On the spectral radius of nonnegative matrices*

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Abstract

We give lower bounds for the spectral radius of nonnegative matrices and nonnegative symmetric matrices, and prove necessary and sufficient conditions to achieve these bounds.

1 Introduction and Preliminaries

In this note, we will be concerned with nonnegative matrices. Let $A$ be an $n \times n$ nonnegative matrix. The spectral radius of $A$ is denoted by $\rho(A)$. Due to the Perron-Frobenius theorem, $\rho(A)$ is an eigenvalue, also known as the Perron root of $A$. For a matrix $X$, $X^t$ denotes the transpose of $X$.

A nonnegative matrix is row-regular if all of its row sums are equal. A matrix $A$ is row-semiregular if there is a permutation matrix $P$ such that $P^tAP = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ where both $B$ and $C$ are row-regular. Column-regular and column-semiregular are defined similarly. The matrix $A$ is regular if $A$ is both row-regular and column-regular. Semiregular is defined similarly.

An $n \times n$ nonnegative matrix $A$ all of whose row sums $d_1, \ldots, d_n$ are positive is almost row-regular if $a_{ij} > 0$ implies that $d_id_j$ is a constant. Almost column-regular is defined similarly. $A$ is almost regular if $A$ is both almost row-regular and almost column-regular.

In this paper, we give lower bounds for the spectral radius of nonnegative matrices and nonnegative symmetric matrices, and prove necessary and sufficient conditions to achieve these bounds.

**Lemma 1.1** Let $A = (a_{ij})$ be an $n \times n$ nonnegative irreducible matrix with positive row sums $d_1, d_2, \ldots, d_n$. Then the following are equivalent:

1. $A$ is almost row-regular;

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(2) A is row-regular or row-semiregular;

(3) for each $1 \leq i \leq n$ and any number $a \neq 0$, \[
\sum_{j=1}^{n} a_{ij} (d_i d_j)^a \frac{i=1}{d_i} \text{ is a constant.}
\]

Proof. Note that (2) \Rightarrow (1) and (1) \Rightarrow (3) are obvious. We need only to prove (3) \Rightarrow (2).

Suppose for each $1 \leq i \leq n$ and any number $a \neq 0$, \[
\sum_{j=1}^{n} a_{ij} (d_i d_j)^a \frac{i=1}{d_i} = r. \]
If all the $d_i$ are equal, then A is row-regular. Otherwise set $\delta = \min_{1 \leq i \leq n} d_i$ and $\Delta = \max_{1 \leq i \leq n} d_i$.

Note that A is irreducible. Choose $u$ and $v$ such that $d_u = \delta$ and $d_v = \Delta$. Suppose without loss of generality that $a > 0$. Then we have

\[
r = \sum_{j=1}^{n} a_{uj} (\delta d_j)^a \frac{j=1}{\delta} \leq (\delta \Delta)^a
\]

and

\[
r = \sum_{j=1}^{n} a_{vj} (\Delta d_j)^a \frac{j=1}{\Delta} \geq (\delta \Delta)^a.
\]
It follows that $r = (\delta \Delta)^a$, and whenever $a_{ij} > 0$, then $d_i = \delta$ and $d_j = \Delta$ or vice versa. Note that $\delta < \Delta$. Then $d_i = d_j$ implies that $a_{ij} = 0$. Set $\alpha = \{i : d_i = \delta\}$ and $\beta = \{i : d_i = \Delta\}$. Then $\alpha \cap \beta = \emptyset$, $\alpha \cup \beta = \{1, \ldots, n\}$, $A[\alpha, \alpha] = 0$, and $A[\beta, \beta] = 0$.

Hence there is a permutation matrix $P$ such that $P^t AP = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ where each row sum of $B$ is $\delta$ and each row sum of $C$ is $\Delta$. We conclude that $A$ is row-semiregular.

