The number of \( h \)-strongly connected digraphs with small diameter

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Abstract

Let \( D_s(n; h, d = k) \) denote the number of \( h \)-strongly connected digraphs of order \( n \) and diameter equal to \( k \). In this paper it is shown that:

i) \( D_s(n; h, d = 3) = 4^{\binom{n}{2}} (3/4 + o(1))^n \) for every fixed \( h \geq 1 \);

ii) \( D_s(n; h, d = 4) = 4^{\binom{n}{2}} (2^{-h-2} + 2^{-2} + o(1))^n \) for every fixed \( h \geq 2 \);

iii) \( D_s(n; h, d = k) = 4^{\binom{n}{2}} ((2^{k+1} - 1)2^{-kh+3h-2} + o(1))^n \) for every fixed \( h \geq 1 \) and \( k \geq 5 \).

Similar asymptotic formulas hold for the number of \( h \)-connected digraphs of order \( n \) and diameter equal to \( k \) when \( n \to \infty \). This extends the corresponding results for \( h \)-connected graphs given in a recent paper by the author.

1 Notation and preliminary results

All digraphs in this paper are finite, labeled, without loops or parallel directed edges. By \( K^*_n \) we denote the complete digraph of order \( n \) such that any two distinct vertices \( x \) and \( y \) are joined by two directed edges \((x, y)\) and \((y, x)\). For a digraph \( G \) the outdegree \( d^+(x) \) of a vertex \( x \) is the number of vertices of \( G \) that are adjacent from \( x \) and the indegree \( d^-(x) \) is the number of vertices of \( G \) adjacent to \( x \). For \( h \geq 2 \), we say that a digraph \( G \) is \( h \)-connected (resp. \( h \)-strongly connected) if either \( G \) is a complete digraph \( K^*_{h+1} \) or else it has at least \( h + 2 \) vertices and for any set of vertices \( X \subset V(G) \), \( |X| = h - 1 \), the digraph \( G - X \) is connected (resp. strongly connected). A connected (resp. strongly connected) digraph is also said to be 1-connected (resp. 1-strongly connected). For a strongly connected digraph \( G \) the distance \( d(x, y) \) from vertex \( x \) to vertex \( y \) is the length of a shortest path of the form \((x, \ldots, y)\). The eccentricity of a vertex \( x \) is \( \text{ecc}(x) = \max_{y \in V(G)} d(x, y) \). The diameter of \( G \), denoted
\(d(G)\) is equal to \(\max_{x,y \in V(G)} d(x, y)\) if \(G\) is strongly connected and \(\infty\) otherwise. By \(D_s(n; h, d = k)\) and \(D_s(n; h, d \geq k)\) (resp. \(D(n; h, d = k)\) and \(D(n; h, d \geq k)\)) we denote the number of \(h\)-strongly connected (resp. \(h\)-connected) digraphs \(G\) of order \(n\) and diameter \(d(G) = k\) and \(d(G) \geq k\), respectively.

It is well known [1, p. 131] that almost all digraphs have diameter two and for every fixed integer \(h \geq 1\) almost all graphs are \(h\)-connected. Also in [2] it was proved that for every fixed integer \(h \geq 1\) almost all digraphs are \(h\)-strongly connected. Hence for every \(h \geq 1\) we have:

\[
D_s(n; h, d = 2) = 4^{\binom{2}{h}}(1 + o(1)) \quad \text{and} \quad D(n; h, d = 2) = 4^{\binom{2}{h}}(1 + o(1)).
\]

If \(\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1\) we denote this by \(f(n) \sim g(n)\), or \(f(n) = g(n)(1 + o(1))\). The following results will be useful in the proofs of the theorems given in the next section.

**Lemma 1.1** ([4]). The number of bipartite digraphs \(G\) whose partite sets are \(A, B\) \((A \cap B = \emptyset, \ | A | = p, \ | B | = q)\) such that \(d^-(x) \geq 1\) for every \(x \in B\) and all edges are directed from \(A\) towards \(B\) is equal to \((2^p - 1)^q\).

