Longest paths through an arc in strongly connected in-tournaments

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Abstract
An in-tournament is an oriented graph such that the in-neighborhood of every vertex induces a tournament. Recently, we have shown that every arc of a strongly connected tournament of order \( n \) is contained in a directed path of order \( \lceil (n + 3)/2 \rceil \). This is no longer valid for strongly connected in-tournaments, because there exist examples containing an arc with the property that the longest directed path through this arc consists of three vertices. But in this paper we shall see that every strongly connected in-tournament has at most one such arc. More general, we shall prove that if a strongly connected in-tournament \( D \) of order \( n \) contains \( m - 2 \leq n - 3 \) arcs \( a_3, a_4, \ldots, a_m \) such that the longest directed path through \( a_k \) consists of \( k \) vertices for \( 3 \leq k \leq m \), then all other arcs of \( D \) belong to directed paths of order at least \( m + 1 \). Furthermore, we shall show that every arc of a strongly connected in-tournament is contained in a directed path of order \( k + 2 \), when \( \max\{\delta^+, \delta^-\} \geq k \), where \( \delta^+ \) and \( \delta^- \) is the minimum outdegree and the minimum indegree, respectively.

1. Terminology and introduction

The vertex set and the arc set of a digraph \( D \) are denoted by \( V(D) \) and \( E(D) \), respectively. The number \( |V(D)| \) is the order of the digraph \( D \). Throughout this paper we will consider digraphs without multiple arcs, loops, or directed cycles of length two. Such digraphs are called oriented graphs. If there is an arc from \( x \) to \( y \) in \( D \), then \( y \) is a positive neighbor of \( x \) and \( x \) is a negative neighbor of \( y \), and we also say that \( x \) dominates \( y \), denoted by \( x \rightarrow y \). More generally, let \( A \) and \( B \) be two disjoint subdigraphs of \( D \) or subsets of \( V(D) \). If \( x \rightarrow y \) for every vertex \( x \) in \( A \) and every vertex \( y \) in \( B \), then we write \( A \rightarrow B \) and say that \( A \) dominates \( B \). Two vertices \( x \) and \( y \) of a digraph are adjacent when \( x \rightarrow y \) or \( y \rightarrow x \). The outset \( N^+(x) \) of a vertex \( x \) is the set of vertices dominated by \( x \), and the inset \( N^-(x) \) is the set of vertices dominating \( x \). The numbers \( d^+(x) = |N^+(x)| \) and \( d^-(x) = |N^-(x)| \) are called outdegree and indegree, respectively. The minimum outdegree \( \delta^+ \) and the minimum
indegree $\delta^-$ of $D$ are given by $\min \{d^+(x) \mid x \in V(D)\}$ and $\min \{d^-(x) \mid x \in V(D)\}$, respectively. For $A \subseteq V(D)$, we define $D[A]$ as the subdigraph induced by $A$. By a cycle (path) we mean a directed cycle (directed path). A cycle or a path of order $m$ is called an $m$-cycle or an $m$-path, respectively. A cycle (path) of a digraph $D$ is Hamiltonian if it includes all the vertices of $D$. We speak of a connected digraph if the underlying graph is connected. A digraph $D$ is said to be strongly connected or just strong, if for every pair $x, y$ of vertices of $D$, there is a path from $x$ to $y$. A strong component of $D$ is a maximal induced strong subdigraph of $D$. A digraph $D$ is $k$-connected if for any set $S$ of at most $k - 1$ vertices, the subdigraph $D - S$ is strong. A minimal separating set of a strong digraph $D$ is a subset $S \subseteq V(D)$ such that $D - S$ is not strong, but $D - S'$ is strong for any $S' \subseteq S$. An in-tournament is an oriented graph with the property that the inset of every vertex induces a tournament, i.e., every pair of distinct vertices that have a common positive neighbor are adjacent. A local tournament is an oriented graph such that the inset as well as the outset of every vertex induces a tournament. Throughout this paper all subscripts are taken modulo the corresponding number.

Local tournaments were introduced by Bang-Jensen [1] in 1990 and there exists extensive literature on this class of digraphs, e.g., the survey paper of Bang-Jensen and Gutin [2]. In particular, the Ph. D. theses of Y. Guo [4] and J. Huang [5] have been devoted to this subject. As a generalization of local tournaments, Bang-Jensen, Huang, and Prisner [3] studied the family of in-tournaments. But in-tournaments have, as yet, received little attention. Except for the above mentioned article of Bang-Jensen, Huang, Prisner [3], these digraphs have only been investigated by Tewes [7], [8], [9], and Tewes, Volkmann [10], [11]. It is the purpose of this paper to give more information about the properties of in-tournaments.

Very recently, we have proved [12] that every arc of a strongly connected tournament of order $n$ (even every arc of a strongly connected $n$-partite tournament) is contained in a directed path of order $[(n + 3)/2]$. The following example shows that this is no longer valid for strongly connected in-tournaments.

**Example 1.1** Let $D$ consist of the cycle $x_1x_2 \ldots x_nx_1$ together with the arcs $x_ix_i$ for $3 \leq i \leq n - 1$. Then it is straightforward to verify that $D$ is a strongly connected in-tournament of order $n$, and that the longest path through the arc $x_1x_{n-1}$ is only of order three.

