On the maximal number of vertices covered by disjoint cycles

Hong Wang*

Department of Mathematics
The University of Idaho
Moscow, Idaho 83844, USA

Abstract

Let \( k, t \) and \( n \) be three integers with \( t \geq 2, k \geq 2t \) and \( n \geq 3t \). We conjecture that if \( G \) is a graph of order \( n \) with minimum degree at least \( k \), then \( G \) contains \( t \) vertex-disjoint cycles covering at least \( \min(2k, n) \) vertices of \( G \). We will show the conjecture to be true for \( t = 2 \).

1 Introduction

We discuss only finite simple graphs and use standard terminology and notation from [1] except as indicated. Let \( k \) be an integer with \( k \geq 2 \). Let \( G \) be a graph of order \( n \geq 3 \). P. Erdős and T. Gallai [5] showed that if \( G \) is 2-connected and every vertex of \( G \) with at most one exception has degree at least \( k \), then \( G \) contains a cycle of length at least \( \min(2k, n) \). We wonder if \( G \) contains at least two vertex-disjoint cycles covering at least \( \min(2k, n) \) vertices of \( G \). This is certainly true if \( k \geq n/2 \) with \( k \geq 4 \) and \( n \geq 6 \) by El-Zahar’s result [4]. El-Zahar proved that if \( n = n_1 + n_2 \) is an integer partition of \( n \) with \( n_1 \geq 3 \) and \( n_2 \geq 3 \) and the minimum degree of \( G \) is at least \( \lceil n_1/2 \rceil + \lceil n_2/2 \rceil \), then \( G \) contains two vertex-disjoint cycles of lengths \( n_1 \) and \( n_2 \), respectively. Corrádi and Hajnal [2] investigated the maximum number of vertex-disjoint cycles in a graph. They proved that if \( G \) is a graph of order at least \( 3t \) with minimum degree at least \( 2t \), then \( G \) contains \( t \) vertex-disjoint cycles. In particular, when the order of \( G \) is exactly \( 3t \), then \( G \) contains \( t \) vertex-disjoint triangles. Motivated by these results, we conjecture the following:

Conjecture A Let \( k, t \) and \( n \) be three integers with \( t \geq 2, k \geq 2t \) and \( n \geq 3t \). Suppose that \( G \) is a graph of order \( n \) with minimum degree at least \( k \). Then \( G \) contains \( t \) vertex-disjoint cycles covering at least \( \min(2k, n) \) vertices of \( G \).

Note that if this conjecture is true, then the condition on the degrees of \( G \) is sharp. This can be seen from the graph \( K_{k-1,n-k+1} \) with \( n > 2(k-1) \). By observing

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$K_{k,n-k}$, we also see that when $n \geq 2k$, $G$ may not contain $t$ vertex-disjoint cycles covering more than $2k$ vertices of $G$.

Erdős and Faudree [6] conjectured that if $G$ is a graph of order $4t$ with minimum degree at least $2t$, then $G$ contains $t$ vertex-disjoint cycles of length 4. With respect to this conjecture, we proved [10] that $G$ contains $t$ vertex-disjoint cycles such that $t - 1$ of them are of length 4. It follows that $G$ contains $t$ vertex-disjoint cycles covering all the vertices of $G$ such that at least $t - 2$ of them are of length 4. Thus Conjecture A is true when $n = 2k = 4t$. In this paper, we will prove the following result.

**Theorem B** Let $k$ and $n$ be two integers with $k \geq 4$ and $n \geq 6$. Let $G$ be a graph of order $n$ with minimum degree at least $k$. Then $G$ contains two vertex-disjoint cycles covering at least $\min(2k, n)$ vertices of $G$.

We shall use the following terminology and notation. Let $G$ be a graph. For a vertex $u \in V(G)$ and a subgraph $H$ of $G$, $N(u, H)$ is the set of neighbors of $u$ contained in $H$, i.e., $N(u, H) = N(u) \cap V(H)$. We let $d(u, H) = |N(u, H)|$. Thus $d(u, G)$ is the degree of $u$ in $G$. For a subset $U$ of $V(G)$, $G[U]$ denotes the subgraph of $G$ induced by $U$. The length of a longest cycle of $G$ is denoted by $c(G)$. We define $c_t(G)$ to be the maximal number of vertices of $G$ covered by a set of $t$ vertex-disjoint cycles of $G$. Thus $c_1(G) = c(G)$.

