1. Introduction.

Let $\mathcal{F}$ denote a simply generated family of rooted trees whose generating function $y = \sum_{n=1}^{\infty} y_n x^n$ satisfies a relation of the form $y = x \Phi(y)$ where $\Phi(t) = 1 + \sum_{m=1}^{\infty} c_m t^m$. If some mild conditions are satisfied, then $y_n \sim c(\Phi(\tau)/\tau)^n \cdot n^{-3/2}$ where $\tau \Phi'(\tau) = \Phi(\tau)$. We say that a node $v$ in a rooted tree $T_n$ has out-degree $k$ if $v$ is incident with $k$ edges that lead away from the root of $T_n$. Our object here is to investigate the behaviour of the maximum out-degree $\Delta = \Delta(T_n)$ of trees $T_n$ in $\mathcal{F}$. After describing briefly in §2 the simply generated families of trees we shall be considering, we obtain bounds in §3 for $Pr\{\Delta < k\}$ in terms of the function $r_k(\tau) = \sum_{m=1}^{\infty} c_m \tau^m$. Then in §4 we obtain certain inequalities involving the functions $r_k(\tau)$, assuming henceforth that the coefficients are reasonably well-behaved. Our main result is in §5 where we show that if $D(n) = \max\{k : nr_k(\tau) \geq 1\}$ then $Pr\{(1 - \epsilon)D(n) < \Delta(T_n) < (1 + \epsilon)D(n)\} \rightarrow 1$ as $n \rightarrow \infty$. We consider the problem of estimating $D(n)$ in §6; we find that $D(n) \sim \log n / \log(R/\tau)$ if $\Phi(t)$ has a finite radius of convergence $R$ while $D(n) = o(\log n)$ if $\Phi(t)$ is an entire function. Finally, in §7 we consider some particular families of trees. For example, if $\Phi(t) = (1 - t)^{-1}$ and $\mathcal{F}$ is the family of plane trees then $D(n) = 1 + \lceil \log_2 n \rceil$; and if $\Phi(t) = e^t$ and $\mathcal{F}$ is the family of rooted labelled trees then $D(n) \sim \log n / \log \log n$. (We remark that the problem of determining the behaviour of $\Delta(T_n)$ for this last family was considered earlier in [6].)
2. Simply generated families.

We recall that plane trees - or ordered trees, as they are called by some authors [2; p. 306] - are rooted trees with an ordering specified for the branches incident with each node. To each such tree $T_n$ we assign a non-negative weight $\omega(T_n)$ satisfying the following condition: there exists a sequence of non-negative constants $c_0(=1), c_1, c_2, \ldots$ such that

$$\omega(T_n) = \prod_{0}^{\infty} c_m d_m(T_n)$$

for every plane tree $T_n$, where $d_m(T_n)$ denotes the number of nodes of out-degree $m$ in $T_n$. The collection of plane trees with such an assignment of weights will be called a simply generated family, henceforth denoted by $\mathcal{F}$. Let $y_n$ denote the number of trees $T_n$ in the family $\mathcal{F}$ where the weights are taken into account (both here and elsewhere); that is

$$y_n = \sum \omega(T_n)$$

where the sum is over all plane trees with $n$ nodes. It is not difficult to see (cf. [3; p. 999] or [9; p. 24]) that if $\mathcal{F}$ is a simply generated family then its generating function $y = \sum_{1}^{\infty} y_n x^n$ satisfies the relation

$$y = x\Phi(y)$$

where $\Phi(t) = 1 + \sum_{1}^{\infty} c_m t^m$.

We shall assume henceforth that $\mathcal{F}$ is some given simply generated family such that the function $\Phi(t)$ appearing in (2.3) is regular when $|t| < R \leq \infty$. We further assume that

$$c_m \geq 0 \quad \text{for} \quad m \geq 1,$$

(2.5) \qquad \gcd\{m : c_m > 0\} = 1, \quad \text{and}$$

(2.6) \quad \tau \Phi'(\tau) = \Phi(\tau) \quad \text{for some } \tau, \quad \text{where} \quad 0 < \tau < R.$$

It follows from these assumptions (see [8; p. 216], [3; p. 999], or [9; p. 32]) that $y(x)$ is regular when $|x| \leq \rho$, $x \neq \rho$, where $\rho = \tau / \Phi(\tau)$; moreover, $y(\rho) = \tau$ and

$$y_n \sim c(\Phi(\tau)/\tau)^n \cdot n^{-3/2}$$

as $n \to \infty$, where $c = (\Phi(\tau)/2\pi \Phi''(\tau))^{1/2}$. 