**Lemma 1.2** ([4]). We have

\[
D_s(n; 1, d = 3) = 4^{\binom{3}{h}}(3/4 + o(1))^n.
\]

Also we need an asymptotic evaluation of the maximum of an arithmetical function. Let

\[
f(n, h; n_1, \ldots, n_k) = \binom{n}{n_1, \ldots, n_k} 2^{\sum_{i=1}^k \binom{n_i}{2}} \prod_{i=1}^{k-1} (2^{n_i} - 1)^{n_{i+1}}
\]

where \(n_1 + \ldots + n_k = n\), \(n_i \geq h\) for every \(1 \leq i \leq k - 1\) and \(n_k \geq 1\). Let us denote

\[
f(n, k) = \max_D f(n, h; n_1, \ldots, n_k),
\]

where \(D\) is defined by: \(n_1 + \ldots + n_k = n\); \(n_i \geq h\) for every \(1 \leq i \leq k - 1\) and \(n_k \geq 1\).

**Theorem 1.3** ([5]). The following equalities hold:

\[
f(n, h, 4) = 2^{\binom{4}{h}}(2^{-h-1} + 2^{-1} + o(1))^n \quad (1)
\]

for every \(h \geq 2\);

\[
f(n, h, k) = 2^{\binom{k}{h}}((2^{h+1} - 1)2^{-kh+3h-1} + o(1))^n \quad (2)
\]

for every \(h \geq 2\) and \(k \geq 5\).

Note that (2) also holds for \(h = 1\) [3]. Moreover, for \(k = 4\), \(f(n, h; n_1, \ldots, n_4)\) can be maximum only if \(n_1 = \alpha_1(n, h, 4)\), \(n_2 = \beta_1(n, h, 4)\), \(n_3 = h\) and \(n_4 = 1\), where

\[
\alpha_1(n, h, 4) = (n - h)\frac{1}{2^{h+1}} - \gamma,
\]

\[
\beta_1(n, h, 4) = (n - h)\frac{2^h}{2^h + 1} - 1 + \gamma,
\]

306
and \( 0 \leq \gamma \leq 1 \).
For \( k \geq 5 \), \( f(n, h, k) = f(n, h; h, \ldots, h, \alpha_0, \beta_0, h, \ldots, h, 1) \), where
\[
\alpha_0(n, h, k) = (n - kh + 3h) \frac{2^h - 1}{2^{h+1} - 1} - \gamma;
\]
\[
\beta_0(n, h, k) = (n - kh + 3h) \frac{2^h}{2^{h+1} - 1} - 1 + \gamma,
\]
and \( 0 \leq \gamma \leq 1 \).

Notice that for \( h = 1 \) the explanation of the asymptotic behavior of the critical function \( f(n, h, k) \), denoted by \( f(n, k) \) was made by a careful analysis in [3].

Lemma 1.4 (i) If \( G \) is an \( h \)-strongly connected digraph, \( x \notin V(G) \) and \( x \) is joined by directed edges in both directions \((x, y)\) and \((y, x)\) with at least \( h \) distinct vertices \( y \) in \( G \), the resulting digraph is \( h \)-strongly connected.

(ii) If \( E \) and \( F \) are two \( h \)-strongly connected digraphs such that \( V(E) \cap V(F) = \emptyset \), joined by directed edges in both directions \((x_i, y_i)\) and \((y_i, x_i)\) \((1 \leq i \leq h)\) which join \( h \) distinct vertices \( x_i \) in \( E \) \((1 \leq i \leq h)\) with \( h \) distinct vertices \( y_j \) in \( F \) \((1 \leq j \leq h)\), the resulting digraph is \( h \)-strongly connected. The property holds even if \( E \) or \( F \) is isomorphic to \( K_h^* \).

Note that this lemma holds if \( h \)-strongly connectedness is replaced by \( h \)-connectedness.

2 Main results

We will deduce an estimation for \( D_s(n; h, d = k) \) for every fixed \( h \geq 2 \) and \( k \geq 3 \) as \( n \to \infty \), by considering first the case \( k = 3 \), when this does not depend on \( h \).

Theorem 2.1 We have
\[
D_s(n; h, d = 3) = 4^{(3)} (3/4 + o(1))^n
\]
for every fixed \( h \geq 1 \).

