On a conjecture of Hilton

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Abstract

We show that if $A_1, A_2, \ldots, A_k$ are collections of distinct subsets from an $n$-element set such that these collections are incomparable and uncomplemented, then $\sum_{i=1}^k |A_i| \leq 2^{n-1}$ under certain conditions. Upper bounds are also given for $\sum_{i=1}^k |A_i|$ with or without the "uncomplemented" condition.

1 Introduction

Let $A_1, \ldots, A_k$ be $k$ collections of distinct subsets of $S = \{1, 2, \ldots, n\}$. These $k$ collections of distinct subsets are called incomparable if $A_i \not\subseteq A_j$ and $A_j \not\subseteq A_i$, $(i \neq j)$, then $A_i \not\subseteq A_j$. A collection of subsets $C$ is called uncomplemented if $A \not\subseteq C$, then $A \not\subseteq C$, where $A = S \setminus A$.

It is well known that if $C$ is a collection of distinct subsets of $\{1, 2, \ldots, n\}$ which are uncomplemented, then $|C| \leq 2^{n-1}$. Hilton extended this result to two incomparable, uncomplemented collections

Theorem 1 [2] If $A_1$ and $A_2$ are collections of distinct subsets of $S$ such that these collections are incomparable and uncomplemented, then

$$|A_1| + |A_2| \leq 2^{n-1}.$$

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He also posed the following conjecture.

**Conjecture 1** [4] If $A_1, A_2, \ldots, A_k$ are collections of distinct subsets of $n$-element set $S$ such that these collections are incomparable and uncomplemented, then

$$\sum_{i=1}^{k} |A_i| \leq 2^{n-1}.$$

In this paper, we will investigate this conjecture. We give an upper bound and show that this conjecture is true under certain conditions. We also discuss the case when $k = 3$.

The following lemma from Kleitman will be used in our proof.

**Lemma 2** [3] Let $U$ and $V$ be collections of subsets of an $n$-element set $S$, such that
(i) if $X \in U$ and $X \subseteq Y \subseteq S$, then $Y \in U$,
(ii) if $X \in V$ and $Y \subseteq X \subseteq S$, then $Y \in V$. Then

$$|U \cap V| \cdot 2^n \leq |U||V|.$$

2  Main results

**Theorem 3** Let $A_1, A_2, \ldots, A_k$ be incomparable collections of distinct subsets of $n$-element set $S$. Then for any $1 \leq j \leq k$,

$$\sum_{i=1}^{k} |A_i| + 2(|A_j| \sum_{i \neq j} |A_i|)^{\frac{1}{2}} \leq 2^n.$$

**Proof.** Without loss of generality, we will show

$$\sum_{i=1}^{k} |A_i| + 2(|A_1| \sum_{i \neq 1} |A_i|)^{\frac{1}{2}} \leq 2^n.$$

Let

$$H = \{Z : \exists A_1 \in A_1, A_1 \subseteq Z, \exists D \in \cup_{i=2}^{k} A_i, D \subseteq Z\},$$

$$I_i = \{Z : \exists A_i \in A_i, A_i \subseteq Z, \forall D \in \cup_{j \neq i} A_j, D \subseteq Z\},$$

$$L = \{Z : \exists A_i \in A_i, A_i \subseteq Z, 1 \leq i \leq k\}.$$

Then clearly, $H \cap L = \emptyset$, $H \cap I_i = \emptyset$, $L \cap I_i = \emptyset$ for $1 \leq i \leq k$ and $I_i \cap I_j = \emptyset$ for any $i \neq j$. Therefore,

$$|H| + \sum_{i=1}^{k} |I_i| + |L| \leq 2^n.$$

Let $U = H \cup I_1$ and $V = L \cup I_1$. We claim that both $U$ and $V$ satisfy the conditions in Lemma 2. Let $X \in U$ and $X \subseteq Y \subseteq S$. Then there exists an $A_1 \in A_1$ such that $A_1 \subseteq X \subseteq Y$ by the definitions of $H$ and $I_1$. If there is a $D \in \cup_{i=2}^{k} A_i$ such that $D \subseteq X \subseteq Y$, then $Y \in H \cup U$. Otherwise, $Y \in I_1 \subseteq U$. 

