Generators of matrix incidence algebras

W. E. Longstaff
Department of Mathematics, The University of Western Australia
Nedlands, Western Australia 6907, Australia
E-mail: longstaff@maths.uwa.edu.au

Peter Rosenthal
Department of Mathematics, University of Toronto
Toronto, Ontario M5S 1A1, Canada
E-mail: rosent@math.toronto.edu

Abstract

Let \( n \in \mathbb{Z}^+ \) and let \( K \) be a field. Let \( \preceq \) be a partial order on \( \{1, 2, \ldots, n\} \). Let \( \mathcal{A}_n(\preceq) \) be the matrix incidence algebra consisting of those \( n \times n \) matrices \( A = (a_{i,j}) \) with entries in \( K \), satisfying \( a_{i,j} = 0 \) whenever \( i \not\preceq j \). For a subset \( \mathcal{E} \subseteq \mathcal{A}_n(\preceq) \), a necessary and sufficient condition that the algebra generated by \( \mathcal{E} \cup \{I\} \) is \( \mathcal{A}_n(\preceq) \) is that (i) for every \( 1 \leq i, j \leq n \) with \( i \not\preceq j \), there exists \( A \in \mathcal{E} \) such that \( a_{i,i} \neq a_{j,j} \) and (ii) for every \( i \preceq j \) with \( j \) covering \( i \), there exists \( B \in \text{span} \mathcal{E} \) such that \( b_{i,i} \neq 0 \) and \( b_{i,i} = b_{i,j} \). If the characteristic of \( K \) is zero or \( > n \), the algebra \( \mathcal{A}_n(\simeq) \) is singly generated and, if \( \preceq \) is not equality, \( \mathcal{A}_n(\preceq) \) has two generators.

1. PRELIMINARIES

Let \( (S, \ll) \) be a locally finite partially ordered set. Here, local finiteness means that every interval \( [x, y] = \{u \in S : x \ll u \ll y\} \) is finite. Let \( K \) be a field. The incidence algebra \( \mathcal{A}(S) \) of \( S \) over \( K \) is the set of all functions \( f : S \times S \rightarrow K \) with the property that \( f(x, y) = 0 \) whenever \( x \not\ll y \). \( \mathcal{A}(S) \) becomes an associative \( K \)-algebra with the pointwise operations of addition and scalar multiplication and with the Dirichlet product:

\[
(f \ast g)(x, y) = \sum_{x \ll u \ll y} f(x, u)g(u, y).
\]

The Kronecker delta function is the multiplicative identity of \( \mathcal{A}(S) \). In [3], Rota proposed the idea of such algebras as a basis for a unified study of combinatorial theory. In [1] (see also [2]), certain subalgebras of \( \mathcal{A}(S) \) were considered. Here we consider generators of \( \mathcal{A}(S) \).

If \( n \in \mathbb{Z}^+ \) and \( \preceq \) is a partial order on \( \{1, 2, \ldots, n\} \), the corresponding incidence algebra can be identified in a natural way with the algebra of \( n \times n \) matrices (with entries in \( K \)) \( A = (a_{i,j}) \) satisfying \( a_{i,j} = 0 \) whenever \( i \not\preceq j \), with the usual matrix operations. (The Dirichlet product becomes matrix multiplication.) We will call such matrix algebras \( \text{matrix incidence algebras} \) and, with a slight change of notation, use \( \mathcal{A}_n(\preceq) \) to denote the matrix incidence algebra described immediately above. If \( \preceq \) is \textit{consistent with the natural order} on \( \{1, 2, \ldots, n\} \) in the sense that \( i \preceq j \) implies \( i \leq j \), then \( \mathcal{A}_n(\preceq) \) is an algebra of upper-triangular matrices. Any incidence algebra arising from a finite partially ordered set is isomorphic to some matrix incidence algebra \( \mathcal{A}_n(\preceq) \) where \( \preceq \) is consistent with the natural order. Indeed, if \( (S, \ll) \) is a finite partially ordered set and \( x_1, x_2, \ldots, x_n \) is an enumeration of \( S \) satisfying: \( x_i \ll x_j \) implies \( i \leq j \), then \( \mathcal{A}(S) \) is isomorphic to \( \mathcal{A}_n(\preceq) \) (where \( \preceq \) is defined by \( i \preceq j \) if \( x_i \ll x_j \)) by the map \( f \mapsto (f(x_i, x_j)) \). It is not unwise therefore, to concentrate attention on matrix algebras of the type \( \mathcal{A}_n(\preceq) \) where \( \preceq \) is consistent with the natural order.