Proof: For \( h = 1 \) this property was shown in [4]. If \( D(n; d \geq k) \) denotes the number of digraphs \( G \) of order \( n \) and diameter \( d(G) \geq k \), from the proof of Lemma 1.3 of [4] it follows that \( D(n; d \geq 4) < (n^2 - n)2^{(3)} + (n^2 - 2)^2 + (5/2)^n - 3 = 4^{(3)} (5/8 + o(1))^n \). Since \( D_s(n; h, d \geq 4) \leq D(n; d \geq 4) \) one gets
\[
D_s(n; h, d \geq 4) < 4^{(3)} (5/8 + o(1))^n. \tag{5}
\]
Let \( A_{ij}^{(k)} \), respectively \( H_{ij}^{(k)} \), denote the set of digraphs (respectively \( h \)-strongly connected digraphs) having vertex set \( \{1, \ldots, n\} \) such that \( d(i, j) \geq k \). In [4] it was shown that \( |A_{ij}^{(3)}| = 3^{n-2} \cdot 2^{(3)} + (n^2) \). Since \( |H_{ij}^{(3)}| \leq |A_{ij}^{(3)}| \) we get
\[
|H_{ij}^{(3)}| \leq 4^{(3)} (3/4 + o(1))^n. \tag{6}
\]
Now a sufficiently large subset of $H_{ij}^{(3)}$ can be constructed as follows: Consider an $h$-strongly connected digraph $F$ with vertex set $\{1, \ldots, n\}\setminus\{i, j\}$ and nonadjacent vertices $i$ and $j$ such that the sets of neighbors $N(i), N(j) \subset V(F)$ satisfy: $|N(i)| = |N(j)| = h$ and $N(i) \cap N(j) = \emptyset$. Vertices $i$ and $j$ are joined by directed edges in both directions with all vertices in $N(i)$ and $N(j)$, respectively. For every vertex $k \in V(F) \setminus \{N(i) \cup N(j)\}$ we suppose that the condition: $(i, k) \in E(G)$ implies $(k, j) \notin E(G)$ is fulfilled, where $G$ denotes the digraph obtained on this way. By Lemma 1.4, $G$ is $h$-strongly connected and the distance $d(i, j) \geq 3$. This implies that for every fixed choice of the subdigraph induced by $\{i, j\}$, for every $k \in V(F) \setminus \{N(i) \cup N(j)\}$ the subdigraph induced by $\{i, j, k\}$ can be chosen in exactly 12 ways. Hence $|H_{i,j}^{(3)}| \geq 12^{n-2h-2}D_s(n - 2, h)$, where $D_s(n, h)$ denotes the number of $h$-strongly connected digraphs of order $n$. Since almost all digraphs of order $n$ are $h$-strongly connected as $n \to \infty$, it follows that $D_s(n - 2, h) \sim 4^{(n-2)}$, which implies $|H_{i,j}^{(3)}| \geq 4^{(n \over 2)}(3/4 + o(1))^n$. Consequently,

$$|H_{i,j}^{(3)}| = 4^{(n \over 2)}(3/4 + o(1))^n$$

for every $1 \leq i, j \leq n$ and $i \neq j$. Because $D_s(n; h, d \geq 3) = |\bigcup_{1 \leq i, j \leq n, i \neq j} H_{i,j}^{(3)}|$ and

$$|H_{i_0,j_0}^{(3)}| \leq \bigcup_{1 \leq i, j \leq n, i \neq j} H_{i,j}^{(3)}| \leq (n^2 - n)|H_{i_0,j_0}^{(3)}|$$

one deduces that

$$D_s(n, h, d \geq 3) = 4^{(n \over 2)}(3/4 + o(1))^n.$$  \hfill (7)

Since $D_s(n; h, d = 3) = D_s(n; h, d \geq 3) - D_s(n; h, d \geq 4)$, the conclusion follows from (5) and (7). \hfill \Box

Because any $h$-strongly connected digraph is also $h$-connected, we get:

**Corollary 2.2** The following equality holds for every fixed $h \geq 1$:

$$D(n; h, d = 3) = 4^{(n \over 2)}(3/4 + o(1))^n$$

**Theorem 2.3** We have:

(i) $D_s(n; h, d = 4) = 4^{(n \over 2)}(2^{-h-2} + 2^{-2} + o(1))^n$

for every fixed $h \geq 2$;

(ii) $D_s(n; h, d = k) = 4^{(n \over 2)}((2^{h+1} - 1)2^{-kh+3h-2} + o(1))^n$

for every fixed $h \geq 1$ and $k \geq 5$. 

