Improvements on inequalities for non-negative matrices*

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Abstract

We prove that there is an integer \( k \leq (n^2 - 2n + 4)/2 \) such that the diagonal entries of \( A^k \) are all positive for any non-negative irreducible \( n \times n \) matrix \( A \), and that there are integers \( i, j \) with \( 0 \leq i < j \leq 3^{n/2} \) such that \( A^i \leq A^j \) for any non-negative \( n \times n \) matrix \( A \) with no entry in \((0,1)\) and \( n \geq 2 \). The results of Wang and Shallit [Linear Algebra Appl. 290 (1999) 135-144] are thus improved.

1. Introduction

In this paper we will be concerned with matrices and vectors with non-negative entries. For a matrix \( A = (a_{ij}) \) and scalar \( c \), by the inequality \( A > c \) we mean that \( a_{ij} > c \) for all \( i, j \), and similarly for the relations \( A \geq c \) and \( A = c \). For matrices \( A \) and \( B \) of the same dimensions, by \( A \geq B \) we mean the inequality holds entrywise. We adopt similar conventions for vectors.

For an \( n \times n \) matrix \( A \), by \( \text{diag}(A) \) we mean the vector containing the diagonal entries of \( A \). Let \( I \) denote the identity matrix.

A square matrix \( A \) is said to be reducible if there is a permutation matrix \( P \) such that

\[
P^T A P = \begin{pmatrix} B & 0 \\ D & C \end{pmatrix},
\]

where the diagonal blocks \( B \) and \( C \) are square matrices. \( A \) is irreducible if it is not reducible.

For an irreducible matrix \( A \), let \( \beta(A) \) be the least integer \( k \geq 1 \) such that \( \text{diag}(A^k) > 0 \). Define \( \beta(n) = \sup \beta(A) \), where the supremum is over all irreducible \( n \times n \) matrices. Recently Wang and Shallit [1] proved that \( \beta(n) \leq n(n-1) \) for \( n \geq 2 \). They posed the problem of determining a more precise upper bound for \( \beta(n) \).

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For a non-negative $n \times n$ matrix $A$ with no entry in $(0, 1)$, let $\alpha(A)$ be the least positive integer $j$ such that there exists an integer $i$ with $0 \leq i < j$ such that $A^i \leq A^j$. Define $\alpha(n) = \sup \alpha(A)$, where the supremum is over all non-negative matrices $A$ with no entry in $(0, 1)$. Wang and Shallit [1] have proved that $\alpha(n) \leq 2^n$. As is remarked in [1], this inequality is almost surely not best possible.

In this paper we prove more precise bounds for $\beta(n)$ and $\alpha(n)$.

2. Bound for $\beta(n)$

The graph of an $n \times n$ matrix $A = (a_{ij})$ is the directed graph on vertices $v_1, v_2, \cdots, v_n$ such that there is an arc from $v_i$ to $v_j$ if and only if $a_{ij} > 0$. We denote the graph of $A$ by $G(A)$. An s-cycle is a (directed) cycle of length $s$.

An irreducible matrix $A$ is primitive if there is a positive integer $l$ such that $A^l > 0$. The least such $l$ is called the exponent of $A$ and is denoted $\gamma(A)$.

For an irreducible matrix $A$, the greatest common divisor of all cycle lengths of $G(A)$ is called the index of imprimitivity of $A$ and is denoted $d(A)$. It is well known (see, e.g., [4]) that a matrix $A$ is irreducible if and only if $G(A)$ is strongly connected and that an irreducible matrix $A$ is primitive if and only if $d(A) = 1$.

We first introduce the following lemmas, which we will use to estimate $\beta(A)$ for an irreducible matrix $A$.

**Lemma 1** [3]. If $A$ is an $n \times n$ primitive matrix whose graph has at least three distinct cycle lengths, then $\gamma(A) \leq \lfloor (n^2 - 2n + 4)/2 \rfloor$.

**Lemma 2** [2]. Suppose $X$ and $Y$ are $r \times t$ and $t \times r$ non-negative matrices and neither has a zero row or column. Then $XY$ is primitive if and only if $YX$ is, and if $XY$ and $YX$ are primitive, then $\gamma(YX) - 1 \leq \gamma(XY) \leq \gamma(YX) + 1$.

**Lemma 3** [5]. If $A$ is an $n \times n$ primitive matrix, then $\gamma(A) \leq (n - 1)^2 + 1$.

Our first theorem refines the bound for $\beta(n)$ obtained in [1].

**Theorem 1.** Let 

$$f(n) = \left\lfloor \frac{n^2 - 2n + 4}{2} \right\rfloor.$$ 

Then $\beta(n) \leq f(n)$.

**Proof.** Let $A$ be an irreducible $n \times n$ matrix with $G = G(A)$. Denote by $L(G)$ the set of cycle lengths of $G$. If $G$ contains an $n$-cycle, then $\beta(A) \leq n \leq f(n)$. Suppose in the following that $G$ contains no $n$-cycle. There are two cases to consider, based on the primitivity of $A$.