**Definition 1.2** If the longest path through an arc $uv$ consists of exactly $m$ vertices, then we call $uv$ an $m$-path arc.

In this paper we shall see that every strongly connected in-tournament of order $n \geq 4$ has at most one 3-path arc. More general, we shall prove that if a strongly connected in-tournament $D$ of order $n$ contains a $k$-path arc for every $3 \leq k \leq m \leq n - 1$, then all other arcs of $D$ belong to paths of order $m + 1$. Also strongly connected in-tournaments without a 3-path arc but containing a 4-path arc, have only one
4-path arc, when the order is at least six. Furthermore, if a strongly connected in-tournament has a $k$-path arc for each $k = 3, 4, \ldots, m$ but no $(m + 1)$-path arc, then it contains no $(m + 2)$-path arc. In addition, we shall prove that every arc of a strongly connected in-tournament is contained in a path of order $k + 2$, when $\max\{\delta^+, \delta^-\} \geq k$. Different examples will show that these results are best possible.

2. Preliminary results

The following known results play an important role in our investigations.


**Theorem 2.2** (Bang-Jensen, Huang, Prisner [3] 1993) An in-tournament has a Hamiltonian cycle if and only if it is strongly connected.

**Theorem 2.3** (Bang-Jensen, Huang, Prisner [3] 1993) Let $D$ be a strongly connected in-tournament and let $S$ be a minimal separating set. Then there exists a unique order $D_1, D_2, \ldots, D_p$ of the strong components of $D - S$ such that there are no arcs from $D_j$ to $D_i$ for $j > i$, and for each $i = 1, 2, \ldots, p - 1$ there exists a vertex $w_i \in V(D_i)$ such that $w_i \rightarrow D_{i+1}$. If in addition, $xy$ is an arc from $D_i$ to $D_j$ for $i < j$, then $x \rightarrow (D_{i+1} \cup D_{i+2} \cup \ldots \cup D_j)$.

**Theorem 2.4** (Bang-Jensen [1] 1990) Let $D$ be a strongly connected local tournament and let $S$ be a minimal separating set. Then there exists a unique order $D_1, D_2, \ldots, D_p$ of the strong components of $D - S$ such that there are no arcs from $D_j$ to $D_i$ for $j > i$, $D_i \rightarrow D_{i+1}$ for $i = 1, 2, \ldots, p - 1$, and $D_i$ is a tournament for $i = 1, 2, \ldots, p$.

The unique order $D_1, D_2, \ldots, D_p$ in Theorem 2.3 as well as in Theorem 2.4 is called the strong decomposition of $D - S$.

3. General results

**Observation 3.1** Let $uv$ be an arbitrary arc of a strongly connected in-tournament $D$. If $D - u$ or $D - v$ is strong, then $D$ contains a Hamiltonian path starting with the arc $uv$ or ending with the arc $uv$, respectively.

**Proof.** If $D - u$ is strong, then by Theorem 2.2, the in-tournament $D - u$ has a Hamiltonian cycle $ux_2x_3 \ldots x_{|V(D)|-1}v$. Therefore, the arc $uv$ is the initial arc of the Hamiltonian path $uvx_2x_3 \ldots x_{|V(D)|-1}$ of $D$. Considering $D - v$ instead of $D - u$, we obtain analogously a Hamiltonian path with the terminal arc $uv$. □

**Theorem 3.2** Let $u$ be a vertex of a strongly connected local tournament $D$ such
that $D - u$ is not strong. If $D_1, D_2, \ldots, D_p$ is the strong decomposition of $D - u$, then the arcs from $D_i$ to $D_{i+1}$ for $1 \leq i \leq p - 1$ and the arcs in $D_i$ for $2 \leq i \leq p - 1$ are contained in a Hamiltonian path.

**Proof.** In view of Theorem 2.2, each strong component $D_i$ with at least three vertices has a Hamiltonian cycle $x^1_i x^2_i \ldots x^{|V(D_i)|}_i x^1_i$ for $1 \leq i \leq p$. Since $D$ is strong, there exists a vertex, say $x^1_i$, in $D_1$ such that $u \rightarrow x^1_i$ and a vertex, say $x^p_i$, in $D_p$ such that $x^p_i \rightarrow u$. By $P_i$ we denote a Hamiltonian path of $D_i$ for $1 \leq i \leq p$. Theorem 2.4 implies $D_i \rightarrow D_{i+1}$ for $i = 1, 2, \ldots, p - 1$. In the following we always use this fact.

**Case 1:** Let $x^i_j x^{i+1}_k$ be an arc from $D_i$ to $D_{i+1}$ for $1 \leq i \leq p - 1$.

**Subcase 1.1:** Let $p \geq 3$. If $i \geq 2$, then

$$ux^1_i x^2_i \ldots x^{|V(D_i)|}_i P_2 \ldots P_{i-1} x^i_j x^{i+1}_j x^2_j \ldots x^i_k x^{i+1}_k x^2_k \ldots x^1_{i-1} P_{i+1} \ldots P_p,$$

and if $i = 1$, then

$$x^1_j x^1_{j+1} \ldots x^2_j x^2_{j+1} \ldots x^2_k x^1_{k+1} \ldots x^2_{k-1} P_3 \ldots P_{p-1} x^p_3 x^p_4 \ldots x^p_i u$$

is a Hamiltonian path of $D$ through the arc $x^i_j x^{i+1}_k$.