308
Proof: For $h = 1$, (ii) was proved in [4]. Let $h \geq 2$, $k \geq 4$ and $G$ be an $h$-strongly connected digraph of order $n$. If $x \in V(G)$ has $\text{ecc}(x) = k$, then

$$V_1(x) \cup \ldots \cup V_k(x)$$

is a partition of $V(G) \setminus \{x\}$, where $V_i(x) = \{y \mid y \in V(G) \text{ and } d(x, y) = i\}$ for $1 \leq i \leq k$. It follows that there are directed edges from $x$ towards all vertices of $V_1(x)$ and for every $2 \leq i \leq k$ and any vertex $z \in V_i(x)$ there exists a directed edge $(t, z)$, where $t \in V_{i-1}(x)$. Also the $h$-strongly connectedness of $G$ implies that $|V_i(x)| \geq h$ for every $i = 1, \ldots, k - 1$. Let $n_i$ be the number of vertices in $V_i(x)$, $1 \leq i \leq k$. By Lemma 1.1 one deduces

$$\sum_{n_1, \ldots, n_{i-1} \geq h} \binom{n-1}{n_1, \ldots, n_k} 4 \sum_{i=1}^{k-1} \prod_{i=1}^{k-1} (2^{n_i} - 1)^{n_{i+1}} \prod_{i=1}^k 2^{n_i(n_{i-1} + \ldots + 1)}$$

$$= 2^{\binom{k}{2}} \sum_{n_1, \ldots, n_k \geq 1} f(n - 1; n_1, \ldots, n_k)$$

because

$$2 \sum_{i=1}^{k} \binom{n}{i} \prod_{i=1}^{k} 2^{n_i(n_{i-1} + \ldots + 1)} = 2^{\binom{k}{2}}. \tag{8}$$

Furthermore

$$\sum_{n_1, \ldots, n_k \geq h} f(n - 1; n_1, \ldots, n_k) \leq \binom{n-2}{k-1} f(n - 1, k)$$

since the number of compositions $n - 1 = n_1 + \ldots + n_k$ having $k$ positive terms equals $\binom{n-2}{k-1}$. Hence $D_s(n; h, d = k) \leq \left| \bigcup_{x \in V(G)} \{G \mid G \text{ is } h \text{-strongly connected, } V(G) = \{1, \ldots, n\} \text{ and } \text{ecc}(x) = k\} \right|$ $\leq n 2^{\binom{n}{2}} \binom{n-2}{k-1} f(n - 1, h, k)$ and this expression equals $4^{\binom{k}{2}} (2^{-h-2} + 2^{-2} + o(1))^n$ for $k = 4$ and $4^{\binom{k}{2}} ((2^{h+1} - 1)2^{-kh+3h-2} + o(1))^n$ for $k \geq 5$ by Theorem 1.3. The proof of the theorem is by double inequality. We shall consider two cases: I $k \geq 5$ and II $k = 4$.