## 2 Lemmas

Let $G = (V, E)$ be a given graph in the following. Lemma 2.1 is an easy observation.

**Lemma 2.1** Let $C$ be a cycle of length $s$ in $G$. Let $P$ be a path of length at least $\lfloor s/2 \rfloor - 1$ in $G - V(C)$. Suppose that $x$ and $y$ are the two endvertices of $P$ with $d(x, C) \geq 1$ and $d(y, C) \geq 1$. Then either $G[V(C \cup P)]$ contains a cycle longer than $C$, or $N(x, C) = N(y, C) = \{u\}$ for some $u \in V(C)$.

**Lemma 2.2** Let $C$ be a cycle of length $s$ in $G$. Let $P$ be a path of length at least 2 in $G - V(C)$. Suppose that $x$ and $y$ are the two endvertices of $P$ and $d(x, C) + d(y, C) > s/2$. Then $G[V(C \cup P)]$ contains a cycle longer than $C$.

**Proof.** Let $C = u_1u_2 \ldots u_su_1$. The subscripts of the $u_i$'s will be reduced modulo $s$ in the following. Clearly, we have

$$2(d(x, C) + d(y, C)) = \sum_{i=1}^{s}(d(x, u_\i+1) + d(y, u_\i+2u_\i+3)) > s.$$  

This implies that there exists $i \in \{1, 2, \ldots, s\}$ such that $d(x, u_\i+1) + d(y, u_\i+2u_\i+3) \geq 2$. The lemma follows.

**Lemma 2.3** [5] Let $C = u_1u_2 \ldots u_su_1$ be a cycle of $G$. Let $i, j \in \{1, 2, \ldots, s\}$ with $i \neq j$. Suppose that $d(u_i, C) + d(u_j, C) \geq s + 1$. Then for each $\varepsilon \in \{-1, 1\}$, $G$ has a path $P$ from $u_{i+\varepsilon}$ to $u_{j+\varepsilon}$ such that $V(P) = V(C)$, where the subscripts are reduced modulo $s$.
Lemma 2.4 [5] Let $s \geq 2$ be an integer. Suppose that $G$ is 2-connected and every vertex of $G$ with at most one exception has degree at least $s$. Then $G$ contains a cycle of length at least $\min(2s, n)$.

3 Proof of Theorem B

Let $k$ and $n$ be two integers with $k \geq 4$ and $n \geq 6$. Let $G = (V, E)$ be a graph of order $n$ with $\delta(G) > k$. Suppose, for a contradiction, that $G$ does not contain two vertex-disjoint cycles covering at least $\min(2k, n)$ vertices of $G$, i.e., $c_2(G) < \min(2k, n)$. By El-Zahar's result, $n > 2k$. Hence $c_2(G) < 2k$. Let $C_0$ be a smallest cycle of $G$; and subject to this, we choose $C_0$ such that the length of a longest cycle of $G - V(C_0)$ is maximal. Let $C_1$ be a longest cycle of $G - V(C_0)$. Subject to the choice of $C_0$ and $C_1$, we choose $C_0$ and $C_1$ such that the length of a longest path of $G - V(C_0 \cup C_1)$ is maximal. Set $H = G - V(C_0)$ and $D = H - V(C_1)$. Let $P_0$ be a longest path in $D$ and set $D_0 = G[V(P_0)]$. We say that a block of $H$ is an endblock if either the block contains exactly one cut-vertex of $H$ or the block is a component of $H$.

We claim that $C_0$ is a triangle. If this is not true, then $d(x, C_0) \leq 2$ for all $x \in V(H)$ for otherwise $G$ contains a smaller cycle than $C_0$. Hence $\delta(H) \geq k - 2$. Let $P = y_1 y_2 \ldots y_m$ be a longest path in $H$. Then $d(y_1, P) \geq k - 2$. As $H$ does not contain a triangle, there exists $y_i$ with $i \geq 2(k - 2)$ such that $y_1 y_i \in E$. Hence $c(H) \geq 2(k - 2)$ and therefore $c_2(G) \geq 2k$, a contradiction. Hence $C_0$ is a triangle. Then it is easy to see that $C_1$ exists.

Let $C_0 = u_1 u_2 u_3 u_1$. We divide our proof into the following two cases: $k = 4$ or $k \geq 5$.

Case 1. $k = 4$.

In this case, $c_2(G) \leq 7$. We break into the following two subcases according to whether $H$ is 2-connected.

