However the following theorem proves a stronger result.

**Theorem 2.2:**
Let \( d \) be a multigraphic sequence. Then there is some realization of \( d \) which does not contain \( P_m \) for any \( m \geq 6 \) and which contains no cycle of order greater than three.

**Proof:**
Without loss of generality we assume that \( d \) is positive. Clearly if \( n \leq 2 \), then the theorem is true, and so we may assume that \( n \geq 3 \).

Let \( m_i = \sum_{k=i}^{n} (-1)^{k-i} d_k \) for all \( i \in \{1, 2, ..., n\} \). Let \( j_0 \) be the smallest value of \( j \) for which \( \sum_{i=3}^{j} m_i \geq m_1 \). Such a value of \( j \) exists since

\[
2 \sum_{i=3}^{n} m_i - m_1 = 2 \sum_{k=0}^{(n-3)/2} d_{2k+3} - \sum_{k=1}^{n} (-1)^{k-1} d_k
\]

\[
= -d_1 + \sum_{k=2}^{n} d_k,
\]

and this expression is non-negative since \( d \) is multigraphic. Define \( \delta = 2 \sum_{i=3}^{j_0} m_i - m_1 \).
Note that \( \delta \) is even since \( m_1 \) is even.

**Case 1:** \( j_0 = 3 \). Then we may realize \( d \) as follows:

- join \( v_1 \) to \( v_2 \) with \( d_2 - \frac{1}{2} \delta \) edges;
- join \( v_1 \) to \( v_3 \) with \( m_3 - \frac{1}{2} \delta \) edges;
- join \( v_2 \) to \( v_3 \) with \( \frac{1}{2} \delta \) edges;
- join \( v_3 \) to \( v_{2k+2} \) with \( d_{2k+2} - d_{2k+3} \) edges for \( k = 1, 2, ..., \left[ \frac{n-3}{2} \right] \);
- join \( v_3 \) to \( v_n \) with \( d_n \) edges if \( n \) is even;
- join \( v_{2k+2} \) to \( v_{2k+3} \) with \( d_{2k+3} \) edges for \( k = 1, 2, ..., \left[ \frac{n-3}{2} \right] \).
The skeleton of this realization is a subgraph of the graph shown in Figure 2.1, when \( n \) is even. Clearly this realization contains no path longer than \( P_5 \).

![Figure 2.1](image)

**Case 2**: \( j_0 > 3 \). Then we may realize \( d \) as follows:

- **join** \( v_i \) to \( v_i \) with \( d_i \) edges for \( i = 3, ..., j_0 - 2 \);
- **join** \( v_i \) to \( v_{j_0 - 1} \) with \( d_{j_0 - 1} - \frac{1}{2} \delta \) edges;
- **join** \( v_i \) to \( v_2 \) with \( d_2 - d_{j_0 + 1} \) edges;
- **join** \( v_i \) to \( v_{j_0 + 2k} \) with \( d_{j_0 + 2k} - d_{j_0 + 2k + 1} \) edges for \( k = 1, 2, ..., \left\lfloor \frac{1}{2}(n - j_0) \right\rfloor \);
- **join** \( v_i \) to \( v_n \) with \( d_n \) edges if \( n \equiv j_0 \) (mod 2);
- **join** \( v_i \) to \( v_{j_0} \) with \( d_{j_0} - \frac{1}{2} \delta \) edges;
- **join** \( v_2 \) to \( v_{j_0 + 1} \) with \( d_{j_0 + 1} \) edges if \( j_0 < n \);
- **join** \( v_{j_0 + 2k} \) to \( v_{j_0 + 2k + 1} \) with \( d_{j_0 + 2k + 1} \) edges for \( k = 1, 2, ..., \left\lfloor \frac{1}{2}(n - j_0 - 1) \right\rfloor \);
- **join** \( v_{j_0 - 1} \) to \( v_{j_0} \) with \( \frac{1}{2} \delta \) edges.

It is clear that this realization of \( d \) contains no path longer than \( P_5 \).

On the other hand we now show that every path from \( P_2 \) to \( P_5 \) must occur in every realization of some degree sequence.

Clearly \( P_2 \) must occur in every realization of every degree sequence where \( \sum d > 0 \).

The sequence \( d = (2^3) \) is uniquely realizable and contains a \( P_3 \) in this realization.
The sequence \( d = (18, 15, 13, 12) \) is such that every realization is connected. This follows since no two terms in the sequence are equal. Inspection of the residue classes modulo 3 of the terms in the sequence reveals that no two distinct subsequences of \( d \) have equal sum and so \( d \) does not have a bipartite realization. Hence every realization of \( d \) contains a triangle. The remaining vertex in any realization of \( d \) must be adjacent to one of the vertices of the triangle since the realization is connected. Thus every realization of \( d \) contains a \( P_4 \).