**Subcase 1.2:** Let $p = 2$. If $u \rightarrow x^1_j$, then $ux^1_j x^1_{j+1} x^2_{j+1} \ldots x^2_j x^2_{k+1} \ldots x^2_{k-1}$ is a desired Hamiltonian path. If $u$ does not dominate $x^1_j$, then let $s \geq 2$ be the smallest integer such that $u \rightarrow x^1_{j+s}$. Then, because $u$ and $x^1_{j+s-1}$ are negative neighbors of $x^1_{j+s}$, we conclude that $x^1_{j+s-1} \rightarrow u$. But now $x^1_j x^2_{k+1}$ is an arc of the Hamiltonian path

$$x^1_{j+1} x^1_j x^1_{j+2} \ldots x^1_{j+s-1} u x^1_{j+s} \ldots x^1_j x^2_k x^1_{k+1} \ldots x^2_{k-1}. $$

**Case 2:** Let $x^i_j x^i_k$ be an arc of the component $D_i$ for $2 \leq i \leq p - 1$. By Theorem 2.4, $D_i$ is a tournament, and thus, $D'_i = D_i - \{x^i_j, x^i_k\}$ is also a tournament. According to Theorem 2.1, $D'_i$ has a Hamiltonian path $P'_i$. Hence, we deduce that $x^i_j x^i_k$ is an arc of the Hamiltonian path

$$x^i_j x^i_k P'_{i+1} P'_{i+2} \ldots P'_{p-1} x^p_k x^p_3 \ldots x^p_i u x^1_j x^1_2 \ldots x^1_{|V(D_i)|} P_2 P_3 \ldots P_{i-1} P'_i. \quad \Box$$

**Example 3.3** Let $T_5$ be the tournament with the cycle $x_1 x_2 x_3 x_4 x_5 x_1$ such that $x_1 \rightarrow \{x_3, x_4\}$, $x_2 \rightarrow \{x_4, x_5\}$, and $x_5 \rightarrow x_3$. Note that the arc $x_1 x_4$ is not contained in a Hamiltonian path of $T_5$. Now let $T_7$ be the tournament consisting of $T_5$ and the two new vertices $u$ and $w$ such that $T_5 \rightarrow u \rightarrow w \rightarrow u \rightarrow x_4$ and $\{x_1, x_2, x_3, x_5\} \rightarrow u$. Then, $T_5$ corresponds to the first component $D_1$ of $T_7 - u$, and it easy to see that the arc $x_1 x_4$ is not contained in a Hamiltonian path of $T_7$.

Using the same method, it is a simple matter to construct strongly connected tournaments $T$ of arbitrarily large order such that the strong components $D_1$ and $D_p$ of $T - u$ have arcs which are not contained in a Hamiltonian path of $T$.

**Remark 3.4** Example 3.3 shows that Theorem 3.2 is not valid for the arcs in $D_1$ or $D_p$, even for tournaments, in general. But if $u \rightarrow D_1$ or $D_p \rightarrow u$, then one can prove
analogously to Case 2 that each arc in $D_1$ or $D_p$ is contained in a Hamiltonian path, respectively.

**Example 3.5** Let $T_p$ be the transitive tournament with the vertex set $\{x_1, x_2, \ldots, x_p\}$ such that $x_i \rightarrow x_j$ for $1 \leq i < j \leq p$. Now let $D$ be the strongly connected local tournament of order $p + 1$ consisting of $T_p$, the new vertex $u$ and the both arcs $x_p u$ and $u x_1$. Then, $D - u = T_p$ has the strong decomposition $D_1, D_2, \ldots, D_p$ such that $V(D_i) = \{x_i\}$ for $1 \leq i \leq p$, and we observe that no arc $x_i x_j$ with $j \geq i + 2$ is contained in a Hamiltonian path.

In view of the Examples 3.3 and 3.5, we see that Theorem 3.2 is best possible.

**Observation 3.6** Let $uv$ be an arc of an in-tournament $D$. If $d^-(u) = m$, then $D$ contains an $(m + 2)$-path with the terminal arc $uv$.

**Proof.** It follows from the definition of an in-tournament that the induced subdigraph $D[N^-(u)]$ is a tournament. Thus, according to Theorem 2.1, there exists a Hamiltonian path $x_1 x_2 \ldots x_m$ of $D[N^-(u)]$. Consequently, $x_1 x_2 \ldots x_m uv$ is path of order $m + 2$ in $D$ with the terminal arc $uv$. □

**Theorem 3.7** Let $uv$ be an arc of a strong in-tournament $D$. If

$$\max\{d^-(u), d^+(v)\} = m,$$

then the arc $uv$ is contained in a path of order $m + 2$.