Case 1.1. $H$ is 2-connected.

Clearly, $c(H) \geq 4$ as $|V(H)| = n - 3 > 4$. Thus $c_2(G) = 7$ and $C_1$ is of length 4. Let $C_1 = x_1 x_2 x_3 x_4 x_1$. As $H$ is 2-connected, for each $x \in V(D)$, there exist two paths from $x$ to two distinct vertices of $C_1$ such that $x$ is the only common vertex of the two paths. Then we see that for each $x \in V(D)$, either $N(x, C_1) = \{x_3, x_4\}$ or $N(x, C_1) = \{x_2, x_4\}$ for otherwise $c(H) \geq 5$. Furthermore, $D$ does not contain any edges. Let $x_0 \in V(D)$. Then $d(x_0, C_0) \geq 2$ and so $C_0 + x_0$ is hamiltonian. Consequently, $c_2(G) \geq 8$, a contradiction.

Case 1.2. $H$ is not 2-connected.

Let $H_1$ and $H_2$ be two endblocks. Moreover, we choose $H_1$ and $H_2$ such that if $H$ has a cut-vertex, then $H_1$ and $H_2$ are in the same component of $H$. For each $i \in \{1, 2\}$, let $x_i \in V(H_i)$ be such that if $H_i$ contains a cut-vertex of $H$ then it is $x_i$. We break into the following two situations.

Case 1.2(a). There exists $y_1 \in V(H_1 - x_1)$ such that $d(y_1, C_0) \geq 2$. Then $C_0 + y_1$ is hamiltonian. Hence $c(H_2) \leq 3$. This implies that $H_2 - x_2$ contains a vertex $z_1$ such
that $d(z_1, C_0) \geq 2$. Therefore $c(H_1) \leq 3$. It follows that $H_i \cong K_2$ or $K_3$ for each $i \in \{1, 2\}$.

First, suppose that either $H_1 \cong K_2$ or $H_2 \cong K_2$. Say w.l.o.g. that $H_1 \cong K_2$. Then $d(y_1, C_0) = 3$. Assume that $H$ has a third endblock $H_3$. Then we also have that $H_3 \cong K_2$ or $K_3$. Let $w_1 \in V(H_3)$ be such that $w_1$ is not a cut-vertex of $H$. Thus $d(w_1, C_0) \geq 2$ and $C_0 + y_1 + w_1$ is hamiltonian. Therefore any block of $H$ other than $H_1$ and $H_3$ is of order 2. In particular, $H_2 \cong K_2$. Similarly, we can readily show that $H_3 \cong K_2$. If $H_1$ and $H_2$ are not in the same component of $H$, then by the choice of $H_1$ and $H_2$, $H$ must consist of independent edges only, and we see that $c_2(G) \geq 9$ as $e(C_0, H_1 \cup H_2 \cup H_3) = 18$, a contradiction. Therefore $H_1$ and $H_2$ are in the same component of $H$. Notice that $d(w_1, C_0) = d(z_1, C_0) = 3$ where $H_2 = x_2z_1$. As $C_0 - u_1 + w_1$ is a triangle in $G$, it follows that $x_1 = x_2$ for otherwise $c(H - w_1 + u_1) \geq 5$. If $H_3$ is in a component $D'$ of $H$ which does not contain $H_1$, then we see that either $D' = H_3$ and so $G[V(H_3) \cup \{u_2, u_3\}] \cong K_4$, or $G[V(D' + u_2)]$ contains a cycle of length at least 4 by applying the above argument to $H_3$ and $H_4$ where $H_4$ is another endblock of $D'$. Thus $c_2(G) \geq 8$, a contradiction. This argument allows us to see that $H$ is connected and conclude that $H \cong K_{1,n-4}$ with $d(x_1, H) = n - 4$. It follows that $d(x, C_0) = 3$ for all $x \in V(H) - \{x_1\}$, and consequently, we readily see that $c_2(G) \geq 8$. Therefore $H$ does not have a third endblock. Then it is easy to see that $H$ is a path and $c_2(G) \geq 8$.

Therefore $H_1 \cong K_3$. Similarly, $H_2 \cong K_3$. Let $H_1 = x_1y_1y_2x_1$. Then we see that $C_0 + y_1 + y_2$ is hamiltonian and so $c_2(G) \geq 8$, a contradiction.

Case 1.2(b). For each $y \in V(H_1 - x_1)$, $d(y, C_0) \leq 1$.