Case 1: $A$ is primitive.

Case 1.1: $|L(G)| = 2$. Suppose $L(G) = \{p, q\}$ with $p < q \leq n - 1$. If $p + q \geq n + 1$, then every $p$-cycle intersects every $q$-cycle, and hence $\beta(A) \leq p + q \leq (n - 2) + (n - 1) = 2n - 3 \leq f(n)$, while if $p + q \leq n$, then $\beta(A) \leq pq \leq ((p + q)/2)^2 \leq n^2/4 \leq f(n)$.

Case 1.2: $|L(G)| \geq 3$. In this case, we have $n \geq 4$. By Lemma 1 we have $\beta(A) \leq \gamma(A) \leq \lfloor (n^2 - 2n + 4)/2 \rfloor = f(n)$.
Case 2: $A$ is not primitive. Suppose $d(A) = d \geq 2$. By classical results on
imprimitive matrices (see [4, pp.71-73]), there is a permutation matrix $P$ such that

$$P^TAP = \begin{pmatrix}
0 & A_1 & 0 & \cdots & 0 \\
0 & 0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_{d-1} \\
A_d & 0 & 0 & \cdots & 0
\end{pmatrix}$$

where the diagonal zero blocks are square and each block $A_i$ has no zero row or
column; furthermore, if $A_i$ is of dimension $n_i \times n_{i+1}$ ($n_{d+1} = n_1$), and we put $B_i = A_iA_{i+1} \cdots A_dA_1 \cdots A_{i-1}$, then

$$P^T A_d P = \begin{pmatrix}
B_1 & 0 & \cdots & 0 \\
0 & B_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_d
\end{pmatrix},$$

where $B_i$ is an $n_i \times n_i$ primitive matrix for each $i$ with $1 \leq i \leq d$.

If $d = n$, then clearly $\beta(A) = n \leq f(n)$. If $n = 3$ and $d = 2$, then $\beta(A) = 2 \leq f(3) = 3$. Suppose $2 \leq d \leq n-1$ and $n \geq 4$.

Let $n_m = \min_{1 \leq m \leq d} n_i$ where $1 \leq m \leq d$ and $\gamma(B_t) = \max_{1 \leq t \leq d} \gamma(B_t)$ where $1 \leq t \leq d$.

We claim that $\gamma(B_t) \leq \gamma(B_m) + 1$. This is obvious if $t = m$. Suppose without
loss of generality that $1 \leq t < m \leq d$. Let $X = A_tA_{t+1} \cdots A_{m-1}$ and $Y = A_mA_{m+1} \cdots A_dA_1 \cdots A_{t-1}$. Then $B_t = XY$ and $B_m = YX$. By Lemma 2, we have

$$\gamma(B_t) = \gamma(XY) \leq \gamma(YX) + 1 = \gamma(B_m) + 1,$$

as desired.

Note that $n_1 + n_2 + \cdots + n_d = n$. We have $n_m \leq n/d$. It follows from Lemma 3
that

$$\max_{1 \leq i \leq d} \gamma(B_i) = \gamma(B_t) \leq \gamma(B_m) + 1$$

$$\leq (n_m - 1)^2 + 1 + 1$$

$$\leq \left(\frac{n}{d} - 1\right)^2 + 2.$$ 

Hence

$$\beta(A) \leq d \max_{1 \leq i \leq d} \gamma(B_i)$$

$$\leq d \left(\frac{n}{d} - 1\right)^2 + 2d$$

$$= \frac{(n-d)^2}{d} + 2d.$$ 

The function $h(d) = (n-d)^2/d + 2d$ is a decreasing function of $d$ in $[2, n/\sqrt{3}]$ and
an increasing function in $[n/\sqrt{3}, n-1]$. Hence it assumes its largest value either for
$d = 2$ or $d = n - 1$. We have

$$h(2) = (n-2)^2/2 + 2, \quad h(n-1) = 2(n-1) + 1/(n-1).$$

It is easy to see that $|h(n-1)| \leq |h(2)| \leq f(n)$ for $n \geq 6$, and $|h(2)| \leq |h(n-1)| \leq f(n)$ for $n = 4$ or 5. Hence

$$\beta(A) \leq h(d) \leq \max\{|h(2)|, |h(n-1)|\} \leq f(n).$$
3. Bound for $\alpha(n)$

For a non-negative $n \times n$ matrix $A$ with no entry in $(0,1)$, Wang and Shallit [1] proved that $\alpha(n) \leq 2^n$ for all $n \geq 1$, and this bound cannot be replaced by $e^{\sqrt{n \log n}}$. We are going to improve this result. First we give a lemma that will be used.