Consider the sequence \( d = (24, 18, 15, 13, 12) \). By previous arguments we see that every realization of \( d \) is connected but none is bipartite. If every realization of \( d \) contains \( C_5 \), then they all contain \( P_5 \). Hence we may suppose that some realization contains a triangle. The only way such a realization does not contain \( P_5 \) is for the two vertices \( v, w \), remaining to be adjacent to the same vertex \( u \) of the triangle. But then \( \deg u > \deg v + \deg w \). However the smallest possible value of \( \deg v + \deg w \) is 25 which exceeds any degree of the sequence. Hence every realization of \( d \) contains a \( P_5 \).

But the longest cycle that can be found in every realization of \( d \) is a triangle and if \( d \) contains a bipartite realization then it will have an acyclic realization as we have already noted in Theorem 2.1.

3. Possible Orders of Cycles and Paths

In this section we find the range of possible cycle orders and path lengths which occur in realizations of the degree sequence \( d \).

**Theorem 3.1**

Let \( d \) be a positive degree sequence. If \( d \neq (2^s) \) for any \( s \geq 3 \), then \( d \) has a realization with a cycle isomorphic to \( C_\ell \) if and only if \( 2 \leq \ell \leq \min \left\{ \frac{1}{2} (\Sigma d - 2d_1 + 4), m \right\} \), where \( m \) is the number of terms of \( d \) which are greater than one. If \( d = (2^s) \) for some \( s \geq 3 \), then \( d \) contains a realization with a cycle isomorphic to \( C_\ell \) whenever \( 2 \leq \ell \leq s-2 \) or \( \ell = s \).
Proof
We may assume that \( d \) is ordered. Suppose \( d \) contains a realization \( G \) with a cycle isomorphic to \( C_\ell \). Then \( 2 \leq \ell \leq m \) and the degree sequence \( d' = d - (2\ell) \), of the graph obtained from \( G \) by removing \( C_\ell \), is graphic.

If \( \max d' = d_1, \) then \( d_1 \leq \sum_{i=2}^{n} d_i - 2\ell \) implies \( \ell \leq \frac{1}{2} (\sum d - 2d_1) < \frac{1}{2} (\sum d - 2d_1 + 4) \).

If \( \max d' = d_1 - 2, \) then \( d_1 - 2 \leq \sum_{i=2}^{n} d_i - 2(\ell - 1) \) implies \( \ell \leq \frac{1}{2} (\sum d - 2d_1 + 4) \).

Finally, if \( \max d' = d_j = d_1 - 1, \) then \( d_j \leq \sum_{i \neq j} d_i - 2\ell \) implies \( \ell \leq \frac{1}{2} (\sum d - 2d_j) \)

\[ = \frac{1}{2} (\sum d - 2d_1 + 2) < \frac{1}{2} (\sum d - 2d_1 + 4). \] Thus we have the stated restriction on \( \ell \).

Suppose now that \( 2 \leq \ell \leq \min \left\lfloor \frac{1}{2} (\sum d - 2d_1 + 4), m \right\rfloor \). Let \( d' = (d_1', \ldots, d_n') \), where

\[
d_{i}' = \begin{cases} d_i - 2 & 1 \leq i \leq \ell \\ d_i & i > \ell \end{cases}
\]

Since \( \sum d \equiv 0 \text{ (mod 2)} \), clearly \( \sum d' \equiv 0 \text{ (mod 2)} \).

If \( d' \) is multigraphic, then \( d \) has a realization with a cycle isomorphic to \( C_\ell \). Hence we may assume that \( d' \) is not multigraphic.

Now \( \max d' \) is either \( d_1 - 2 \) or \( d_{\ell+1} \). If \( \max d' = d_1 - 2 \) we must have

\[ d_1 - 2 > \sum_{i=2}^{n} d_i - 2(\ell - 1) \]

i.e.

\[ d_1 > \sum_{i=2}^{n} d_i - 2\ell + 4 \]

i.e.

\[ \ell > \frac{1}{2} (\sum d - 2d_1 + 4). \]

This contradicts our choice of \( \ell \).
If \( \max d' = d_{\ell+1} \) we must have

\[
d_{\ell} \geq d_{\ell+1} > \sum_{i=1}^{\ell} d_i + \sum_{i=\ell+2}^{n} d_i - 2\ell \\
\geq \sum_{i=1}^{\ell} d_i - 2\ell.
\]

Hence

\[
\sum_{i=1}^{\ell-1} d_i < 2\ell.
\]

Now \( d_i \geq 2 \) for \( 1 \leq i \leq \ell \), since \( \ell \leq m \). Therefore either \( d_i = 2 \) for \( 1 \leq i \leq \ell - 1 \) (and \( d_\ell = 2 \) since \( \ell \leq m \)) or \( d_1 = 3 \) and \( d_i = 2 \) for \( 2 \leq i \leq \ell \). So either \( d = (2^n) \) for \( n \geq \ell \), \( d = (2^\alpha, 1^\beta) \) for \( \alpha \geq \ell, \beta > 0 \) and \( \beta \) even, or \( d = (3, 2^\alpha, 1^\beta) \) for \( \alpha \geq \ell - 1 \) and \( \beta \) odd. Since \( d' \) is not multigraphic, \( d = (2^{\ell+1}) \), the one degree sequence not covered by the theorem.