Similarly, we must have that $d(z, C_0) \leq 1$ for all $z \in V(H_2 - x_2)$. Thus for each $i \in \{1, 2\}$, $d(v, H_i) \geq 3$ for all $v \in V(H_i - x_i)$. Clearly, $c(H_1) \geq 4$ and $c(H_2) \geq 4$. On the other hand, we must have $c(H) \leq 4$ and so $c(H_1) = c(H_2) = 4$. Thus $x_1 = x_2$. Let $P = v_1v_2\ldots v_m$ be a longest path of $H_1$ with $v_1 \neq x_1$. Then $N(v_1, H_1) = \{v_2, v_3, v_4\}$ and $d(v_1, C_0) = 1$. It is easy to see that $H_1 \cong K_4$ for otherwise we readily see that either $c(H_1) \geq 5$ or $H_1$ has a path longer than $P$. Similarly, $H_2 \cong K_4$. Clearly, $G[V(C_0 \cup H_1 - x_1)]$ contains a cycle of length at least 4. We obtain that $c_2(G) \geq 8$, a contradiction.

Case 2. $k \geq 5$.

Let $C_1 = x_1x_2\ldots x_{s-1}$. Then $s \leq 2k - 4$. We break into the following two cases: $s \geq 2k - 6$ or $s \leq 2k - 7$.

Case 2.1. $s \geq 2k - 6$.

Thus $s \in \{2k - 6, 2k - 5, 2k - 4\}$. Let $P_0 = y_1y_2\ldots y_r$. As $s = c(H)$, we clearly have

$$d(y, C_1) \leq \lfloor s/2 \rfloor \text{ for all } y \in V(D).$$  \hspace{1cm} (1)

We claim

$$r \geq 4.$$  \hspace{1cm} (2)

Proof of (2). On the contrary, suppose $r \leq 3$. First, assume $r = 1$. Then by (1), $d(y, C_0) \geq 2$ for all $y \in V(D)$. Thus $C_0 + y_1$ is hamiltonian and so $s \leq 2k - 5$. Then
by (1) again, \( d(y, C_0) \geq 3 \) for all \( y \in V(D) \). Clearly, adding any three vertices of \( D \) to \( C_0 \) will result in a hamiltonian subgraph of \( G \). Consequently, \( c_2(G) \geq 2k \), a contradiction.

Next, assume \( r = 2 \). If \( d(y_1, C_0) + d(y_2, C_0) \leq 2 \), then \( d(y_1, C_1) + d(y_2, C_1) \geq 2k - 4 \). By (1), we must have that \( d(y_1, C_1) = d(y_2, C_1) = k - 2 \). It is easy to see that \( C_1 + y_1 + y_2 \) contains a cycle of length \( s + 1 \) or \( s + 2 \), a contradiction. Hence \( d(y_1, C_0) + d(y_2, C_0) \geq 3 \). Thus \( C_0 + y_1 + y_2 \) contains a cycle of length at least \( 4 \), and so \( s \leq 2k - 5 \). If \( d(y_1, C_0) + d(y_2, C_0) = 3 \), then \( d(y_1, C_1) + d(y_2, C_1) \geq 2k - 5 \), and consequently, either \( d(y_1, C_1) \geq k - 2 \) or \( d(y_2, C_1) \geq k - 2 \), contradicting (1). So \( d(y_1, C_0) + d(y_2, C_0) \geq 4 \). Thus \( C_0 + y_1 + y_2 \) is hamiltonian, and so \( s = 2k - 6 \). If \( d(y_1, C_0) + d(y_2, C_0) = 4 \), then we have, by (1), that \( d(y_1, C_1) = d(y_2, C_1) = k - 3 \). Again, we readily see that \( C_1 + y_1 + y_2 \) contains a cycle longer than \( C \), a contradiction. Hence \( d(y_1, C_0) + d(y_2, C_0) \geq 5 \). Let \( y' \) be a third vertex of \( D \). Then \( d(y', D) \leq 1 \) as \( r = 2 \). Thus \( d(y', C_0) \geq 2 \) by (1), and consequently, \( C_0 + y_1 + y_2 + y' \) is hamiltonian. It follows that \( c_2(G) \geq 2k \).