Case I. In order to produce a suitable lower bound for $D(n; h, d = k)$ in the case $k \geq 5$ we shall generate a large class of $h$-strongly connected digraphs of order $n$ and diameter equal to $k$ as follows: Let $x \in \{1, \ldots, n\}$ be a fixed vertex and $X_1 \cup \ldots \cup X_k$ be a partition of $\{1, \ldots, n\} \setminus \{x\}$ such that $|X_1| = |X_2| = \ldots = |X_{k-4}| = h, |X_{k-3}| = \alpha_0, |X_{k-2}| = \beta_0, |X_{k-1}| = h$ and $|X_k| = 1$, where $\alpha_0 = \alpha_0(n - 1, h, k)$ and $\beta_0 = \beta_0(n - 1, h, k)$ are given by (4). Vertex $x$ is joined by directed edges in both directions with all vertices of $X_1$ and the unique vertex of $X_k$ is joined by directed edges in both directions with all vertices of $X_{k-1}$. Let us denote $X_i = \{x_i^1, \ldots, x_i^h\}$ for every $1 \leq i \leq k - 4$ and $i = k - 1$. We choose an $h$-element subset $Y_{k-3} = \{x_{k-3}^1, \ldots, x_{k-3}^h\} \subset X_{k-3}$ and an $h$-element subset $\{x_{k-2}^1, \ldots, x_{k-2}^h\} \subset X_{k-2}$. Now for every $1 \leq i \leq k - 2$ we join vertex $x_i^j$ with $x_{i+1}^j$ by directed edges $(x_i^j, x_{i+1}^j)$ and $(x_{i+1}^j, x_i^j)$ for every $j = 1, \ldots, h$. Every $X_1, X_2, \ldots, X_{k-4}$ and $X_{k-1}$ induces a subdigraph isomorphic to $K^*_h$ and subdigraphs induced by $X_{k-3}$ and $X_{k-2}$ are $h$-strongly connected and have diameter equal to two. Also for any vertex $u \in X_{k-3}$
there exists at least one directed edge \((s, u)\), where \(s \in X_{k-4}\) and for any vertex \(v \in X_{k-2}\) there exists at least one directed edge \((t, v)\), where \(t \in X_{k-3}\). If \(G\) denotes a digraph generated by this procedure, it is easy to see that \(|V(G)| = n\), ecc\((x) = k\) and \(d(G) = k\); by Lemma 1.4 it follows that \(G\) is \(h\)-strongly connected. The number of directed edges oriented from classes \(X_j\) towards classes \(X_i\) where \(i < j\) is a function \(\varphi(k, h)\) which does not depend on \(n\).

The number of digraphs generated in this way is greater than or equal to
\[
\binom{n-1}{\alpha_0} \binom{n-1}{\beta_0} \cdot \binom{\binom{n}{2} - \varphi(k, h) - \binom{\alpha_0}{2} - \binom{\beta_0}{2}}{2} \cdot D_s(\alpha_0; h, d = 2) \cdot D_s(\beta_0; h, d = 2) (2^h - 1) \cdot \frac{\alpha_0 - h (2^{\alpha_0 - 1} \beta_0 - h 2^{h - 1} 2^{h (\alpha_0 - 1)} 2^{h (\beta_0 - 1)}))}{2} \cdot \text{ by Lemma 1.1 and (8)}.
\]
Indeed, each vertex \(z \in X_{k-3} \setminus \{x^1_{k-3}, \ldots, x^h_{k-3}\}\) must have at least one incoming edge from some vertex in \(X_{k-4}\), hence there are \(2^h - 1\) choices for the set of incoming edges to any such vertex. If \(z = x^i_{k-3}\) (\(1 \leq i \leq h\)), there exists the directed edge \((x^i_{k-4}, x^i_{k-3})\); hence there are \(2^{i-1}\) choices for the set of incoming edges to any vertex in \(\{x^1_{k-3}, \ldots, x^h_{k-3}\}\). So the number of choices for the set of incoming edges to \(X_{k-3}\) is equal to \((2^h - 1)^{\alpha_0 - h 2^{h - 1}}\). In a similar way we find the number of choices for the set of incoming edges to \(X_{k-2}\) and \(X_{k-1}\). Since \(D_s(\alpha; h, d = 2) \sim 4^{\binom{n}{2}}\) as \(h \to \infty\), this expression is equal to
\[
2^{(\frac{n}{2})} f(n - 1, h, k)(1 + o(1))^n = 4^{\binom{n}{2}} ((2^{h+1} - 1) 2^{-kh + 3h - 2} + o(1))^n
\]
by Theorem 1.3. Hence \(D_s(n; h, d = k) \geq 4^{\binom{n}{2}} ((2^{h+1} - 1) 2^{-kh + 3h - 2} + o(1))^n\) and the proof is complete in this case.