266
Now let \( X \in \mathcal{V} \) and \( Y \subseteq X \). If there is no \( A_i \in \mathcal{A}_1 \) such that \( A_i \subseteq Y \), then \( Y \in \mathcal{L} \) and hence \( Y \in \mathcal{V} \). Otherwise, \( X \in \mathcal{I}_1 \). This implies that there is no \( D \in \bigcup_{j \neq 1} \mathcal{A}_j \) with \( D \subseteq Y \). Therefore, \( Y \in \mathcal{I}_1 \subseteq \mathcal{V} \).

By Lemma 2, we have
\[
|U \cap V| 2^n \leq |U| |V|.
\]
That is,
\[
|I_1| \cdot 2^n \leq (|H| + |I_1|)(|L| + |I_1|).
\]
Then
\[
|I_1|(|H| + \sum_{i=1}^k |I_i| + |L|) \leq (|H| + |I_1|)(|L| + |I_1|).
\]
Simplify,
\[
|I_1|(\sum_{i=2}^k |I_i|) \leq |H||L| \leq \left(\frac{|H| + |L|}{2}\right)^2 \leq \frac{2^n - \sum_{i=1}^k |I_i|}{2}.
\]
Therefore,
\[
\sum_{i=1}^k |I_i| + 2[|I_1| \sum_{i \neq 1} |I_i|]^{\frac{1}{2}} \leq 2^n.
\]
We note that \( A_i \subseteq \mathcal{I}_i \) for any \( i = 1, \ldots, k \) as \( A_1, \ldots, A_k \) are incomparable collections. Hence \( |A_i| \leq |I_i| \) for \( 1 \leq i \leq k \). Therefore,
\[
\sum_{i=1}^k |A_i| + 2[|A_1| \sum_{i \neq 1} |A_i|]^{\frac{1}{2}} \leq 2^n.
\]
This completes the proof. \( \blacksquare \)

**Corollary 4** Let \( A_1, A_2, \ldots, A_k \) be incomparable collections of distinct subsets of \( n \)-element set \( S \). Let \( I \) and \( J \) be any partition of \( \{1, \ldots, k\} \). Then
\[
\sum_{i=1}^k |A_i| + 2[\sum_{j \in J} |A_j| \sum_{i \in I} |A_i|]^{\frac{1}{2}} \leq 2^n.
\]

**Proof.** The corollary follows from the fact that \( \bigcup_{i \in I} A_i \) and \( \bigcup_{j \in J} A_j \) are incomparable.

\( \blacksquare \)

The following theorem gives an upper bound if there is no \( A_i \) having its cardinality too large.

**Theorem 5** Let \( A_1, A_2, \ldots, A_k \) be incomparable collections of distinct subsets of \( n \)-element set \( S \). If there is an \( I \subseteq \{1, \ldots, k\} \) such that \( \sum_{i=1}^k |A_i| \leq \frac{2^n}{\sqrt{k} - 1 + k} \), then
\[
\sum_{i=1}^k |A_i| \leq \frac{k}{2\sqrt{k} - 1 + k} 2^n.
\]

267
Proof. From Corollary 4, 
\[ \sum_{i=1}^{k} |A_i| + 2[\sum_{j \in J} |A_j| \left( \sum_{i \in I} |A_i| \right) \frac{1}{2} \leq 2^n, \]
where \( J = \{1, \ldots, k\} - I \). That is,
\[ \sum_{i=1}^{k} |A_i| + 2[\sum_{i \in I} |A_i|\left( \sum_{i=1}^{k} |A_i| - \sum_{i \in I} |A_i| \right) \frac{1}{2} \leq 2^n. \]
The function \( f(x) = \sqrt{x(a-x)} \), where \( a = \sum_{i=1}^{k} |A_i| \) is a constant, is an increasing function for \( 0 \leq x \leq \frac{a}{2} \). Therefore, we can replace \( \sum_{i \in I} |A_i| \) by the average \( \frac{\sum_{i=1}^{k} |A_i|}{k} \) in the above inequality. We have
\[ \sum_{i=1}^{k} |A_i| + 2\left[ \frac{\sum_{i=1}^{k} |A_i|}{k} \left( \sum_{i=1}^{k} |A_i| - \frac{\sum_{i=1}^{k} |A_i|}{k} \right) \right] \frac{1}{2} \leq 2^n. \]
Solving for \( \sum_{i=1}^{k} |A_i| \) yields
\[ \sum_{i=1}^{k} |A_i| \leq \frac{k}{2\sqrt{k-1} + k} 2^n. \]
This completes the proof. \( \blacksquare \)