Finally, we assume that \( r = 3 \). By Lemma 2.2, \( d(y_1, C_1) + d(y_3, C_1) \leq \lfloor s/2 \rfloor \). We must have that \( d(y_1, C_0) + d(y_3, C_0) \leq 3 \) for otherwise \( C_0 + y_1 + y_2 + y_3 \) is hamiltonian. This implies that \( d(y_1) + d(y_3) \leq \lfloor s/2 \rfloor + 3 + 4 \). Furthermore, if \( d(y_1, C_0) + d(y_3, C_0) = 3 \), then \( C_0 + y_1 + y_3 \) contains a cycle of length at least \( 4 \), and so we must have that \( s \leq 2k - 5 \). It follows that \( d(y_1) + d(y_3) < 2k \), a contradiction. So (2) holds.

By (2) and Lemma 2.2, we obtain

\[
d(y_1, C_0) + d(y_r, C_0) \leq 3 \text{ and } d(y_1, C_1) + d(y_r, C_1) \leq \lfloor s/2 \rfloor.
\] (3)

Note that if \( \max(d(y_1, C_0), d(y_r, C_0)) \geq 2 \), then \( C_0 + y_1 + y_r \) contains a cycle of length at least \( 4 \) and so \( s \leq 2k - 5 \). Together with (3), we obtain

\[
d(y_1, P_0) + d(y_r, P_0) \geq k.
\] (4)

By (4), we see that either \( d(y_1, P_0) \geq \lceil k/2 \rceil \) or \( d(y_r, P_0) \geq \lceil k/2 \rceil \), and so \( c(D_0) \geq \lceil k/2 \rceil + 1 \). As \( c_2(H) < 2k \), \( 4 \leq \lceil k/2 \rceil + 1 \leq 5 \). It follows

\[
k \in \{5, 6, 7, 8\} \text{ and } s \in \{2k - 6, 2k - 5\}.
\] (5)

We now break into the following two situations.

Case 2.1(a): \( s = 2k - 5 \).

Then \( c(G - V(C_1)) \leq 4 \). W.l.o.g., say \( d(y_1, P_0) \geq d(y_r, P_0) \). Then we must have that \( k \in \{5, 6\} \) and \( N(y_1, P_0) = \{y_2, y_3, y_4\} \). Then \( D_0 \) has a hamiltonian path from \( y_1 \) to \( y_r \) for each \( i \in \{1, 2, 3\} \). By Lemma 2.2, \( d(y_1, C_1) + d(y_3, C_1) \leq k - 3 \). First, suppose that \( d(y_r, C_0) \geq 1 \). Then we must have that \( d(y_1, C_0) = 0 \) for each \( i \in \{1, 2, 3\} \). Consequently, \( d(y_1, P_0) + d(y_3, P_0) \geq k + 3 \). It follows that \( c(D_0) \geq 5 \), a contradiction. Therefore, we must have that \( d(y_r, C_0) = 0 \). By (1), \( d(y_r, C_1) \leq k - 3 \) and so \( d(y_r, P_0) = 3 \), too. Similarly, we can readily show that \( d(y_1, C_0) = 0 \), \( d(y_1, P_0) + d(y_r, P_0) \geq k + 3 \) and \( c(D_0) \geq 5 \), a contradiction.
Case 2.1(b). \( s = 2k - 6 \).

Note that \( 4 \leq s \leq 10 \) by (5). First, suppose that \( d(y_1, C_0) \geq 1 \) and \( d(y_r, C_0) \geq 1 \). Then we must have that \( N(y_1, C_0) = N(y_r, C_0) = \{u_i\} \) for some \( i \in \{1, 2, 3\} \) and \( r = 4 \) for otherwise \( c(G[V(C_0 \cup P_0)]) \geq 6 \). If \( d(y_1, C_1) = 0 \), then \( d(y_1, P_0) \geq k - 1 \geq 4 \) and so \( r \geq 5 \), a contradiction. Hence \( d(y_1, C_1) \geq 1 \), and similarly, \( d(y_r, C_1) \geq 1 \). Then we see that \( c(H) \geq 5 \) and so \( k \geq 6 \) by the maximality of \( s \). It is easy to see that if either \( \max(d(y_1, C_1), d(y_r, C_1)) \geq 2 \) or \( N(y_1, C_1) \neq N(y_r, C_1) \), then \( k = 8 \) and \( \max(d(y_1, C_1), d(y_r, C_1)) \leq 2 \) for otherwise \( c(H) > s \). Hence \( d(y_1, C_1) = 1 \) for otherwise \( d(y_1, P_0) \geq 5 \) and so \( c(D_0) \geq 6 \), a contradiction. It follows that \( d(y_1, P_0) \geq k - 2 \geq 4 \) and so \( r \geq 5 \), a contradiction.