The proof is now completed. \qed

For a nonnegative symmetric matrix, the same argument as in the proof of Lemma 1.1 will apply to all irreducible components of $A$. Hence we have the following.

Lemma 1.2 Let $A = (a_{ij})$ be an $n \times n$ nonnegative symmetric matrix with positive row sums $d_1, d_2, \ldots, d_n$. Then the following are equivalent:

(1) $A$ is almost regular;

(2) $A$ is regular or semiregular;

(3) for each $1 \leq i \leq n$ and any number $a \neq 0$, \[
\sum_{j=1}^{n} a_{ij} (d_i d_j)^a \frac{j=1}{d_i} \text{ is a constant.}
\]

2 Nonnegative matrices and digraphs

The following Lemma is contained in [1].

Lemma 2.1 Let $A$ be an $n \times n$ nonnegative matrix. Then

\[
\rho(A) \geq \min \frac{(Ax)_i}{x_i}. \tag{2.1}
\]

302
If \( A \) is irreducible, then equality holds in (2.1) if and only if \( x \) is an eigenvector corresponding to \( \rho(A) \).

**Theorem 2.2** Let \( A = (a_{ij}) \) be an \( n \times n \) nonnegative matrix with positive row sums \( d_1, d_2, \ldots, d_n \). Then

\[
\rho(A) \geq \min_{1 \leq i \leq n} \left( \sum_{j=1}^{n} a_{ij}d_j \right).
\]  

If \( A \) is reducible, then equality holds in (2.2) if and only if \( A \) is row-regular or row-semi-regular.

**Proof.** Let \( A^2 = B = (b_{ij}) \). On setting \( x = (1, \ldots, 1)^t \), by Lemma 2.1 we obtain

\[
\rho(B) \geq \min_{1 \leq i \leq n} \left( \frac{(Bx)_i}{x_i} \right) = \min_{1 \leq i \leq n} \sum_{s=1}^{n} a_{is} = \min_{1 \leq i \leq n} \sum_{j=1}^{n} a_{ij}d_j.
\]

Observe that \( \rho(A) = \sqrt{\rho(B)} \). From (2.3) we have (2.2), as desired.

Suppose that \( A \) is irreducible. Now we are going to prove the second part of Theorem 2.2.

First suppose that \( A \) is row-regular or row-semi-regular. By Lemma 1.1, for each \( 1 \leq i \leq n \), \( \sum_{j=1}^{n} a_{ij}d_j \) is a constant. Then for some constant \( r \) and each \( i \),

\[
\sum_{s=1}^{n} b_{is} = \sum_{s=1}^{n} a_{is}a_{js} = \sum_{j=1}^{n} a_{ij}d_j = r.
\]

It follows from (2.4) that \( r \) is an eigenvalue of \( B \) corresponding to \( x = (1, \ldots, 1)^t \), which implies that \( \rho(B) = r \). It follows that

\[
\rho(A) = \sqrt{\rho(B)} = \sqrt{\sum_{j=1}^{n} a_{ij}d_j}.
\]

Conversely, suppose equality in (2.2) holds. Then by Lemma 2.1,

\[
B(1, \ldots, 1)^t = \rho(B)(1, \ldots, 1)^t.
\]

Hence

\[
\sum_{j=1}^{n} a_{ij}d_j = \sum_{s=1}^{n} b_{is} = \rho(B) = \rho(A)^2
\]

is a constant for each \( i \). By Lemma 1.1 (let \( a=1 \)), \( A \) is row-regular or row-semi-regular. This completes the proof of Theorem 2.2. \( \Box \)

Let \( D = (V, E) \) be a directed graph with vertex set \( V = \{1, 2, \ldots, n\} \) and arc set \( E \). The adjacency matrix of \( D \) is the \( n \times n \) (0-1) matrix \( A = (a_{ij}) \) in which \( a_{ij} = 1 \) if and only if vertex \( i \) is adjacent to vertex \( j \) (that is, there is an arc from vertex \( i \)}