Corollary 6 If \( A_1 \) and \( A_2 \) are incomparable collections of distinct subsets of \( n \)-element set \( S \) with \( |A_1| = |A_2| \), then
\[ |A_1| + |A_2| \leq 2^{n-1}. \]

Corollary 7 If \( A_1, A_2, \ldots, A_k \) are incomparable and uncomplemented collections of distinct subsets of \( n \)-element set \( S \), then either
\[ \sum_{i=1}^{k} |A_i| \leq \frac{k}{2\sqrt{k-1} + k} 2^n \]
\[ \text{or} \]
\[ \sum_{i=1}^{k} |A_i| < \frac{k}{k-1} 2^{n-1}. \]

Proof. Without loss of generality, we assume \( |A_1| \leq |A_2| \leq \cdots \leq |A_k| \). If \( |A_k| \leq \sum_{i=1}^{k-1} |A_i| \), then we take \( I = \{k\} \) in Theorem 5, we have \( \sum_{i=1}^{k} \frac{|A_i|}{k} \leq \sum_{i \in I} |A_i| \leq \sum_{i=1}^{k} |A_i| \). Therefore,
\[ \sum_{i=1}^{k} |A_i| \leq \frac{k}{2\sqrt{k-1} + k} 2^n. \]
Thus, we may assume $|\mathcal{A}_k| > \sum_{i=1}^{k-1} |\mathcal{A}_i|$. If $\sum_{i=1}^{k-1} |\mathcal{A}_i| \leq \sum_{i=1}^{k-1} |\mathcal{A}_i|$, then we take $I = \{1, \ldots, k-1\}$ and have $\sum_{i=1}^{k-1} |\mathcal{A}_i| \leq \sum_{i \in I} |\mathcal{A}_i| \leq \frac{\sum_{i=1}^{k-1} |\mathcal{A}_i|}{2}$. Hence

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \frac{k}{2\sqrt{k-1} + k} 2^n$$

by Theorem 5 again.

Therefore, $\sum_{i=1}^{k} |\mathcal{A}_i| > \sum_{i=1}^{k-1} |\mathcal{A}_i|$. That is $\sum_{i=1}^{k} |\mathcal{A}_i| > k \sum_{i=1}^{k-1} |\mathcal{A}_i|$. Thus, $|\mathcal{A}_k| > (k-1) \sum_{i=1}^{k-1} |\mathcal{A}_i|$. This is equivalent to $k |\mathcal{A}_k| > (k-1) \sum_{i=1}^{k-1} |\mathcal{A}_i|$. But $|\mathcal{A}_k| \leq 2^{n-1}$ as $\mathcal{A}_k$ is uncomplemented. Therefore,

$$\sum_{i=1}^{k} |\mathcal{A}_i| < \frac{k}{k-1} 2^{n-1}.$$

This completes the proof. ■

In [5], Seymour proved the following result.

**Theorem 8** If $\mathcal{A}$ is a collection of subsets of $n$-set $S$ such that for all $A, B \in \mathcal{A}$, $A \cap B \neq \emptyset$ and $A \cup B \neq S$, then $|\mathcal{A}| \leq 2^{n-2}$.

Combining Theorems 5 and 8, we have the following result.

**Theorem 9** Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ be incomparable collections of distinct subsets of $n$-element set $S$. If for each $\mathcal{A}_i$, $A, B \in \mathcal{A}_i$, $A \cap B \neq \emptyset$ and $A \cup B \neq S$, then

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \frac{k}{2\sqrt{k-1} + k} 2^n.$$ 

**Proof.** We have that for each $i$, $|\mathcal{A}_i| \leq 2^{n-2}$ by Seymour's result. Let $a = \sum_{i=1}^{k} |\mathcal{A}_i|$. If $a \leq 2^{n-1}$, then we are done. Otherwise, we have that for any $i$, $|\mathcal{A}_i| \leq \frac{a}{2}$ from Theorem 8. Therefore,

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \frac{k}{2\sqrt{k-1} + k} 2^n,$$

by Theorem 5. ■

**Lemma 10** Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be collections of distinct subsets of $n$-element set $S$ such that $\mathcal{A}_1$ and $\mathcal{A}_2$ are incomparable and $\mathcal{A}_1$ is uncomplemented. Then

(a) $|\mathcal{A}_1| \leq 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lfloor \frac{n}{3} \rfloor} + 2$ if $\mathcal{A}_2$ contains a pair of complemented sets.