**Proof.** If $d^-(u) = m$, then we are done by Observation 3.6. Now assume that $d^+(v) = m$ and let $|V(D)| = n$. By Theorem 2.2, $D$ has a Hamiltonian cycle, and hence the in-tournament $D - v$ contains a Hamiltonian path $x_1 x_2 \ldots x_{n-1}$. Let $u = x_k$ for some $1 \leq k \leq n - 1$. If $k \geq m + 1$, then $x_1 x_2 \ldots x_k v$ is a path of order $k + 1 \geq m + 2$ through the arc $uv$. If $k \leq m$, then, because of $d^+(v) = m$, the vertex $v$ has at least $m - (k - 1)$ positive neighbors in the vertex set $\{x_{k+1}, x_{k+2}, \ldots, x_{n-1}\}$. If $j \geq k + 1$ is the smallest index such that $v \rightarrow x_j$, then $j \leq n - m + k - 1$. Therefore, the path $x_1 x_2 \ldots x_k v x_j x_{j+1} \ldots x_{n-1}$ through $uv$ consists of at least $k + 1 + (n - 1) - j + 1 \geq n + k + 1 - (n - m + k - 1) = m + 2$ vertices. □

**Corollary 3.8** Let $D$ be a strongly connected in-tournament. If

$$\max\{\delta^+, \delta^-\} \geq m,$$

then every arc of $D$ is contained in a path of order $m + 2$.

The next example will demonstrate that Theorem 3.7 is best possible, even for the family of local tournaments.

**Example 3.9** Let $T_k$ be a strong tournament and let $T_{m+1}$ be a transitive tournament with the vertex set $\{v, x_1, x_2, \ldots, x_m\}$ such that $x_i \rightarrow x_j$ for $1 \leq i < j \leq m$
and \( v \rightarrow \{x_1, x_2, \ldots, x_m\} \). If the digraph \( D \) consists of the tournaments \( T_k \) and \( T_{m+1} \) and the vertex \( u \) such that \( u \rightarrow (V(T_k) \cup \{v\}) \), \( \{x_1, x_2, \ldots, x_m\} \rightarrow u \), and \( T_k \rightarrow v \), then it is a simple matter to verify that \( D \) is a strongly connected local tournament with \( d^+(v) = d^-(u) = m \) containing the \((m + 2)\)-path arc \( uv \).

4. Strong in-tournaments containing a 3-path arc

First, we present a structure result of strongly connected in-tournaments containing a 3-path arc, which implies that only one such arc exists.

**Theorem 4.1** Let \( D \) be a strongly connected in-tournament of order \( n \geq 4 \) containing a 3-path arc \( uv \). Then, \( D \) has no further 3-path arc, \( D - u \) is not strong, and the strong decomposition \( D_1, D_2, \ldots, D_p \) of \( D - u \) has the following properties. The strong component \( D_p \) consists of a single vertex, say \( w_p \), such that \( w_p \rightarrow u \), \( V(D_{p-1}) = \{v\} \), \( N^-(w_p) = \{v\} \), and \( u \rightarrow (D_1 \cup D_2 \cup \ldots \cup D_{p-1}) \).

**Proof.** From Observation 3.1 it follows that \( D - u \) is not strong. If \( D_1, D_2, \ldots, D_p \) are the strong components of \( D - u \), then in view of Theorem 2.3, there are no arcs from \( D_j \) to \( D_i \) for \( j > i \), and for each \( i = 1, 2, \ldots, p - 1 \) there exists a vertex \( w_i \in V(D_i) \) such that \( w_i \rightarrow D_{i+1} \). Since \( D \) is strong, there is a vertex \( w_p \in V(D_p) \) with \( w_p \rightarrow u \). First, we show that \( v \in V(D_j) \) implies \( V(D_j) = \{v\} \). Because otherwise, the strong component \( D_j \) consists of at least three vertices, and according to Theorem 2.2, \( D_j \) has a Hamiltonian cycle, say \( vx_1x_2 \ldots x_tv \) with \( t \geq 2 \). Then the arc \( uv \) belongs to the 4-path \( uvx_1x_2 \), a contradiction to the hypothesis that \( uv \) is a 3-path arc.

Since \( w_p \in V(D_p) \) with \( w_p \rightarrow u \), we conclude that \( j \neq p \). Analogously, we can show that \( V(D_p) = \{w_p\} \). Furthermore, if we assume that \( j \leq p - 2 \), then \( v = w_j \rightarrow D_{j+1} \), and hence, \( uww_{j+1}w_{j+2} \) is a 4-path containing the arc \( uv \), a contradiction. Consequently, \( j = p - 1 \), and therefore \( V(D_{p-1}) = \{v\} \).

Next we note that there are no arcs \( xu \) and \( xw_p \) such that \( x \) is a vertex of \( D_1 \cup D_2 \cup \ldots \cup D_{p-2} \), because otherwise, \( xuw_p \) and \( xw_puv \) would be a 4-path through \( uv \), respectively. This implies, in particular that \( N^-(w_p) = \{v\} \). The vertices \( u \) and \( w_{p-2} \) are negative neighbors of \( v \), and thus they are adjacent. Since there is no arc from \( w_{p-2} \) to \( u \), we deduce that \( u \rightarrow w_{p-2} \). If \( D_{p-2} \) consists only of the single vertex \( w_{p-2} \), then \( u \rightarrow D_{p-2} \). In the other case we use the facts that \( D_{p-2} \) has a Hamiltonian cycle, that there is no arc from \( D_{p-2} \) to \( u \), and \( u \rightarrow w_{p-2} \), to verify that \( u \rightarrow V(D_{p-2}) \). If we continue this process, we finally arrive at \( u \rightarrow (D_1 \cup D_2 \cup \ldots \cup D_{p-1}) \). We notice that all other arcs of \( D \) do not influence the property that \( uv \) is a 3-path arc.