Therefore we may assume w.l.o.g. that \( d(y_r, C_0) = 0 \). Then \( d(y_r, C_1) \geq 1 \) for otherwise we readily see that \( D_0 \geq 6 \). We claim that \( d(y_1, C_1) = 0 \). If this is not true, then \( c(H) \geq 5 \) and so \( k \geq 6 \). As \( c(D_0) \leq 5 \), \( d(y_r, P_0) \leq 4 \) and so \( d(y_r, C_1) \geq 2 \). Then again, we must have that \( k = 8 \) and \( d(y_r, C_1) = 2 \) for otherwise \( c(H) > s \). Hence \( d(y_r, P_0) \geq 6 \) and so \( c(D_0) \geq 7 \), a contradiction. So \( d(y_1, C_1) = 0 \). Hence \( d(y_1, C_0) \geq 1 \) for otherwise \( c(D_0) \geq 6 \).

As \( k \geq 5 \) and \( d(y_1, C_1) = 0 \), \( d(y_1, P_0) \geq 2 \). Let \( j + 1 \) be the greatest integer in \( \{2, 3, \ldots, r\} \) such that \( y_iy_{i+1} \in E \). Then \( D_0 \) has a Hamiltonian path from \( y_j \) to \( y_r \). Similarly, we must have that \( d(y_j, C_1) = 0 \) and \( d(y_j, C_0) \geq 1 \). As \( y_iy_{i+1}y_j \) is a path of \( G \), we see that \( d(y_1, C_0) = d(y_j, C_0) = 1 \) for otherwise \( c(G[V(C_0 \cup D_0)]) \geq 6 \). This yields that \( d(y_1, P_0) + d(y_j, P_0) \geq 2k - 2 \), and consequently, \( c(D_0) \geq k \). It follows that \( k = 5 \). But then \( s = 4 \), contradicting the maximality of \( s \).

Case 2.2. \( s \leq 2k - 7 \).

Clearly, we have that \( \delta(H) \geq k - 3 \). If \( H \) is 2-connected, then \( c(H) \geq 2k - 6 \) by Lemma 2.4, a contradiction. Hence \( H \) is not 2-connected. Let \( H_1 \) and \( H_2 \) be two arbitrary endblocks of \( H \). Set \( n_1 = |V(H_1)| \) and \( n_2 = |V(H_2)| \). As \( \delta(H) \geq k - 3 \) and by Lemma 2.4, we must have

\[
k - 2 \leq n_1 \leq 2k - 7 \quad \text{and} \quad k - 2 \leq n_2 \leq 2k - 7.
\]

By Lemma 2.4, both \( H_1 \) and \( H_2 \) are Hamiltonian. Let \( Q_1 = z_1z_2 \ldots z_{n_1}z_1 \) and \( Q_2 = y_1y_2 \ldots y_{n_2}y_1 \) be two Hamiltonian cycles of \( H_1 \) and \( H_2 \), respectively such that every \( v \in V(H_1 \cup H_2) - \{z_1, y_1\} \) is not a cut-vertex of \( H \).

First, suppose that for each \( i \in \{1, 2\} \), \( G \) does not have two independent edges between \( C_0 \) and \( H_i \). As \( \delta(G) \geq k \), this implies that \( n_1 \geq k \) and \( n_2 \geq k \). Therefore we must have that \( z_1 = y_1 \) for otherwise \( c_2(H) \geq 2k \). As \( 2k - 7 \geq n_1 \geq k \), \( k \geq 7 \). As \( \delta(H) \geq k - 3 \), we have that \( \delta(H_i - z_i) \geq k - 4 \geq (n_i - 1)/2 \) for each \( i \in \{1, 2\} \). Therefore both \( H_1 - z_1 \) and \( H_2 - z_1 \) are Hamiltonian. Hence we must have that \( n_1 = n_2 = k \) for otherwise \( c_2(H) \geq 2k \). Therefore \( d(z_i, C_0) \geq 1 \) for all \( i \in \{2, 3, \ldots, k\} \). As there exist no two independent edges between \( C_0 \) and \( H_1 \), we obtain that \( d(z_i, C_0) = 1 \) and \( d(z_i, H_1) = k - 1 \) for all \( i \in \{2, 3, \ldots, k\} \). Consequently, \( H_1 \cong K_2 \), and we readily see that \( c(G[V(C_0 \cup H_1 - z_1)]) \geq k \), and so \( c_2(G) \geq 2k \), a contradiction.