Notice that \( \mathcal{A}_n(=) \) is the algebra of diagonal matrices, and \( \mathcal{A}_n(\preceq) \) is the algebra of upper-triangular matrices. Throughout, \( K \) will denote a fixed but arbitrary field, and all matrices will be assumed to have entries in \( K \). Also, \( I \) will denote the identity matrix and span \( \mathcal{E} \) will denote the linear span of \( \mathcal{E} \).

2. MAIN THEOREM

First we need a lemma. In what follows, we will use the notation \((A)_{i,j}\) to denote the \( i,j \)-entry of a matrix \( A \).

**LEMMA.** Let \( n \in \mathbb{Z}^+ \) and let \( T_1, T_2, \ldots, T_n \) be \( n \times n \) upper-triangular matrices satisfying \((T_i)_{i,i} = 0, i = 1, 2, \ldots, n \). Then \( T_1 T_2 \cdots T_n = 0 \).

**Proof.** The result is true when \( n = 1 \). Assume that it is true for \( n \) and let \( T_1, T_2, \ldots, T_{n+1} \) be \((n+1) \times (n+1)\) upper-triangular matrices satisfying \((T_i)_{i,i} = 0, i = 1, 2, \ldots, n+1 \). By the assumption, \( T_1 T_2 \cdots T_n \) is of the form \( \begin{pmatrix} 0 & X \\ 0 & \lambda \end{pmatrix} \), for some \( n \times 1 \) matrix \( X \) and \( \lambda \in K \). Since \( T_{n+1} = \begin{pmatrix} Y & Z \\ 0 & 0 \end{pmatrix} \), for some \( n \times n \) matrix \( Y \) and \( n \times 1 \) matrix \( Z \) we have

\[ T_1 T_2 \cdots T_n T_{n+1} = \begin{pmatrix} 0 & X \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} Y & Z \\ 0 & 0 \end{pmatrix} = 0. \]

The proof is completed by induction.

In the above lemma the order of the factors is important. For example, over any field of characteristic zero, only one product of the following three matrices is zero:

\[ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \]

**THEOREM.** Let \( n \in \mathbb{Z}^+ \) and let \( K \) be a field. Let \( \preceq \) be a partial order on \( \{1, 2, \ldots, n\} \) and let \( \mathcal{A}_n(\preceq) \) be the corresponding matrix incidence algebra over \( K \).
Let $\mathcal{E}$ be a subset of $\mathcal{A}_n(\preceq)$. The algebra generated by $\mathcal{E} \cup \{I\}$ is $\mathcal{A}_n(\preceq)$ if and only if

(i) for every $1 \leq i, j \leq n$ with $i \neq j$, there exists $A \in \mathcal{E}$ such that $(A)_{i,i} \neq (A)_{j,j}$, and

(ii) for every $1 \leq i, j \leq n$ with $i \preceq j$ and with $j$ covering $i$, there exists $B \in \text{span } \mathcal{E}$ such that $(B)_{i,j} \neq 0$ and $(B)_{i,i} = (B)_{j,j}$.

Proof. Suppose first that the partial order $\preceq$ is consistent with the natural order. Then, as noted earlier, $\mathcal{A}_n(\preceq)$ is an algebra of upper-triangular matrices.