Finally, we show that all arcs different from \( uv \) are contained in a 4-path. If \( ux_i \) is an arc with \( x_i \in V(D_i) \) for \( 1 \leq i \leq p - 2 \), then \( vwx_u \) is a 4-path through \( ux_i \) as well as through \( vuv \) and \( u \). Each arc \( x_iy_i \) of \( D_i \) for \( 1 \leq i \leq p - 2 \) belongs to the 4-path \( w_pux_iy_i \). In the case that \( x_iy_j \) is an arc from \( D_1 \) to \( D_j \) for \( 1 \leq i < j \leq p - 1 \), we see that \( x_ix_j \) is an arc of the 4-path \( w_pux_iy_j \). Since we have discussed all possible arcs, the proof is complete. \( \square \)
Theorem 4.2 Let $D$ be a strongly connected in-tournament of order $n \geq 5$ containing a 3-path arc but no 4-path arc. Then $n \geq 6$ and $D$ contains no 5-path arc.

Proof. Let $uv$ be the 3-path arc of $D$, and let $D_1, D_2, \ldots, D_p$ be the strong decomposition of $D - u$. Then, Theorem 4.1 implies $V(D_p) = \{w_p\}$, $w_p \rightarrow u$, $V(D_{p-1}) = \{v\}$, $N^-(w_p) = \{v\}$, and $u \rightarrow (D_1 \cup D_2 \cup \ldots \cup D_{p-1})$. Since $D$ contains no 4-path arc, we deduce that $|V(D_{p-2})| \geq 3$ and thus, $n \geq 6$. By Theorem 2.2, there exists a Hamiltonian cycle $b_1b_2 \ldots b_{t}b_1$ of $D_{p-2}$ with $t \geq 3$, and in view of Theorem 2.3, we assume without loss of generality that $w_{p-2} = b_1 \rightarrow v$.

First, we show that all arcs, different from $uv$, of the subdigraph induced by the vertices $u, v, w_p, b_1, b_2, \ldots, b_t$ are contained in a 6-path. The path $vw_pu_b_{i+1} \ldots b_{i-1}$ for $1 \leq i \leq t$ shows that the arcs $ub_i$, $vw_p$, and $w_pu$ are contained in a path of order at least 6. If $b_i \rightarrow v$ for any $1 \leq i \leq t$, then $b_iu$ is an arc of the path $b_{i-1}v_w_pu_{b_{i+1}} \ldots b_{i-1}$ which is of order at least 6. Each arc $b_iu$ with $i, j \neq 1$ belongs to the 6-path $b_1v_{w_p}u_{b_{i+1}}$. An arc $b_iu$ with $i \geq 3$ is contained in the 6-path $b_{i-1}v_{w_p}u_{b_{i-1}}$, and an arc $b_iu$ with $i \neq t$ is contained in the 6-path $v_{w_p}u_{b_{i+1}}$.

Using in particular the fact that $u \rightarrow (D_1 \cup D_2 \cup \ldots \cup D_{p-1})$, we now prove that the other arcs are also contained in a 6-path, when $p \geq 4$. Each arc $ux_{i}$ with $i \in V(D_i)$ for $1 \leq i \leq p - 3$ belongs to the 6-path $b_1v_{w_p}ux_{i}$. If $x_iz_{i}$ is an arc of $D_i$ for $1 \leq i \leq p - 3$, then it is contained in the 6-path $b_1v_{w_p}ux_{i}z_{i}$. Finally, let $x_iz_{j}$ be an arc from $D_i$ to $D_j$ for $1 \leq i \leq p - 3$ and $i < j < p - 1$. If $j < p - 3$, then $b_1v_{w_p}ux_{i}z_{j}$ is a 6-path through the arc $x_iz_{j}$. In the case $j = p - 2$, we observe that $y_j = b_i$ for any $1 \leq s \leq t$, and obviously, $v_{w_p}ux_{i}z_{j}b_{i+1}$ is such a desired 6-path. In the remaining case $j = p - 1$, we have $y_j = v$. Since $i \leq p - 3$, we observe that $x_iz_{j}v_{w_p}u_{b_1}b_1$ is a 6-path with the initial arc $x_iz_{j}$. Consequently, $D$ contains no 5-path arc, and the proof is complete. □

Next we will show that Theorem 4.2 is sharp in the sense that there exist in-tournaments containing a 3-path arc, without 4 or 5-path arcs, however with 6-path arcs.

Example 4.3 Let $D$ be consists of the cycle $C = b_1b_2 \ldots b_nb_1$, the arcs $b_iu$ for $3 \leq i \leq n - 1$, and the vertices $u,v$ and $w_3$ such that $b_1 \rightarrow v \rightarrow w_3 \rightarrow u \rightarrow (V(C) \cup \{v\})$. Then, it is straightforward to verify that $D$ is a strongly connected in-tournament of order $n + 3$ with the 3-path arc $uv$, without a 4 or a 5-path arc, but $D$ contains the 6-path arc $b_1b_{n-1}$.