**Corollary 3.2**

The positive degree sequence \( d \) has no realization with a cycle of order greater than two if and only if \( d_1 = \sum_{i=2}^{n} d_i \) or \( d \) has at most two terms greater than one.

**Proof**

If \( d_1 = \sum_{i=2}^{n} d_i \), then \( d \) has a unique realization whose skeleton is a star. Thus no cycle has order greater than two. If \( d \) has at most two terms greater than one, then no realization can contain a cycle larger than \( C_2 \).

Suppose \( d \) has no realization with a cycle longer than \( C_2 \). Then \( d \neq (2^s) \) for any \( s \geq 3 \) and so by the theorem we must have \( m \leq 2 \) or \( \frac{1}{2} (\sum d - 2d_1 + 4) \leq 2 \). The latter condition gives \( d_1 \geq \sum_{i=2}^{n} d_i \), but since \( d \) is multigraphic we must have \( d_1 = \sum_{i=2}^{n} d_i \). The corollary then follows.
Corollary 3.3

The positive degree sequence $d$ contains a realization with a Hamiltonian cycle if and only if $2d_1 \leq \sum d - 2n + 4$ and $m = n$.

Theorem 3.4

The positive degree sequence $d$ has a realization with a path isomorphic to $P_\ell$ if and only if $1 \leq \ell \leq \min\left\{ \frac{1}{2} (\sum d - 2d_1 + 6), n, m+2 \right\}$.

Proof

The argument is analogous to that of Theorem 3.1 and we only sketch it here.

If $d$ contains a path isomorphic to $P_\ell$, then clearly $1 \leq \ell \leq \min(n, m+2)$. Further $d - (2\ell-2,1^2)$ is graphic and similar inequalities to those of the proof of Theorem 3.1 show that $\ell \leq \frac{1}{2} (\sum d - 2d_1 + 6)$. We note however that there are four cases to consider since we may subtract 1 from $d_1$ in forming $d - (2\ell-2,1^2)$ from $d$.

Suppose then that $1 \leq \ell \leq \min\left\{ \frac{1}{2} (\sum d - 2d_1 + 6), n, m+2 \right\}$. Here we need $d' = (d_1', \ldots, d_n')$ defined by $d_i' = \begin{cases} d_i - 2 & 1 \leq i \leq \ell-2 \\ d_i & \ell-1 \leq i \leq n-2 \\ d_i - 1 & n-1 \leq i \leq n \end{cases}$.

If $d'$ is multigraphic the theorem follows. Suppose therefore that $d'$ is not multigraphic. Arguing as in the proof of Theorem 3.1 we obtain a contradiction unless $\max d' = d_{\ell-1}$ where $\ell < n$, or $\max d' = d_{n-1} - 1$.

Note that any realization of $d$ contains a $P_\ell$ if $\ell \leq 2$, since $d$ is positive. Suppose $\max d' = d_{\ell-1}$ and $2 < \ell < n$. Since $d'$ is not multigraphic, we then have

$$d_{\ell-2} \geq d_{\ell-1} + \sum_{i=1}^{\ell-2} d_i + \sum_{i=\ell}^{n} d_i - 2\ell + 2. \quad (1)$$

Thus

$$d_{\ell-2} > \sum_{i=1}^{\ell-2} d_i - 2\ell + 4, \quad (2)$$
since
\[ \sum_{i=1}^{n} d_i \geq d_{n-1} + d_n \geq 2. \]

On the other hand, if \( \max d' = d_{n-1} - 1 \) then \( \ell = n \) and (1) is replaced by
\[ d_{\ell-2} \geq d_{\ell-1} - 1 > \sum_{i=1}^{\ell-2} d_i + d_n - 2\ell + 3 \]
and (2) again follows. Thus in either case
\[ \sum_{i=1}^{\ell-3} d_i < 2\ell - 4. \]

Now, arguing as in the proof of Theorem 3.1, we find that either \( \bar{d} = (2^\alpha, 1^\beta) \) for some \( \alpha \geq \ell - 2 \) and some even \( \beta \geq 0 \) or \( \bar{d} = (3, 2\alpha, 1^\beta) \) for some \( \alpha \geq \ell - 3 \) and some odd \( \beta \). As \( d' \) is not multigraphic, these sequences force \( d'_{\ell-1} = 2 \) and \( d_i' = 0 \) for all \( i \neq \ell - 1 \). Hence \( \bar{d} = (2^{\ell-1}, 1^2) \) and \( \bar{d} \) has a realization with a path isomorphic to \( P_{\ell+1} \).

**Corollary 3.5**
The positive degree sequence \( \bar{d} \) contains a realization with a Hamiltonian path if and only if \( 2d_1 \leq \sum d - 2n + 6 \) and \( m \geq n-2 \).

**REFERENCES**