Case II. If \(k = 4\) the construction is somewhat similar to the case \(k \geq 5\):
We consider a partition \(X_1 \cup X_2 \cup X_3 \cup X_4\) of \(\{1, \ldots, n\}\) such that \(|X_1| = \alpha_1(n - 1, h, 4), \ |X_2| = \beta_1(n - 1, h, 4)\) (given by (3)), \(|X_3| = h\) and \(|X_4| = 1\). Let \(X_4 = \{w\}\).
We choose any vertex \(t \in X_2\) and join \(t\) with \(x\) by a directed edge \((t, x)\). By choosing \(Y_1 \subset X_1\) and \(Y_2 \subset X_2\) the remaining adjacencies are defined as for the case \(k \geq 5\).
Let us denote the set of \(h\)-strongly connected digraphs of order \(n\) produced in this way by \(G\). If \(G \in \mathcal{G}\), we have \(d(x, w) = 4\); also \(d(u, v) \leq 4\) for every \(u, v \in V(G)\) unless \(u \in X_1\) and \(v = w\), when we have only \(d(u, w) \leq 5\). If \(G \in \mathcal{G}\) has \(d(G) = 5\) we define the digraph \(\varphi(G)\) deduced from \(G\) by deleting directed edges joining \(w\) in both directions with vertices of \(X_3\) and replacing them by directed edges joining \(w\) in both directions with the \(h\) vertices of \(Y_2 \subset X_2\). We have \(d\varphi(G)(x, w) = 3\). If \(u \in X_1\) has \(d_G(u, w) = 5\) then \(d_G(u, Y_2) = 3\), which implies \(d\varphi(G) = 4\), hence \(\varphi(G)\) has diameter equal to four. If the vertex \(w\) in \(X_4\) is fixed, the ordered partition \(X_1 \cup X_2 \cup X_3\) can be generated in
\[
\binom{n}{\alpha_1} \binom{n-2}{\beta_1} = \frac{(n-1)!}{\alpha_1! \beta_1!} (1 + o(1))^n
\]
ways. In this case \(\varphi\) is injective and for every \(F, G \in \mathcal{G}\) we have \(\varphi(G) \neq F\) since \(d_F(x, w) = 4\) but \(d\varphi(G)(x, w) = 3\).
Hence we can generate a class consisting of \(|\mathcal{G}| \ h\)-strongly connected digraphs of order \(n\) and diameter equal to four as follows: we choose a digraph \(G \in \mathcal{G}\) if \(d(G) = 4\); otherwise we choose the digraph \(\varphi(G)\).
It follows that the number of digraphs generated in this way is equal to
\[ |G| = \frac{(n-1)!}{\alpha_1!\beta_1!} 2^{\binom{n}{2}} - \varphi(h) - \binom{n}{2} - (\binom{\beta}{2}) \]
\[ D_s(\alpha_1; h, d = 2) D_s(\beta_1; h, d = 2)(2^{\alpha_1 - 1})^{\beta_1 - h} 2^{h(\alpha_1 - 1)}(1 + o(1))^n, \]
where \( \varphi(k, h) \) was defined in the case \( k \geq 5 \). As for the case I
the last expression equal to
\[ 2^{\binom{n}{2}} f(n - 1, h, 4)(1 + o(1))^n = 4^{\binom{n}{2}} (2^{-h} + 2^{-2} + o(1))^n \]
which concludes the proof. \( \square \)

**Corollary 2.4** Equalities (i) and (ii) also hold for the numbers \( D(n; h, d = 4) \) and
\( D(n; h, d = k) \) of \( h \)-connected digraphs \( G \) of order \( n \) and diameter \( d(G) = 4 \), respecti-
vively \( d(G) = k \geq 5 \).

**Corollary 2.5** For every fixed \( h \geq 1 \) and \( k \geq 2 \) we have
\[ \lim_{n \to \infty} \frac{D_s(n; h, d = k)}{D_s(n; h, d = k + 1)} = \lim_{n \to \infty} \frac{D(n; h, d = k)}{D(n; h, d = k + 1)} = \infty. \]

**Corollary 2.6** The following equalities
\[ \lim_{n \to \infty} \frac{D_s(n; h, d = k)}{D_s(n; h + 1, d = k)} = \lim_{n \to \infty} \frac{D(n; h, d = k)}{D(n; h + 1, d = k)} = \infty \]
hold for every fixed \( h \geq 1 \) and \( k \geq 4 \).

**References**


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