Example 4.4 Let $D$ be consists of the cycle $C = b_1b_2b_3b_1$, the vertices $u,v,w_1$ and $w_4$ such that $w_1 \rightarrow C$ and $b_1 \rightarrow v \rightarrow w_4 \rightarrow u \rightarrow (V(C) \cup \{v,w_1\})$. Then, $D$ is a strongly connected in-tournament of order 7 containing the 3-path arc $uv$, without a 4 or a 5-path arc, but with the two 6-path arcs $ub_1$ and $ub_3$.

Example 4.5 Let $D$ be consists of the cycle $C = b_1b_2b_3b_1$, the vertices $u,v,w_4$, and an arbitrary tournament $T_1$ such that $b_1 \rightarrow v \rightarrow w_4 \rightarrow u \rightarrow (V(C) \cup V(T_1) \cup \{v\})$ and $T_1 \rightarrow C$. Then, $D$ is a strongly connected in-tournament with the 3-path arc.
Theorem 4.6 Let $D$ be a strongly connected in-tournament of order $n \geq 5$ containing the 3-path arc $uv$. If $D_1, D_2, \ldots, D_p$ is the strong decomposition of $D - u$, and if $D$ contains an $k$-path arc for each $4 \leq k \leq m \leq n - 1$, then $V(D_{p+2-k}) = \{w_{p+2-k}\}$, $N^{-}(w_{p+3-k}) = \{u, w_{p+2-k}\}$, and $uw_{p+2-k}$ is the unique $k$-path arc for $4 \leq k \leq m$.

Proof. We proceed by induction on $m$, using the structure of $D$, described in Theorem 4.1 and parts of the proof of Theorem 4.2.

Let $m = 4$. Suppose first that $|V(D_{p-2})| \geq 3$. Then, by the proof of Theorem 4.2, we see that all arcs different from $uv$ of the subdigraph induced by the vertices $u, v, w_{p}$ and the vertex set $V(D_{p-2})$ are contained in a 6-path. But since $D$ contains 4-path arc, we deduce that $p \geq 4$.

Next we prove that all arcs different from $uv$ and $ux$ with $x \in V(D_{p-2})$ belong to a 5-path of $D$, independently from the order of $D_{p-2}$. Every arc $ux_i$ with $x_i \in V(D_i)$ is contained in the path $w_{p-2}vw_{p}ux_i$ for $1 \leq i \leq p - 3$. Thus, $vw_{p}$ and $w_{p}u$ are also arcs of a 5-path. Every arc $xiy_j$ of $D_i$ is contained in the path $vw_{p}uxiy_j$ for $1 \leq i \leq p - 2$. Now let $xiy_j$ be an arc from $D_i$ to $D_j$ for $1 \leq i < j \leq p - 1$. If $j \leq p - 2$, then the 5-path $vw_{p}uxiy_j$ has the terminal arc $x_iy_j$. If $j = p - 1$, then $x_j = v$, and $x_1v$ belongs to the 5-path $x_1vw_{p}ux_i$ with $s \neq i, p - 1, p$ and $x_1 \in V(D_s)$.

All together we see there exists at most a 4-path arc in $D$, if $p \geq 4$ and if $D_{p-2}$ consists of the single vertex $w_{p-2}$. But in this case, certainly, $uw_{p-2}$ is the only 4-path arc of $D$, when $N^{-}(v) = N^{-}(w_{p-1}) = \{u, w_{p-2}\}$.

Now let $5 \leq m \leq n - 1$ and assume that $D$ contains a $k$-path arc for each $4 \leq k \leq m$. Then, $D$ contains a $k$-path arc for each $4 \leq k \leq m - 1$, and by the induction hypothesis $V(D_{p+2-k}) = \{w_{p+2-k}\}$, $N^{-}(w_{p+3-k}) = \{u, w_{p+2-k}\}$, and $uw_{p+2-k}$ is the unique $k$-path arc for $4 \leq k \leq m - 1$. Analogously to the case $m = 4$, one can prove that $|V(D_{p+2-m})| \geq 3$ is not possible, and thus $p \geq m$, and that all arcs different from $uv$, $uw_{p+2-k}$ for $4 \leq k \leq m - 1$ and $ux$ with $x \in V(D_{p+2-m})$ are contained in an $(m+1)$-path, independently from the order of $D_{p+2-m}$. Consequently, $D_{p+2-m}$ consists of the single vertex $w_{p+2-m}$. In addition, from the hypothesis that $D$ has an $m$-path arc, it follows that $N^{-}(w_{p+3-m}) = \{u, w_{p+2-m}\}$ and this implies that $uw_{p-2}$ is the only $m$-path arc of $D$. □

Using Theorem 4.6, it is no problem to obtain the next result, analogously to Theorem 4.2.

Theorem 4.7 Let $D$ be a strongly connected in-tournament of order $n \geq m + 2$ containing a $k$-path arc for each $k = 3, 4, \ldots, m$ but no $(m+1)$-path arc. Then $n \geq m + 3$ and $D$ contains no $(m + 2)$-path arc.

Theorem 4.8 Let $D$ be a strongly connected local tournament of order $n \geq 4$ with the 3-path arc $uv$. Then all arcs of $D$ which are not incident with $u$ are contained in a Hamiltonian path.