(b) $|\mathcal{A}_1| \leq 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lfloor \frac{n}{3} \rfloor} - 2^{\lfloor \frac{n}{4} \rfloor} - 1 + 2$ if $\mathcal{A}_2$ contains more than one pair of complemented sets.

**Proof.** (a) Let $\mathcal{A}_{1i}$ ($1 \leq i \leq 2$) and $\mathcal{A}_{2j}$ ($1 \leq j \leq 3$) be such that

$$\begin{align*}
\mathcal{A}_1 &= \mathcal{A}_{11} \cup \mathcal{A}_{12}, \\
\mathcal{A}_2 &= \mathcal{A}_{21} \cup \mathcal{A}_{22} \cup \mathcal{A}_{23} \cup \bar{\mathcal{A}}_{23},
\end{align*}$$

269
where $A_{12} = \bar{A}_{21}$, $A_{11} \cap A_{12} = \emptyset$, $A_{2i} \cap A_{2j} = \emptyset$ for $i \neq j$ and $1 \leq i, j \leq 3$, $A_{2i} \cap \bar{A}_{23} = \emptyset$ for $1 \leq i \leq 3$, and $A_{2} \cap \bar{A}_{2} = \emptyset$.

Since $|A_{23}| \neq 0$, we can choose $A_{23} \in A_{23}$. Clearly, $S = A_{23} \cup \bar{A}_{23}$. We let $A_{23} = \{a_1, \ldots, a_k\}$ and $\bar{A}_{23} = \{a_{k+1}, \ldots, a_n\}$. Without loss of generality, we assume that $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

We have that, for any $A_1 \in A_1$,

\[
\begin{cases}
A_1 \cap A_{23} \neq \emptyset, \\
A_1 \cap \bar{A}_{23} \neq \emptyset, \\
\bar{A}_1 \cap A_{23} \neq \emptyset, \\
\bar{A}_1 \cap \bar{A}_{23} \neq \emptyset.
\end{cases}
\]  

(\ast)

This claim is true since otherwise $A_1$ and $A_2$ are not incomparable, which contradicts our assumption.

It follows from (\ast) that any element $A_1$ of $A_1$ can be written as $A_1 = A_{11} \cup A_{12}$, where $A_{11}$ and $A_{12}$ are proper subsets of $A_{23}$ and $\bar{A}_{23}$, respectively. Obviously, $1 \leq |A_{11}| \leq k - 1$ and $1 \leq |A_{21}| \leq n - k - 1$. It is easy to see that there are at most

\[
\sum_{j=1}^{n-k-1} \sum_{i=1}^{k-1} \binom{k}{i} \binom{n-k}{j} = (2^k - 2)(2^{n-k} - 2)
\]

such subsets satisfying the property (\ast).

Note that if $A_1 = A_{11} \cup A_{12}$, where $A_{11} \subset A_{23}$ and $A_{12} \subset \bar{A}_{23}$, then $(A_{23} \setminus A_{11}) \cup (\bar{A}_{23} \setminus A_{12})$ is also a subset satisfying the property (\ast). Since $A_1$ is uncomplemented, we have that

\[
|A_1| \leq \frac{1}{2}(2^k - 2)(2^{n-k} - 2) = 2^{n-1} - 2^k - 2^{n-k} + 2.
\]  

(\ast\ast)

It is easy to verify that the function $2^x + 2^{n-x}$ is a decreasing function if $1 \leq x \leq \lfloor \frac{n}{2} \rfloor$. Therefore, taking $x = \lfloor \frac{n}{2} \rfloor$, we have

\[
|A_1| \leq 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{n-\lfloor \frac{n}{2} \rfloor} + 2 = 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lfloor \frac{n}{2} \rfloor} + 2.
\]

This completes the proof of (a).