Let $\mathcal{B}$ denote the algebra generated by $\mathcal{E} \cup \{I\}$ and put $\mathcal{A} = \mathcal{A}_n(\preceq)$.

Suppose that conditions (i) and (ii) are satisfied. We proceed by induction. The result is true for $n = 1$. Assume that the result is true for $n$. Consider the situation for $n + 1$. Temporarily, for any $(n + 1) \times (n + 1)$ matrix $A$ let $\hat{A}$ denote the $n \times n$ matrix occurring in the top left-hand corner of $A$. Since $\text{span } \hat{\mathcal{E}} = \text{span } \hat{\mathcal{E}}$, the induction assumption gives that the algebra generated by $\hat{\mathcal{E}} \cup \{I\}$ is $\hat{\mathcal{A}}$. But the algebra generated by $\hat{\mathcal{E}} \cup \{I\}$ is $\hat{\mathcal{B}}$, so $\hat{\mathcal{A}} = \hat{\mathcal{B}}$. Thus, for every $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $\hat{A} = \hat{B}$.

By condition (i), for every $1 \leq i \leq n$, there exists $A_i \in \mathcal{B}$ such that $(A_i)_{i,i} = 0$ and $(A_i)_{n+1,n+1} = 1$. Then $A_0$ defined by $A_0 = A_1A_2 \ldots A_n$ belongs to $\mathcal{B}$ and, by the lemma, satisfies $\hat{A}_0 = 0$ and $(A_0)_{n+1,n+1} = 1$.

If $n + 1$ covers no element of $\{1, 2, \ldots, n\}$, every entry in the last column of any element of $\mathcal{A}$ is zero, except possibly for the $n+1, n+1$- entry. In this case it easily follows that $\mathcal{B} = \mathcal{A}$. For then, if $A \in \mathcal{A}$ and $B \in \mathcal{B}$ satisfy $\hat{A} = \hat{B}$, then $B + \mu A_0 = A$, for some $\mu \in K$. So $A \in \mathcal{B}$.

On the other hand, suppose that $n+1$ covers at least one element of $\{1, 2, \ldots, n\}$. Let $E_{k,l}$ denote the $(n + 1) \times (n + 1)$ matrix having $k, l$-th entry equal to 1 and all other entries zero. We show that $E_{i,n+1} \in \mathcal{B}$, for every $i$ satisfying $i \preceq n+1$. From this, together with the fact that $\hat{A} = \hat{B}$, it readily follows that $\mathcal{A} = \mathcal{B}$.

Let the elements of $\{1, 2, \ldots, n\}$ covered by $n+1$ be $i_1, i_2, \ldots, i_q$, where $1 \leq q \leq n$ and $i_1 < i_2 < \ldots < i_q < n + 1$. By condition (ii), for every $1 \leq p \leq q$, there exists $B_p \in \mathcal{B}$ such that $(B_p)_{i_p,n+1} = 1$ and $(B_p)_{i_p,i_p} = (B_p)_{n+1,n+1} = 0$. For every $1 \leq p \leq q$, there exists $F_p \in \mathcal{B}$ such that $\hat{E}_{i_p,i_p} = \hat{F}_p$. All the rows of $F_pB_p$ are zero except the $i_p$-th and this is the same as the $i_p$-th row of $B_p$. Note that, for every $1 \leq p \leq q$ and $X, Y \in \mathcal{A}$ we have

$\sum_{j=1}^{n} (X)_{i_p,j} (Y)_{j,n+1} = (X)_{i_p,i_p} (Y)_{i_p,n+1}$

since $n + 1$ covers $i_p$. Using this (recalling that $(B_p)_{i_p,i_p} = 0$) gives $F_pB_pA_0 = E_{i_p,n+1}$, so $E_{i_p,n+1} \in \mathcal{B}$ for $1 \leq p \leq q$.