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Proof. By Observation 3.1, we assume without loss of generality that $D - u$ is not strong. If $D_1, D_2, \ldots, D_p$ is the strong decomposition of $D - u$, then by Theorem 4.1, $V(D_p) = \{w_p\}$, $w_p \to u$, $V(D_{p-1}) = \{v\}$, $N^-(w_p) = \{v\}$, and $u \to (D_1 \cup D_2 \cup \ldots \cup D_{p-1})$. Furthermore, Theorem 2.4 implies $D_i \to D_{i+1}$ for $i = 1, 2, \ldots, p-1$. In the following let $x_1^i x_2^i \ldots x_k^{i|V(D_i)|} x_1^i$ be a Hamiltonian cycle of the strong component $D_i$ for $1 \leq i \leq p - 2$, when $|V(D_i)| \geq 3$, and define by $P_i$ a Hamiltonian path of $D_i$. By Theorem 3.2 and Remark 3.4, every arc from $D_i$ to $D_{i+1}$ for $1 \leq i \leq p - 1$, and each arc of the component $D_i$ for $1 \leq i \leq p - 2$ is contained in a Hamiltonian path of $D$. Now, let $x_j^t x_k^t$ be an arc from $D_i$ to $D_t$ for $1 \leq i \leq p - 2$ and $i+2 \leq t \leq p-1$. Then, because of $D_i \to D_{i+1}$ for $i = 1, 2, \ldots, p-1$, we deduce that

$P_1 P_2 \ldots P_{i-1} x_j^t x_{j+1}^t x_{j+2}^t \ldots x_k^t x_k^{i+1} x_{k+1}^t \ldots x_{k-1}^t P_{t+1} \ldots P_{t-1} = vw_p u P_{i-1} \ldots P_{t-1}$

is a Hamiltonian path through $x_j^t x_k^t$, and this completes the proof. □

Obviously, in Theorem 4.8, the arc $w_p u$ and all arcs from $u$ to $D_1$ are also contained in a Hamiltonian path, even in a Hamiltonian cycle. Example 4.3 shows that Theorem 4.8 is no longer valid for in-tournaments in general.

5. Strong in-tournaments without a 3-path arc

Next we describe the structure of strongly connected in-tournaments containing a 4-path arc but no 3-path arc. We shall see that such in-tournaments have only one 4-path arc, when the order is at least six.

Theorem 5.1 Let $D$ be a strongly connected in-tournament of order $n \geq 6$ containing a 4-path arc $uv$ but no 3-path arc. If $D_1, D_2, \ldots, D_p$ is the strong decomposition of $D - u$, then $p \geq 4$, $D_p$ consists of a single vertex, say $w_p$ such that $w_p \to u$, $V(D_{p-1}) = \{w_{p-1}\}$, $w_{p-1} \to (D_1 \cup D_2 \cup \ldots \cup D_{p-2})$, $v \in V(D_{p-1}) \cup V(D_{p-2})$, and $D$ has no further 4-path arc. In addition:

If $v \in V(D_{p-1})$, then $V(D_{p-1}) = \{v\}$, $V(D_1) = \{v_1\}$, and $N^-(w_p) = \{v_1, v\}$.

If $v \in V(D_{p-2})$, then $V(D_{p-2}) = \{v\}$ and there are no arcs from $D_j$ to the vertices $w_{p-1}$ or $w_p$ for $1 \leq j \leq p - 3$. Furthermore, if $u \to w_{p-1}$, then $v \to w_p$.

Proof. From Observation 3.1 it follows that $D - u$ is not strong. Since $D$ is strong, there exists a vertex $w_p \in V(D_p)$ with $w_p \to u$.

Suppose first that $v \in V(D_p)$. Since the vertex $w_p \neq v$ is also in $D_p$, the strong component $D_p$ consists of at least three vertices, and according to Theorem 2.2, $D_p$ has a Hamiltonian cycle, say $vx_1 x_2 \ldots x_tv$, with $t \geq 2$. If $t \geq 3$, then $uwx_1x_2x_3$ is a 5-path through $uv$, a contradiction. Thus, $t = 2$. The vertices $u$ and $x_2$ are adjacent, since they are negative neighbors of $v$. If $x_2 \to u$, then $w_{p-1}x_1x_2uv$ is a 5-path, a contradiction. Consequently, $u \to x_2$ and $w_p = x_1 \to u$. But now it follows easily from the hypothesis $n \geq 6$ that $uv$ is not a 4-path arc, a contradiction.

Second, let $v \in D_{p-1}$. If $|V(D_{p-1})| \geq 3$, then there exists a Hamiltonian cycle $vx_1 x_2 \ldots x_tv$ of $D_{p-1}$, and $w_puvx_1x_2$ is a 5-path through $uv$, a contradiction. This implies $V(D_{p-1}) = \{v\}$, and similarly we find that $V(D_p) = \{w_p\}$. Since $uv$ is a
4-path arc, there exist an arc $wu$ or $ww_p$ with $w \in V(D_j)$ for $1 \leq j \leq p - 2$. In both cases we deduce that $w \in V(D_1)$ and $|V(D_1)| = 1$. This is a contradiction if $wu$ is an arc of $D$, because there is also an arc from $u$ to $D_1$. In the other case we see that $w = w_1$, $N^-(w_p) = \{w_1, v\}$, and $p \geq 4$. Analogously to the proof of Theorem 4.1, we obtain $u \rightarrow (D_1 \cup D_2 \cup \ldots \cup D_{p-2})$, and this implies that $D$ has no further 4-path arc.