(b) We divide the proof of (b) into two cases.

Case 1. $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$.

Taking $x = \lfloor \frac{n}{2} \rfloor - 1$ in (\ast\ast), we have

\[
|A_1| \leq 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor - 1} - 2^{n-\lfloor \frac{n}{2} \rfloor + 1} + 2 = 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor - 1} - 2^{\lfloor \frac{n}{2} \rfloor + 1} + 2 \leq 2^{n-1} - 2^{n/2} - 2^{n/2} - 2^{n/2-1} + 2.
\]

Case 2. $k = \lfloor \frac{n}{2} \rfloor$.  

270
In this case, we have, that for any $A_{23} \in A_{23}$, $|A_{23}| = \lfloor \frac{n}{2} \rfloor$. We pick a $B_1 \in A_{23}$ and a $B_2 \in A_{23}$ where $B_1 \neq \overline{B_2}$, $B_1 \neq B_2$, and $|B_1| = |B_2| = \lfloor \frac{n}{2} \rfloor$.

**Case 2.1.** $n \equiv 1 \ (\text{mod} \ 2)$.

Observe that $B_1 \cap B_2 < \lfloor \frac{n}{2} \rfloor$. Otherwise we would have a contradiction. First we assume that $1 \leq |B_1 \cap B_2| = x < \lfloor \frac{n}{2} \rfloor$. Then $|B_1 \cap \overline{B_2}| = \lfloor \frac{n}{2} \rfloor - x$. By repeating the argument in the proof of (a) we deduce that the number of $A$'s which intersect $B_1$ and $\overline{B_1}$ properly, and do not contain all, is $(2^{\lfloor \frac{n}{2} \rfloor} - 2)(2^{n-\lfloor \frac{n}{2} \rfloor} - 2)$. The number of these $A$'s contained in $B_2$ is $(2^x - 1)(2^{\lfloor \frac{n}{2} \rfloor - x} - 1)$. The number of these $A$'s containing $B_2$ is $(2^{x+1} - 1)(2^{\lfloor \frac{n}{2} \rfloor - x} - 1)$. The number of these $A$'s contained in $\overline{B_2}$ is $(2^{x+1} - 1)(2^{\lfloor \frac{n}{2} \rfloor - x} - 1)$. Therefore,

$$|A_1| \leq \frac{1}{2} \{(2^{\lfloor \frac{n}{2} \rfloor} - 2)(2^{n-\lfloor \frac{n}{2} \rfloor} - 2) - 2(2^x - 1)(2^{\lfloor \frac{n}{2} \rfloor - x} - 1)$$

$$- 2(2^{x+1} - 1)(2^{\lfloor \frac{n}{2} \rfloor - x} - 1)\}$$

$$\leq 2^{n-1} - 2^{\lfloor n/2 \rfloor} - 2^{n-\lfloor n/2 \rfloor} - 2^{\lfloor n/2 \rfloor - 1} + 2.$$

Next we assume $|B_1 \cap B_2| = 0$. Then $B_2 \subset \overline{B_1}$ and $|\overline{B_1} \cap \overline{B_2}| = 1$. It follows that $|\overline{B_1}| = |\overline{B_2}| = \lfloor \frac{n}{2} \rfloor + 1$. Repeating the proof in the above, we deduce that the number of $A$'s which intersect $B_1$ and $\overline{B_1}$ properly, and do not contain all, is $(2^{\lfloor \frac{n}{2} \rfloor} - 2)(2^{n-\lfloor \frac{n}{2} \rfloor} - 2)$. The number of these $A$'s contained in $\overline{B_2}$ is $2^{\lfloor n/2 \rfloor} - 2$. The number of these $A$'s containing $B_2$ is $2^{\lfloor n/2 \rfloor} - 2$. Therefore,

$$|A_1| \leq \frac{1}{2} \{(2^{\lfloor \frac{n}{2} \rfloor} - 2)(2^{n-\lfloor \frac{n}{2} \rfloor} - 2) - 2(2^{\lfloor n/2 \rfloor} - 2)\}$$

$$\leq 2^{n-1} - 2^{\lfloor n/2 \rfloor} - 2^{n-\lfloor n/2 \rfloor} - 2^{\lfloor n/2 \rfloor - 1} + 2.$$

