Uniform coverings of 2-paths by 4-paths

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Abstract

We construct a uniform covering of 2-paths by 4-paths in $K_n$ for all $n \geq 5$, i.e., we construct a set $S$ of 4-paths in $K_n$ having the property that each 2-path in $K_n$ lies in exactly one 4-path in $S$ for all $n \geq 5$.

1 Introduction

Let $K_n$ be the complete graph on $n$ vertices. A $k$-path is a path of length $k$ and a $k$-cycle is a cycle of length $k$, where the length of a path [cycle] is the number of edges in the path [cycle]. Note that paths and cycles are undirected. A uniform covering of the 2-paths in $K_n$ by $k$-paths [$k$-cycles] is a set $S$ of $k$-paths [$k$-cycles] having the property that each 2-path in $K_n$ lies in exactly one $k$-path [$k$-cycle] in $S$. Only the following cases of the problem of constructing a uniform covering of the 2-paths in $K_n$ by $k$-paths or $k$-cycles have been solved [2, 8];

1. by 3-cycles,
2. by 3-paths,
3. by 4-cycles,
4. by $n$-cycles (Hamilton cycles) when $n$ is even.

When $n$ is odd, a uniform covering of the 2-paths in $K_n$ by Hamilton cycles has only been constructed for a few cases: $n = 2^e + 1$, where $e$ is a natural number [7], $n = p + 2$, where $p$ is an odd prime and 2 is a generator of the multiplicative group of $GF(p)$ [1], and some other infinite cases [3, 5]. But in general the problem when $n$ is odd is still open.

In this paper, we solve the problem in the case of 4-paths, that is, we prove,

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Theorem 1.1 Let \( n \geq 5 \). Then there exists a set \( S \) of 4-paths in \( K_n \) having the property that each 2-path in \( K_n \) lies in exactly one path in \( S \).

Finally, we mention the problem in the case of \((n-1)\)-paths (Hamilton paths).

Lemma 1.2 Let \( n \geq 3 \). If there is a uniform covering of 2-paths by Hamilton cycles in \( K_{n+1} \), there is a uniform covering of 2-paths by Hamilton paths in \( K_n \).

Proof. Let \( V_{n+1} = \{v_0,v_1,\ldots,v_n\} \) be the vertex set of \( K_{n+1} \) and let \( \mathcal{D} \) be a uniform covering of 2-paths by Hamilton cycles in \( K_{n+1} \). Let \( K_n \) be the complete graph with the vertex set \( V_n = V_{n+1} \setminus \{v_0\} \). For each Hamilton cycle \( H \in \mathcal{D} \), we obtain a Hamilton path in \( K_n \) by removing the point \( v_0 \) and the two edges incident to \( v_0 \) from \( H \). We denote it by \( H' \). Put \( \mathcal{D}' = \{H' \mid H \in \mathcal{D}\} \), then \( \mathcal{D}' \) is a uniform covering of 2-paths by Hamilton paths in \( K_n \). \( \square \)

The proof of Theorem 1.3 is immediate from Lemma 1.2 and the existence of a uniform covering of 2-paths by Hamilton cycles in \( K_n \) when \( n \) is even \( \geq 4 \).

Theorem 1.3 [8] Let \( n \) be an odd integer \( \geq 3 \). Then there exists a set \( S \) of Hamilton paths in \( K_n \) having the property that each 2-path in \( K_n \) lies in exactly one path in \( S \).

When \( n \) is even, the problem of Theorem 1.3 is still open, but Verrall constructed a double covering of 2-paths by Hamilton paths:

Theorem 1.4 [8] Let \( n \) be an even integer \( \geq 4 \). Then there exists a set \( S \) of Hamilton paths in \( K_n \) having the property that each 2-path in \( K_n \) lies in exactly two paths in \( S \).

2 Proof of Theorem 1.1

There are \( n(n-1)(n-2)/2 \) 2-paths in \( K_n \) and three 2-paths in a 4-path, so \( n(n-1)(n-2)/6 \) 4-paths are needed to cover the 2-paths in \( K_n \). This is an integer for \( n \geq 3 \).

When \( n = 3 \) or 4, \( K_n \) has 2-paths but doesn’t have 4-paths, so there is no uniform covering of 2-paths by 4-paths in \( K_n \). We consider the case \( n \geq 5 \).