303
to vertex \( j \)). Then the \( i \)-th row sum \( d_i \) of \( A \) is just the out-degree of vertex \( i \) in \( D \). The directed graph \( D \) is out-regular (out-semiregular) if the adjacency matrix is row-
regular (row-semiregular). Clearly \( G \) is out-semiregular if and only \( D \) is bipartite, and each vertex in the same part of the bipartition has the same out-degree. The
spectral radius of \( D \), denoted by \( \rho(G) \), is defined to be the spectral radius of its adjacency matrix \( A \). An immediate corollary of Theorem 2.2 is given as follows.

**Corollary 2.3** Let \( G \) be a directed graph of order \( n \) with positive out-degree sequence \( d_1, d_2, \ldots, d_n \). Then

\[
\rho(G) \geq \min_{1 \leq i \leq n} \sqrt{\sum_{(i,j) \in E} d_j}.
\]

If \( G \) is strongly connected, then equality holds in (2.6) if and only if \( D \) is out-regular or out-semiregular.

3 Symmetric matrices and graphs

We need the following lemma.

**Lemma 3.1** Let \( A \) be an \( n \times n \) nonnegative symmetric matrix. Then

\[
\rho(A) \geq \frac{x^t(Ax)}{x^tx}
\]  

with equality if and only if \( x \) is an eigenvector corresponding to \( \rho(A) \).

The proof of Lemma 3.1 is a routine exercise in linear algebra.
We are now ready to prove the main result of this section.

**Theorem 3.2** Let \( A = (a_{ij}) \) be an \( n \times n \) nonnegative symmetric matrix with positive row sums \( d_1, d_2, \ldots, d_n \). Then

\[
\rho(A) \geq \sqrt{\frac{\sum_{i=1}^n d_i^2}{n}}
\]

with equality if and only if \( A \) is regular or semiregular.

**Proof.** Let \( A^2 = B = (b_{ij}) \). On setting \( x = (1, \ldots, 1)^t \), by Lemma 3.1 we obtain

\[
\rho(B) \geq \frac{x^tBx}{x^tx} = \frac{\sum_{i=1}^n \sum_{s=1}^n a_{ji}a_{is}}{n} = \frac{\sum_{i=1}^n a_{ij} \sum_{s=1}^n a_{is}}{n} = \frac{\sum_{i=1}^n d_i^2}{n}.
\]

Note that \( \rho(A) = \sqrt{\rho(B)} \). We have (3.2).
Now we are going to prove the second part of Theorem 3.2.
First suppose that $A$ is regular or semiregular. Then by Lemma 1.2 (let $a=1$), for some constant $r$ and each $j$,

$$\sum_{s=1}^{n} b_{js} = \sum_{i=1}^{n} a_{ji} \sum_{s=1}^{n} a_{is} = \sum_{i=1}^{n} a_{ji} d_i = r. \tag{3.3}$$

Hence $\rho(B) = r$. On the other hand, by (3.3) we also have $\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} d_i = nr$,

i.e., $\frac{\sum_{i=1}^{n} d_i}{n} = r$. Hence

$$\rho(A) = \sqrt{\rho(B)} = \sqrt{r} = \sqrt{\frac{\sum_{i=1}^{n} d_i}{n}}.$$

Conversely suppose equality in (3.2) holds. Then by Lemma 3.1, $(1, \ldots, 1)^t$ is an eigenvector of $B$ corresponding to $\rho(B)$. Hence for each $i$ $\sum_{j=1}^{n} a_{ij} d_j = \frac{\sum_{s=1}^{n} b_{is}}{n} = \rho(B) = \rho(A)^2$. By Lemma 1.2 (let $a=1$), $A$ is regular or semiregular. This completes the proof of Theorem 3.2. \hfill \Box