148
3. Bounds for $Pr\{\Delta(T_n) < k\}$.

Let $r_k(t) = \sum_{k=0}^{\infty} c_m t^m$ for $k = 0, 1, \ldots$. We now establish bounds for $Pr\{\Delta(T_n) < k\}$, where $\Delta(T_n)$ denotes the maximum out-degree of nodes in the tree $T_n$, assuming that any given tree $T_n$ is selected from the trees in $\mathcal{F}$ with $n$ nodes with probability $\omega(T_n)/y_n$.

**Lemma 1.** Let $A = 2r/c$ and $B = 1/\Phi(\tau)$. Then

\begin{equation}
Pr\{\Delta(T_n) < k\} < An^{1/2} \cdot e^{-Bnr_k(\tau)}
\end{equation}

for $k = 1, 2, \ldots$ and all sufficiently large values of $n$.

**Proof:** Let $\bar{y}_k = \bar{y}_k(x) = \sum \bar{y}_{kn} x^n$ where $\bar{y}_{kn}$ denotes the number of trees $T_n$ in $\mathcal{F}$ such that $\Delta(T_n) < k$. Then it is not difficult to see that $\bar{y}_k$ satisfies the relation $\bar{y}_k = x\Phi_k(\bar{y}_k)$ where $\Phi_k(t) = \Phi(t) - r_k(t)$. Thus it follows from Lagrange's inversion formula (see, e.g., [1; 148]) that

$$
\bar{y}_{kn} = n^{-1} \cdot C_{n-1} \left\{ \left( \Phi_k(t) \right)^n \right\}
$$

where $C_m \{f(t)\}$ denotes the coefficient of $t^m$ in the power series expansion of $f(t)$. Therefore,

$$
\bar{y}_{kn} \tau^{n-1} \leq n^{-1} \cdot (\Phi_k(\tau))^n
$$

$$
= n^{-1} \cdot \left( \Phi(\tau) - r_k(\tau) \right)^n
$$

$$
\leq n^{-1} \Phi^n(\tau) \cdot e^{-Bnr_k(\tau)}.
$$

This implies inequality (3.1) since $Pr\{\Delta(T_n) < k\} = \bar{y}_{kn}/y_n$ and $y_n \sim c(\Phi(\tau)/\tau)^n \cdot n^{-3/2}$.

Notice that if $\Phi(t)$ is a polynomial of exact degree $h$, then it follows from Lemma 1 that $Pr\{\Delta(T_n) = h\} \to 1$ as $n \to \infty$. So we will assume henceforth that $c_m > 0$ for infinitely many $m$, i.e., that $r_k(\tau) > 0$ for all $k$.

We shall need two known results in proving the next inequality. Firstly, let $e_k(n)$ denote the expected number of nodes of out-degree at least $k$ in a tree $T_n$ in $\mathcal{F}$, where the expectation is taken over all trees $T_n$ in $\mathcal{F}$; then it follows from [4; Theorem 4] (see also [9; p. 26]) that

\begin{equation}
\sum_{1}^{\infty} e_k(n) y_n x^n = x^2 r_k(y(x)) y'(x)/y(x)
\end{equation}
for \( k = 0, 1, \ldots \). Secondly, it was shown in [5] (see also [7; p. 305]) that for each family \( \mathcal{F} \) there exists a positive constant \( Q \) such that if \( y = \tau e^{i\theta} \), where \( |\theta| \leq \pi \), then

\[
|\Phi(y)| \leq \Phi(\tau)e^{-Q\theta^2}.
\]

\[\text{Lemma 2.} \quad \text{Let } K = \rho(c^2 \pi Q)^{-1/2}. \text{ Then}\]

\[
Pr\{\Delta(T_n) \geq k\} \leq Knr_k(\tau)
\]

for \( k = 0, 1, \ldots \) and all sufficiently large values of \( n \).