Lemma 2.1 There is a uniform covering of 2-paths by 4-paths in \( K_n \) when \( n = 5 \).

Proof. Let \( \{0,1,2,3,4\} \) be the vertex set of \( K_5 \). Let \( S \) be a set of 4-paths:

\[
S = \{[2,4,0,1,3], \ [3,0,1,2,4], \ [4,1,2,3,0], \ [0,2,3,4,1],
    [1,3,4,0,2], \ [1,2,0,3,4], \ [2,3,1,4,0], \ [3,4,2,0,1],
    [4,0,3,1,2], \ [0,1,4,2,3]\},
\]

then \( S \) is a uniform covering of 2-paths by 4-paths in \( K_5 \). \( \square \)

Now we prove Theorem 1.1. We use induction on \( n \). When \( n = 5 \) there is a uniform covering of 2-paths by 4-paths in \( K_n \) from Lemma 2.1. Let \( n \geq 6 \) and assume that there is a uniform covering of 2-paths by 4-paths in \( K_{n-1} \).

Put \( m = n - 1 \). Let \( K_n \) be the complete graph with vertex set \( V = \{x\} \cup V' \), where \( |V'| = m \). Let \( K_m \) be the complete graph with vertex set \( V' \). By the induction
hypothesis, there is a uniform covering \( S' \) of the 2-paths in \( K_m \) by 4-paths. Let \( T \) and \( T' \) be the sets of all 2-paths in \( K_n \) and \( K_m \), respectively.

Put \( T_1 = \{(a, b, x) \mid a, b \in V', \ a \neq b\} \), \( T_2 = \{(a, x, b) \mid a, b \in V', \ a \neq b\} \), and \( T'' = T_1 \cup T_2 \), where \( (a, b, x), (a, x, b) \) are 2-paths. Then we have \( T = T' \cup T'' \). We already covered the 2-paths in \( T' \) by \( S' \), so we will construct a set \( S'' \) of 4-paths in \( K_n \) that will cover the 2-paths in \( T'' \).

We will construct 4-paths of type \((a, b, x, c, d)\) to cover \( T'' \), where \( a, b, c, d(\in V') \) are all different. Note that \( |T_1| = m(m-1) \) and \( |T_2| = m(m-1)/2 \). We will construct \( S'' \) by considering the two cases of \( m \) odd and \( m \) even separately.

(Case 1) \( m \) is odd.

There is a Hamilton cycle decomposition \( \mathcal{H} \) in \( K_m \), that is, there is a set \( \mathcal{H} \) of Hamilton cycles in \( K_m \) such that each edge of \( K_m \) lies in exactly one cycle in \( \mathcal{H} \). \(|\mathcal{H}| = (m - 1)/2\). For each Hamilton cycle \( H = (v_1, v_2, \ldots, v_m) \) in \( \mathcal{H} \), define a set \( S(H) \) of 4-paths:

\[
S(H) = \{ [v_1, v_2, x, v_3, v_4], \quad [v_2, v_3, x, v_4, v_5], \quad \ldots \quad [v_{m-1}, v_m, x, v_1, v_2], \quad [v_m, v_1, x, v_2, v_3] \}.
\]

Define \( S'' = \bigcup_{H \in \mathcal{H}} S(H) \). We will show that \( S'' \) covers each 2-path in \( T'' \) exactly once.

(i) Let \((a, b, x)\) be any 2-path in \( T_1 \). There is a Hamilton cycle \( H = (v_1, v_2, \ldots, v_m) \in \mathcal{H} \) which contains the edge \((a, b)\). So we can write \( a = v_i, \ b = v_{i+1} \) or \( a = v_{i+1}, \ b = v_i \), for some \( i, 1 \leq i \leq m \), where subscripts are calculated modulo \( m \). In either case, the 2-path \((a, b, x)\) is in some 4-path in \( S(H) \).