Next, suppose that $1 \leq i \leq n$ and $i \leq n + 1$, but $n + 1$ does not cover $i$. Then $i \preceq i_p$, for some $1 \leq p \leq q$ and there exists $G_{i,i_p} \in \mathcal{B}$ such that $\hat{E}_{i,i_p} = \hat{G}_{i,i_p}$. Then $G_{i,i_p}F_pB_pA_0 = E_{i,n+1}$, so $E_{i,n+1} \in \mathcal{B}$.

Since $A_0 \in \mathcal{B}$ and $E_{i,n+1} \in \mathcal{B}$ whenever $1 \leq i \leq n$ and $i \leq n + 1$, we get that $E_{n+1,n+1} \in \mathcal{B}$. It now follows that $\mathcal{A} = \mathcal{B}$ and the proof is completed by induction.

Conversely, let $\mathcal{A} = \mathcal{B}$. Since $\{A \in \mathcal{A} : (A)_{i,i} = (A)_{j,j}\}$ is a proper subalgebra of $\mathcal{A}$, for every $1 \leq i, j \leq n$ with $i \neq j$, condition (i) holds.

119
Suppose that condition (ii) does not hold. Then there exist $1 \leq i, j \leq n$ with $i \leq j$ and with $j$ covering $i$, such that, for every $B \in \mathcal{E}$, $(B)_{i,j} = 0$ or $(B)_{i,i} \neq (B)_{j,j}$. This conclusion is, in fact, valid for every $B \in \text{span}(\mathcal{E} \cup \{I\})$. Let $\mathcal{F} = \text{span}(\mathcal{E} \cup \{I\})$. Temporarily, for any $n \times n$ matrix $A$, let $\tilde{A}$ be the $2 \times 2$ matrix given by $\tilde{A} = \begin{pmatrix} (A)_{i,i} & (A)_{i,j} \\ (A)_{j,i} & (A)_{j,j} \end{pmatrix}$. Then $\tilde{F} = \text{span}(\tilde{\mathcal{F}} \cup \{I\})$ is a subspace of the vector space of $2 \times 2$ upper-triangular matrices, of dimension $< 3$. It cannot be the case that $(B)_{i,j} = 0$, for every $B \in \mathcal{E}$ since $\{A \in \mathcal{A} : (A)_{i,j} = 0\}$ is a proper subalgebra of $\mathcal{A}$ containing $I$. (Note that, since $j$ covers $i$, for every $X, Y \in \mathcal{A}$, we have $(XY)_{i,j} = (X)_{i,i}(Y)_{i,j} + (X)_{i,j}(Y)_{j,j}$.) Hence there exists $B_0 \in \mathcal{E}$ such that $\tilde{B} = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}$ with $\gamma \neq 0$. Since $\alpha \neq \beta$, $\begin{pmatrix} 0 & \delta \\ 0 & 1 \end{pmatrix} \in \tilde{F}$, for some $\delta \neq 0$. It follows that $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \delta \\ 0 & 1 \end{pmatrix} \right\}$ is a basis for $\tilde{F}$. But the linear span of $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \delta \\ 0 & 1 \end{pmatrix} \right\}$ is an algebra (since $\begin{pmatrix} 0 & \delta \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 0 & \delta \\ 0 & 1 \end{pmatrix}$). Thus $\tilde{F}$ is a proper subalgebra of the algebra of all $2 \times 2$ upper-triangular matrices. The set of matrices $\{A \in \mathcal{A} : \tilde{A} \in \tilde{F}\}$ is therefore a proper subalgebra of $\mathcal{A}$ containing $\mathcal{E} \cup \{I\}$. (Note that $\tilde{X}Y = \tilde{X}\tilde{Y}$, for every $X, Y \in \mathcal{A}$.) This contradicts the fact that $\mathcal{B} = \mathcal{A}$ and the proof for the case where $\leq$ is consistent with the natural order is complete.