Let $G = (V, E)$ be an undirected simple graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E$. The adjacency matrix of $G$ is the $n \times n$ (0-1) symmetric matrix $A = (a_{ij})$ in which $a_{ij} = 1$ if and only if vertex $i$ is adjacent to vertex $j$ (that is, there is an edge between vertices $i$ and $j$). Then the $i$-th row sum $d_i$ of $A$ is just the degree of vertex $i$ in $G$. The graph $G$ is regular (semiregular) if the adjacency matrix is regular (semiregular). Clearly $G$ is semiregular if and only $G$ is bipartite, and each vertex in the same part of bipartition has the same degree. The spectral radius of $G$, denoted by $\rho(G)$, is defined to be the spectral radius of its adjacency matrix $A$. An immediate corollary of Theorem 3.2 is given as follows.

**Corollary 3.3** Let $G$ be an undirected simple graph of order $n$ with positive degree sequence $d_1, d_2, \ldots, d_n$. Then

$$\rho(G) \geq \sqrt{\frac{\sum_{i=1}^{n} d_i}{n}} \tag{3.4}$$

with equality if and only if $G$ is regular or semiregular.

Let $A$ be an $n \times n$ nonnegative symmetric matrix with positive row sums $d_1, \ldots, d_n$. Hoffman, Wolfe and Hofmeister [3] have proved that

$$\rho(A) \geq \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \sqrt{d_i d_j}}{\sum_{i=1}^{n} d_i} \tag{3.5}.$$

By Lemma 1.2 and the result in [3], equality holds in (3.5) if and only if $A$ is regular or semiregular.
It is natural to wonder how (3.2) compares with (3.5); the examples below shows that (3.2) is sharper than (3.5) in many cases.

First, for any $n \geq 4$, let $A$ be the $n \times n$ matrix with $a_{1,i} = 1$ for $2 \leq i \leq n$, $a_{i,i+1} = a_{i+1,i} = a_{2,n} = a_{n,2} = 1$ for $2 \leq i \leq n-1$ and all other entries 0. Note that $A$ is the adjacency matrix of a wheel of order $n$, and $d_1 = n - 1$, $d_2 = \ldots = d_{n-1} = 3$. The right hand side of (3.2) is \( \sqrt{\frac{(n-1)(n+8)}{n}} \). The right hand side of (3.5) is $\sqrt{\frac{3(n-1)+3}{2}}$. Since

$$\lim_{n \to \infty} \frac{\sqrt{\frac{(n-1)(n+8)}{n}}}{\sqrt{\frac{3(n-1)+3}{2}}} = \frac{2}{\sqrt{3}} > 1,$$

for the matrix $A$, (3.2) is a sharper bound than (3.5) if $n$ is sufficiently large.

Next, we give another example. For $n \geq 5$, Let $A$ be the $n \times n$ matrix with $a_{1,i} = a_{2,i} = 1$ for $3 \leq i \leq n$, $a_{i,i+1} = a_{i+1,i} = 1$ for $3 \leq i \leq n$ and all other entries 0. Then $d_1 = d_2 = n - 2$, $d_3 = d_4 = 3$, $d_5 = \ldots = d_n = 4$. The right hand side of (3.2) is \( \sqrt{\frac{2(n-2)^2+16(n-4)+9}{n}} \). However, the right hand side of (3.5) is

$$\frac{2(n-4)\sqrt{4(n-2)+4\sqrt{3(n-2)+4(n-5)+4\sqrt{3}}}}{3n-7}.$$

Note that

$$\lim_{n \to \infty} \frac{\sqrt{\frac{2(n-2)^2+16(n-4)+9}{n}}}{\frac{2(n-4)\sqrt{4(n-2)+4\sqrt{3(n-2)+4(n-5)+4\sqrt{3}}}}{3n-7}} = \frac{3\sqrt{2}}{4} > 1.$$

We see that for $A$, (3.2) is sharper than (3.5) again if $n$ is sufficiently large.

References


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306