(ii) Let \((a, x, b)\) be any 2-path in \( T_2 \). There is a Hamilton cycle \( H = (v_1, v_2, \ldots, v_m) \in \mathcal{H} \) which contains the edge \((a, b)\). So we can write \( a = v_i, \ b = v_{i+1} \) or \( a = v_{i+1}, \ b = v_i \), for some \( i, 1 \leq i < m \). In either case, the 2-path \((a, x, b)\) is in a 4-path \([v_{i-1}, v_i, x, v_{i+1}, v_{i+2}] \) in \( S(H) \).

Since the numbers of 2-paths in \( T'' \) and in \( S'' \) are equal, \( S'' \) covers each 2-path in \( T'' \) exactly once.

(Case 2) \( m \) is even.

Label the vertices in \( V' \) as \( \infty, 0, 1, \ldots, m - 2 \). Put \( r = (m - 2)/2 \). Let \( \sigma \) be the following permutation of the vertices of \( K_{m+1} \): \( \sigma = (\infty)(x)(0\ 1\ 2\ \ldots\ m-2) \), and put \( \Sigma = \langle \sigma \rangle = \{ \sigma^j \mid 0 \leq j \leq m - 2 \} \). Define the set \( S^0 \) of 4-paths:

\[
S^0 = \{ [r + 1, \infty, x, 0, 1], \quad [0, 1, x, m - 2, 2], \quad [m - 2, 2, x, m - 3, 3], \quad [m - 3, 3, x, m - 4, 4], \quad \ldots \quad [r + 3, r - 1, x, r + 2, r], \quad [r + 2, r, x, r + 1, \infty] \}.
\]

Note that the set of edges \( \{(u_2, u_3) \mid [u_1, u_2, x, u_3, u_4] \in S^0 \} \) is \( F_0 \) and the set of arcs \( \{(u_1, u_2) \mid [u_1, u_2, x, u_3, u_4] \in S^0 \} \) which equals the set \( \{(u_4, u_3) \mid [u_1, u_2, x, u_3, u_4] \in S^0 \} \) is \( F_{r+1}^* \), where

\[
F_0 = \{ (\infty, 0) \} \cup \{ (u, v) \mid u + v \equiv 0 \pmod{m-1}, \ u, v \in V', \ u, v \neq \infty, \ u \neq v \} \quad F_{r+1}^* = \{ (\infty, r + 1), (r + 1, \infty) \} \cup \{ (u, v) \mid u + v \equiv 1 \pmod{m-1}, \ u, v \in V', \ u, v \neq \infty, \ u \neq v \}.
\]

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Put $S'' = \Sigma S_0^0 = \{ P^{\sigma_j} \mid P \in S_0^0, \ 0 \leq j \leq m - 2 \}$. We will show that $S''$ is a set of 4-paths in $K_n$ that covers each 2-path in $T''$ exactly once.

(i) Let $(a, b, x)$ be any 2-path in $T_1$. Then there is an arc $(u, v) \in F_{r+1}^*$ such that $(a, b) = (u, v)^{\sigma_j}$ for some $j$. Since $\{(u_1, u_2) \mid [u_1, u_2, x, u_3, u_4] \in S_0^0 \} = F_{r+1}^*$, $[u, v, x, u_3, u_4] \in S_0^0$ for some $u_3, u_4 \in V'$. Therefore $[u, v, x, u_3, u_4]^{\sigma_j} = [a, b, x, u_3^{\sigma_j}, u_4^{\sigma_j}] \in S''$. Thus $S''$ covers the 2-path $(a, b, x)$.

(ii) Let $(a, x, b)$ be any 2-path in $T_2$. There is an edge $\{u, v\} \in F_0$ such that $(a, b) = \{u, v\}^{\sigma_j}$ for some $j$. Since $\{(u_2, u_3) \mid [u_1, u_2, x, u_3, u_4] \in S_0^0 \} = F_0, [u_1, u, x, v, u_4] \in S_0^0$ for some $u_1, u_4 \in V'$. Therefore $[u_1, u, x, v, u_4]^{\sigma_j} = [u_1^{\sigma_j}, a, x, b, u_4^{\sigma_j}] \in S''$. Thus $S''$ covers the 2-path $(a, x, b)$.

Hence $S''$ covers each 2-path in $T''$ exactly once.

Put $S = S' \cup S''$, then $S$ is a set of 4-paths with the property that each 2-path in $T$ lies in exactly one path in $S$. This completes the proof of Theorem 1.1. □

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References


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