**Case 2.2.** $n \equiv 0 \ (\text{mod} \ 2)$.

In this case we only have that $1 \leq |B_1 \cap B_2| \leq \lfloor n/2 \rfloor - 1$. By repeating the argument of Case 2.1, we conclude that

$$|A_1| \leq \frac{1}{2} \{(2^{\lfloor \frac{n}{2} \rfloor} - 2)(2^{n-\lfloor \frac{n}{2} \rfloor} - 2) - 2(2^x - 1)(2^{\lfloor \frac{n}{2} \rfloor - x} - 1)$$

$$- 2(2^{x+1} - 1)(2^{\lfloor \frac{n}{2} \rfloor - x} - 1)\}$$

$$\leq 2^{n-1} - 2^{\lfloor n/2 \rfloor} - 2^{n-\lfloor n/2 \rfloor} - 2^{\lfloor n/2 \rfloor - 1} + 2.$$

This completes the proof. ■

**Theorem 11** If $A_1$, $A_2$, $\ldots$, $A_k$ are collections of distinct subsets of $n$-element set $S$ such that these collections are incomparable and uncomplemented, then

$$\sum_{i=1}^{k} |A_i| \leq 2^{n-1},$$

if $\max_{1 \leq i \leq k} |A_i| > 2^{n-1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lfloor \frac{n}{2} \rfloor} + 2$.
Proof. Without loss of generality, we assume that $|\mathcal{A}_1| = \max_{1 \leq i \leq k} \{|\mathcal{A}_i|\}$. Let $\mathcal{B} = \bigcup_{i=2}^{n} A_i$. Then $\mathcal{A}_1$ and $\mathcal{B}$ are incomparable. If $\mathcal{B}$ is not uncomplemented, then $|\mathcal{A}_1| \leq 2^{n-1} - 2^{l_{\frac{3}{2}}} - 2^{l_{\frac{3}{2}}} - 2^{l_{\frac{3}{2}}} - 2$ by Lemma 10 (a), which is a contradiction. Therefore, both $\mathcal{A}_1$ and $\mathcal{B}$ are uncomplemented and hence $|\mathcal{A}_1| + |\mathcal{B}| \leq 2^{n-1}$ by Theorem 1. That is, $\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1}$. ■

Theorem 12 If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ are collections of distinct subsets of $n$-elements set $S$ such that these collections are incomparable and uncomplemented, then

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1} + 1,$$

if $\max_{1 \leq i \leq k} \{|\mathcal{A}_i|\} > 2^{n-1} - 2^{l_{\frac{3}{2}}} - 2^{l_{\frac{3}{2}}} - 2^{l_{\frac{3}{2}}}$. ■

Proof. Without loss of generality, we assume that $|\mathcal{A}_1| = \max_{1 \leq i \leq k} \{|\mathcal{A}_i|\}$. Let $\mathcal{B} = \bigcup_{i=2}^{n} A_i$. Then $\mathcal{A}_1$ and $\mathcal{B}$ are incomparable. If $\mathcal{B}$ contains more than one pair of complemented sets, then $|\mathcal{A}_1| \leq 2^{n-1} - 2^{l_{\frac{3}{2}}} - 2^{l_{\frac{3}{2}}} - 2^{l_{\frac{3}{2}}} - 2$ by Lemma 10 (b), which is a contradiction. Therefore, $\mathcal{B}$ contains at most one pair of complemented sets. Let $U$ be one of the set in the pair. Then $\mathcal{B} - U$ is uncomplemented, therefore, $|\mathcal{A}_1| + |\mathcal{B} - U| \leq 2^{n-1}$. That is, $\sum_{i=1}^{k} |\mathcal{A}_i| \leq 2^{n-1} + 1$. ■

3 The case $k = 3$

Let $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ be collections of distinct subsets of $n$-element set $S$ such that these collections are incomparable and uncomplemented. Then we can partition $\mathcal{A}_1$ into $\mathcal{A}_{11}, \mathcal{A}_{12}, \mathcal{A}_{13}$ such that $\overline{\mathcal{A}}_{12}$ is contained in $\mathcal{A}_2$ and $\overline{\mathcal{A}}_{13}$ is contained in $\mathcal{A}_3$. The similar partition applies to $\mathcal{A}_2$ and $\mathcal{A}_3$. Therefore, we have the following partitions:

$\mathcal{A}_1 = \mathcal{A}_{11} \cup \mathcal{A}_{12} \cup \mathcal{A}_{13}, \mathcal{A}_2 = \mathcal{A}_{21} \cup \mathcal{A}_{22} \cup \mathcal{A}_{23}, \mathcal{A}_3 = \mathcal{A}_{31} \cup \mathcal{A}_{32} \cup \mathcal{A}_{33},$

such that $\overline{\mathcal{A}}_{i,j} = \mathcal{A}_{j,i}$ for $i \neq j$.

We have the following result.

Theorem 13 Let $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ be collections of distinct subsets of $n$-element set $S$ such that these collections are incomparable and uncomplemented. Then for any $1 \leq i, j \leq 3,$

$$|\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| \leq 2^{n-1} + |\mathcal{A}_{i,j}|.$$

Proof. Without loss of generality, we need only to show that

$$|\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| \leq 2^{n-1} + |\mathcal{A}_{3,j}|,$$

for $j = 1, 2, 3$. There are three cases.

Case 1. $j = 1$.

Let $\mathcal{B}_1 = \mathcal{A}_1 \cup \mathcal{A}_{32} \cup \mathcal{A}_{33}$ and $\mathcal{B}_2 = \mathcal{A}_2$. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are collections of uncomplemented and incomparable. By Theorem 1,

$$|\mathcal{B}_1| + |\mathcal{B}_2| \leq 2^{n-1}.$$
Therefore,
\[ |A_1| + |A_2| + |A_3| \leq 2^{n-1} + |A_{3,1}|. \]

**Case 2.** \( j = 2. \)

The proof is similar to Case 1.

**Case 3.** \( j = 3. \)

Let \( C_1 = A_1 \cup A_{3,2} \) and \( C_2 = A_2 \cup A_{3,1}. \) Then \( C_1 \) and \( C_2 \) are collections of uncomplemented subsets from \( S. \) To show that they are incomparable, we need to show that if \( A \in C_1, A = A_{3,2} \in A_{3,2}, \) and \( B \in C_2, B = A_{3,1} \in A_{3,1}, \) then \( A \not\subseteq B \) and \( B \not\subseteq A. \) We observe that \( A_{3,2} \not\subseteq A_{3,1}. \) Otherwise, \( A_{3,1} \subset A_{3,2}. \) But \( A_{3,1} \) is in \( A_{1,3} \) and \( A_{3,2} \) is in \( A_{2,3}, \) which contradicts the fact that \( A_1 \) and \( A_2 \) are incomparable. Similarly, \( A_{3,1} \not\subseteq A_{3,2}. \) Therefore, \( C_1 \) and \( C_2 \) are incomparable.

By Theorem 1 again,
\[ |C_1| + |C_2| \leq 2^{n-1}. \]

Therefore,
\[ |A_1| + |A_2| + |A_3| \leq 2^{n-1} + |A_{3,3}|. \]

This completes the proof. \( \blacksquare \)

**Remark** We note that in many cases, \( \min\{ |A_{ij}| : 1 \leq i, j \leq 3 \} \) is zero.

**Corollary 14** Let \( A_1, A_2 \) and \( A_3 \) be collections of distinct subsets of \( n \)-element set \( S \) such that these collections are incomparable and uncomplemented. Then
\[ |A_1| + |A_2| + |A_3| \leq \frac{9}{8} \cdot 2^{n-1}. \]

**Proof.** By Theorem 13, we have that for any \( 1 \leq i, j \leq 3, \)
\[ |A_1| + |A_2| + |A_3| \leq 2^{n-1} + |A_{i,j}|. \]

Summing up over all \( 1 \leq i, j \leq 3, \) we have
\[ 9(|A_1| + |A_2| + |A_3|) = 9 \times 2^{n-1} + (|A_1| + |A_2| + |A_3|). \]

Therefore,
\[ |A_1| + |A_2| + |A_3| \leq \frac{9}{8} \cdot 2^{n-1}. \]

This completes the proof. \( \blacksquare \)

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References


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