Finally, let $\preceq$ be any partial order defined on $\{1, 2, \ldots, n\}$. Choose a permutation $\tau$ of $\{1, 2, \ldots, n\}$ satisfying $\tau(i) \preceq \tau(j)$ implies $i \leq j$. The partial order $\ll$, defined on $\{1, 2, \ldots, n\}$ by $i \ll j$ if $\tau(i) \preceq \tau(j)$, is consistent with the natural order. Let $V$ be the $n \times n$ matrix having all its $i$, $\tau(i)$-entries equal to $1$ ($i = 1, 2, \ldots, n$) and all other entries zero. Then $V$ is invertible and $V^{-1}$ is the transpose of $V$ (it has all its $\tau(i)$, $i$- entries equal to $1$ and zeros elsewhere). Then $A \mapsto VAV^{-1}$ is an algebra isomorphism of $\mathcal{A}_n(\preceq)$ onto $\mathcal{A}_n(\ll)$. Let $\mathcal{E}$ be as in the statement of the theorem. Then $\mathcal{E} \cup \{I\}$ generates $\mathcal{A}_n(\preceq)$ if and only if $\mathcal{E}V^{-1} \cup \{I\}$ generates $\mathcal{A}_n(\ll)$. It is easy to check that $\mathcal{E}V^{-1}$ satisfies conditions (i) and (ii), with $\preceq$ replaced by $\ll$, if and only if $\mathcal{E}$ satisfies them as they stand. This completes the proof. \qed

The above theorem includes results about generating sets for the algebra of diagonal matrices and for the algebra of upper-triangular matrices: take $\preceq$ to be, respectively, equality or the natural order.

COROLLARY 1. If $\mathcal{E}$ is a set of diagonal $n \times n$ matrices, then the algebra generated by $\mathcal{E} \cup \{I\}$ is the algebra of all diagonal matrices if and only if, for every $1 \leq i, j \leq n$ with $i \neq j$, there exists $A \in \mathcal{E}$ such that $(A)_{i,i} \neq (A)_{j,j}$. If the characteristic of the field $K$ is zero or $> n$, any diagonal matrix with non-zero distinct diagonal entries generates the algebra of $n \times n$ diagonal matrices over $K$.

Proof. The first part of the statement of this corollary follows from the theorem. Suppose that the characteristic of $K$ is zero or $> n$, and let $A$ be any diagonal matrix with non-zero distinct diagonal entries. Then $\{A, I\}$ generates the algebra of diagonal matrices by the theorem. But $p(A) = I$ for some polynomial
$p(x)$ satisfying $p(0) = 0$. It follows that $\{A\}$ generates the algebra of diagonal matrices.

The proof of the following corollary is obvious.

**COROLLARY 2.** If $\leq$ is different from equality, and if the characteristic of the field $K$ is zero or $> n$, then $A(\leq)$ has two generators. Indeed, in this case, any diagonal matrix with non-zero distinct diagonal entries, and any matrix having a non-zero $i,j$-entry whenever $j$ covers $i$ and all its other entries equal to zero, together generate $A(\leq)$.

**REMARKS.** 1. Clearly $\langle E \rangle$ can be replaced by ‘span $E$’ in condition (i) of the statement of the theorem. However, ‘span $E$’ cannot be replaced by $\langle E \rangle$ in condition (ii). For example, if $E = \left\{ \begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \right\}$, then $E \cup \{I\}$ generates the algebra of upper-triangular $2 \times 2$ matrices since $\begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{span } E$, but $E$ itself does not satisfy condition (ii) of the theorem.
2. In the statement of the theorem ‘$\langle E \cup \{I\} \rangle$’ cannot be replaced by ‘$E$’. For example,

$$E = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

does not generate the algebra of $3 \times 3$ upper-triangular matrices.

**REFERENCES**


(Received 23/9/99)