Therefore we may assume w.l.o.g. that there exist two independent edges between \( C_0 \) and \( H_1 \). Say \( \{u_1z_i, u_2z_j\} \subseteq E \) for some \( 1 \leq i < j \leq n_1 \). If \( \{z_i, z_j\} = \{z_2, z_{n_1}\} \),

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then \( c(G[V(C_0 \cup H_1 - z_1)]) \geq k \). Then \( n_2 \leq k - 1 \) for otherwise \( c_2(G) \geq 2k \). Hence \( d(y_i, C_0) \geq 2 \) for all \( i \in \{2, 3, \ldots, n_2\} \). As \( \delta(H) \geq k - 3 \) and \( n_2 \leq 2k - 7 \), it is easy to prove that \( H_2 \) contains a triangle. Therefore \( 2k - 7 \geq k \) by the maximality of \( s \), and so \( k \geq 7 \). It follows that there are two independent edges between \( C_0 \) and \( H_2 \) which are not incident with any of \( y_2 \) and \( y_{n_2} \). Therefore by abusing notation, we may assume in the first place that \( \{z_i, z_j\} \neq \{z_2, z_{n_1}\} \). Then either \( z_1 \notin \{z_i, z_{i-1}\} \) or \( z_1 \notin \{z_{i+1}, z_{j+1}\} \) where the subscripts are taken modulo \( n_1 \). We show \( k \geq 7 \) as follows. As \( \delta(H) \geq k - 3 \), \( n_1 \leq 2k - 7 \) and by Lemma 2.3, \( H_1 \) has a hamiltonian path from \( z_i \) to \( z_j \) and so \( c(G[V(C_0 \cup H_1)]) \geq k + 1 \). As before, we readily see that if \( H_2 - y_1 \) contains a triangle, then \( k \geq 8 \). If \( H_2 - y_1 \) does not contain a triangle, then we must have that \( d(y_2, C_0) = d(y_3, C_0) = 3 \) and therefore \( u_3y_2y_3u_3 \) is a triangle. Clearly, \( c(H_1 + u_1 + u_2) \geq k \). Then we obtain \( k \geq 7 \) as \( 2k - 7 \geq k \) by the maximality of \( s \).

Suppose \( z_1 \neq y_1 \). Then we must have \( n_2 = k - 2 \) by (6) for otherwise \( c_2(G) \geq 2k \). Consequently, we see that \( H_2 \cong K_{k-2} \) and \( d(y_i, C_0) = 3 \) for each \( i \in \{2, 3, \ldots, k-2\} \). Similarly, we must have that \( H_1 \cong K_{k-2} \) and \( d(z_i, C_0) = 3 \) for all \( i \in \{2, 3, \ldots, k-2\} \). Then we see that \( H \) does not have a path of length at least 2 from \( z_1 \) to \( y_1 \) for otherwise \( c_2(G) \geq 2k \). Thus \( H \) must have a third endblock \( H_3 \). Then we may assume that \( H_1 \cap H_3 = \emptyset \) and repeat the above argument with \( H_3 \) replacing the role of \( H_2 \). Clearly, we see that \( c_2(G) \geq 3(k-2) + 2 > 2k \), a contradiction.

Therefore \( z_1 = y_1 \). As \( n > 2k \), \( H \) has a third endblock \( H_3 \), too. Set \( n_3 = |V(H_3)| \). Similarly, we can show that \( z_1 \in V(H_3), k-2 \leq n_3 \leq k-1 \) and \( H_3 - z_1 \) is hamiltonian. Clearly, \( d(y_2, C_2) \geq 2 \). As before, using Lemma 2.3, we see that \( H_1 \) has a hamiltonian path from \( z_1 \) to each \( z \in V(H_1) - \{z_1\} \). In particular, \( H_1 \) has a hamiltonian path from \( z_1 \) to a vertex \( z' \in \{z_i, z_j\} \). Then we see that \( G[V(C_0 \cup H_1 \cup H_2)] \) is hamiltonian. Hence \( c_2(G) \geq 3k - 5 > 2k \), a contradiction. This proves the theorem.

4 References


[10] H. Wang, On quadrilaterals in a graph, manuscript.

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