**Lemma 4 [1].** Suppose $A \geq 0$ is an $n \times n$ matrix of the form

$$A = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$

where $B$, $D$ are square matrices with $D \geq I$. For integers $l \geq 0$, define the matrices $C_l$ by

$$A^l = \begin{pmatrix} B^l & 0 \\ C_l & D^l \end{pmatrix}.$$

Then for all $l \geq 0$, we have $C_l \leq C_{l+1}$ and $D^l \leq D^{l+1}$.

An easily verified fact is that $f(n) = [(n^2 - 2n + 4)/2] \leq 3^{n/2}$ for all $n \geq 2$.

**Theorem 2.** For all $n \geq 2$, we have $\alpha(n) \leq 3^{n/2}$.

**Proof.** Let $A$ be a non-negative $n \times n$ matrix with no entry in $(0,1)$. We use induction on $n$ to prove the theorem. For $n = 2$, if $A$ is irreducible, then clearly $A^0 = I \leq A^2$, while if $A$ is reducible, then we have either $A = A^2$ or $A^2 = A^3 = 0$. Hence $\alpha(A) \leq 3$ for $n = 2$.

Assume $n \geq 3$ and the result holds for all $m$ with $2 \leq m < n$. The proof is now divided into the following two cases.

Case 1: $A$ is irreducible. By Theorem 1, there is an integer $k$, $1 \leq k \leq f(n)$, such that $\text{diag}(A^k) > 0$. Note that every positive diagonal entry of $A^k$ is $\geq 1$. We have $I = A^0 \leq A^k$. Hence $\alpha(A) \leq k \leq f(n) \leq 3^{n/2}$.

Case 2: $A$ is reducible. There is a permutation matrix $P$ such that

$$P^T A P = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{t1} & A_{t2} & \cdots & A_{tt} \end{pmatrix},$$

where $A_{11}, A_{22}, \cdots, A_{tt}$ are square matrices that are either 0 or irreducible.

Case 2.1: $A_{tt} = 0$. The last column of $A$ is 0. We write

$$A = \begin{pmatrix} B & 0 \\ x & 0 \end{pmatrix},$$

where $x$ is a vector of dimension $n - 1$. Note that $n - 1 \geq 2$. By induction, $\alpha(B) \leq 3^{(n-1)/2}$, i.e., there are integers $i,j$ with $0 \leq i < j \leq 3^{(n-1)/2}$ such that $B^i \leq B^j$. It follows that

$$A^{i+1} = \begin{pmatrix} B^{i+1} & 0 \\ xB^i & 0 \end{pmatrix} \leq \begin{pmatrix} B^{j+1} & 0 \\ xB^j & 0 \end{pmatrix} = A^{j+1},$$

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and \(1 \leq i + 1 < j + 1 \leq 3^{(n-1)/2} + 1 \leq 3^{n/2}\). Hence \(\alpha(A) \leq 3^{n/2}\).

Case 2.2: \(A_{tt}\) is irreducible. Suppose \(A_{tt}\) is of dimension \(m \times m\) with \(1 \leq m \leq n-1\). By Theorem 1, there is an integer \(k\) with \(1 \leq k \leq f(m) \leq 3^{m/2}\) such that \(A_{tt}^k \geq I\). We write

\[
A = \begin{pmatrix} B & 0 \\ C & A_{tt} \end{pmatrix}.
\]

Case 2.2.1: \(B\) is 0 of dimension \(1 \times 1\). Then \(C\) is a column vector of dimension \(n - 1\). By similar arguments as in Case 2.1, we have

\[
A^{i+1} = \begin{pmatrix} 0 & 0 \\ A_{tt}^i C & A_{tt}^{i+1} \end{pmatrix} \leq \begin{pmatrix} 0 & 0 \\ A_{tt}^i & A_{tt}^{i+1} \end{pmatrix} = A^{i+1},
\]

and \(1 \leq i + 1 < j + 1 \leq 3^{(n-1)/2} + 1 \leq 3^{n/2}\). Hence \(\alpha(A) \leq 3^{n/2}\).

Case 2.2.2: \(B\) is not 0 of dimension \(1 \times 1\). Then we have either \(m \leq n - 2\) or \(B\) is of dimension \(1 \times 1\) but not 0. In the former case, we know by the induction hypothesis applied to \(B^k\) that there are integers \(i, j\) with \(0 \leq i < j \leq 3^{(n-m)/2}\) such that \((B^k)^i \leq (B^k)^j\), while in the latter case we have \((B^k)^i \leq (B^k)^j\) where \(i = 0\) and \(j = 1\). Note that

\[
A^k = \begin{pmatrix} B^k & 0 \\ C_k & A_{tt}^k \end{pmatrix}
\]

for some \(C_k\). By Lemma 4, \((A^k)^i \leq (A^k)^j\) and \(0 \leq ki < kj \leq 3^{m/2}3^{(n-m)/2} = 3^{n/2}\). Hence \(\alpha(A) \leq 3^{n/2}\).

The proof is now completed. \(\square\)

References


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