Suppose third that $v \in V(D_j)$ for any $j \leq p - 2$. The cases $j \leq p - 3$ or $j = p - 2$ and $|V(D_{p-2})| \geq 3$ lead to a contradiction, and thus, $V(D_{p-2}) = \{v\}$. This implies immediately $V(D_p) = \{w_p\}$, $V(D_{p-1}) = \{w_{p-1}\}$, and $p \geq 4$. Next we note that there are no arcs $x_iw_{p-1}$ or $x_iw_p$ with $x_i \in V(D_i)$ for $1 \leq i \leq p - 3$, because otherwise $x_iw_{p-1}uv$ or $x_iw_puwv_{p-1}$ would be 5-paths through $uv$. Obviously, there is no arc from $D_j$ to $u$ for $j \leq p - 3$, and hence, analogously to the proof of Theorem 4.1, we obtain $u \rightarrow (D_1 \cup D_2 \cup \ldots \cup D_{p-2})$. If $u \rightarrow w_{p-1}$, then it follows that $v \rightarrow w_p$, because otherwise $uw_{p-1}$ would be a 3-path arc. With help of the hypothesis $n \geq 6$, it is straightforward to verify that there is no further 4-path arc in $D$. □

**Remark 5.2** For $n = 5$ there exist exactly three non isomorphic strongly connected in-tournaments containing a 4-path arc but no 3-path arc. Let $C = uw_1w_2w_3w_4u$ be a 5-cycle.

If $T_5$ is the tournament consisting of $C$ such that $u \rightarrow \{w_2, w_3\}$, $w_1 \rightarrow \{w_3, w_4\}$, and $w_4 \rightarrow w_2$, then $T_5$ contains the the unique 4-path arc $uw_3$.

If $D_5$ is the in-tournament consisting of $C$ such that $u \rightarrow w_2$ and $w_3 \rightarrow u$, then $D_5$ has even the two 4-path arcs $uw_2$ and $w_3u$. If we add in $D_5$ the arc $w_2w_4$, then we obtain an in-tournament with the unique 4-path arc $uw_2$.

With help of Theorem 3.2 and Theorem 5.1, one can prove the next result, analogously to Theorem 4.8.

**Theorem 5.3** Let $D$ be a strongly connected local tournament of order $n \geq 6$ with the 4-path arc $uv$ but without a 3-path arc. Then all arcs of $D$ which are not incident with $u$ are contained in a Hamiltonian path, with exception of the arc $vv_p$, when $v = w_{p-2}$ and $u$ and $w_{p-1}$ are not adjacent. But in this situation the arc $vv_p$ is contained in an $(n-1)$-path.

Our next example shows that Theorem 5.3 is not valid for strong in-tournaments in general.

**Example 5.4** Let $D$ be consists of the cycle $C = b_1b_2 \ldots b_nb_1$, the arcs $b_ib_i$ for $3 \leq i \leq n - 1$, and the vertices $u, v, w_2$ and $w_3$ such that $b_1 \rightarrow v \rightarrow w_2 \rightarrow w_3 \rightarrow u \rightarrow (V(C) \cup \{v\})$ and $v \rightarrow w_3$. Then, $D$ is a strongly connected in-tournament of order $n + 4$ without a 3-path arc containing the 4-path arc $uv$. We observe that $D$ has the $(n + 3)$-path arc $vv_3$ and the 7-path arc $b_1b_{n-1}$, so that $b_1b_{n-1}$ is not contained in a Hamiltonian path, when $n \geq 4$.

We also have a corresponding result to the Theorems 4.1 and 5.1, when $uv$ is a
5-path arc and $D$ contains neither a 3-path arc nor a 4-path arc. Since the description of such in-tournaments is long and not very transparent, we omit it here. But especially, we have found the following uniqueness theorem.

**Theorem 5.5** Let $D$ be a strong in-tournament of order $n \geq 8$ containing a 5-path arc but neither a 3-path arc nor a 4-path arc. Then, $D$ has exactly one 5-path arc.

Next we present an example that demonstrates that the condition $n \geq 8$ in Theorem 5.5 is necessary.

**Example 5.6** Let $m \geq 5$ be an integer, and let the strongly connected in-tournament $D$ consists of the cycle $x_1x_2 \ldots x_{m-2}x_{m-1}y_1y_2 \ldots y_{m-2}x_1$ such that $x_1 \rightarrow \{x_3, x_4, \ldots, x_{m-1}\}$ and $x_{m-1} \rightarrow \{y_2, y_3, \ldots, y_{m-2}\}$. Then, $D$ is of order $2m - 3$ with the two $m$-path arcs $x_1x_{m-1}$ and $x_{m-1}y_{m-2}$.

Theorems 4.1, 5.1, 5.5, and Example 5.6 leads us to the following conjecture.

**Conjecture 5.7** Let $m \geq 6$ be an integer, and let $D$ be a strongly connected in-tournament of order $n \geq 2m - 2$. If $D$ has an $m$-path arc but no $k$-path arc for $3 \leq k \leq m - 1$, then there exists exactly one $m$-path arc.

Example 5.6 shows that the condition $n \geq 2m - 2$ in Conjecture 5.7 would be best possible.

**References**


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