\[\text{Proof:} \quad \text{We first observe that } Pr\{\Delta(T_n) \geq k\} \leq e_k(n), \text{ by Boole's inequality. Hence it follows from (3.2) and Cauchy's theorem that}\]

\[
Pr\{\Delta(T_n) \geq k\}y_n \leq (2\pi)^{-1}\int r_k(y(x))x^{1-n}(y'(x)/y(x))dx
\]

where the integration is along a small circle around \( x = 0 \). Since \( y'(0) = 1 \) we may change variables by letting \( x = y/\Phi(y) \); consequently

\[
Pr\{\Delta(T_n) \geq k\}y_n \leq (2\pi)^{-1}\int r_k(y)\Phi^{-1}(y) \cdot y^{-n}dy
\]

where now the integration can be taken around the circle \( |y| = \tau \). But if \( y = \tau e^{i\theta} \), where \( |\theta| \leq \pi \), then \( |r_k(y)| \leq r_k(\tau) \). Therefore, taking absolute values and applying (3.3), we find that

\[
Pr\{\Delta(T_n) \geq k\}y_n \leq (2\pi)^{-1}r_k(\tau) \cdot (\Phi(\tau)/\tau)^{n-1}\int_{-\pi}^{\pi} e^{-(n-1)Q\theta^2}d\theta
\]

\[
\leq (4\pi(n - 1)Q)^{-1/2} \cdot (\Phi(\tau)/\tau)^{n-1} \cdot r_k(\tau).
\]

This implies inequality (3.4) since \( y_n \sim c(\Phi(\tau)/\tau)^n \cdot n^{-3/2} \).
4. Some inequalities for $r_k(\tau)$.

In the last section we obtained bounds for $Pr\{\Delta(T_n) < k\}$ in terms of the function $r_k(\tau) = \sum_{m=1}^{\infty} c_m \tau^m$. We need some information about the behaviour of $r_k = r_k(\tau)$ as $k \to \infty$ in order to exploit these bounds. In the following two lemmas we consider separately the cases when $R$, the radius of convergence of the function $\Phi(t)$, is finite or is infinite.

**Lemma 3.** Suppose that $R < \infty$ and that

$$c_m^{1/m} \to R^{-1}$$

as $m \to \infty$. Then for every $\delta > 0$ there exists a number $J$ such that if $j \geq J$ and $\ell \geq (1 + 2\delta)j$, then

$$r_{\ell} < r_j^{1+\delta}.$$

**Proof:** Let $\epsilon$ denote a positive constant such that

$$\epsilon < \frac{1}{2}(R - \tau)$$

and

$$\{((R + \epsilon)/(R - \epsilon))^{1+\delta} \cdot (\tau/(R - \epsilon))^\delta < 1.$$  

It follows from assumption (4.1) that there exists an integer $N$ such that if $m \geq N$, then

$$(R + \epsilon)^{-m} < c_m < (R - \epsilon)^{-m}.$$  

Consequently, if $k \geq N$ then

$$\tau/(R + \epsilon))^k < r_k \leq \sum_{k=1}^{\infty} (\tau/(R - \epsilon))^m < U \cdot (\tau/(R - \epsilon))^k$$

where $U = 2R/(R - \tau)$, using (4.3) at the last step.
Now let $\ell$ and $j$ be such that $\ell \geq (1 + 2\delta)j$. Then, appealing to (4.5) twice, we find that

$$r_\ell < U \cdot (\tau/(R - \epsilon))^\ell$$

$$< U \cdot \left( \left( \frac{R + \epsilon}{R - \epsilon} \right)^{(1+\delta)} \cdot \left( \frac{\tau}{R - \epsilon} \right)^{\ell - (1+\delta)j} \cdot \left( \frac{\tau}{R + \epsilon} \right)^{(1+\delta)} \right)^j \cdot r_j^{1+\delta}$$

if $j \geq N$. Condition (4.4) implies that the coefficient of $r_j^{1+\delta}$ in this last expression is less than one when $j \geq M$ for sufficiently large $M$. Hence $r_\ell < r_j^{1+\delta}$ when $j \geq J = \max\{M, N\}$, as required.

**Lemma 4.** Suppose that $R = \infty$ so that

$$(4.6) \quad c_m^{1/m} \to 0$$

as $m \to \infty$. In addition, suppose that

$$(4.7) \quad c_{k+1}^{1/(k+1)} \leq c_k^{1/k}$$

when $k \geq N$. Then for every $\delta > 0$ there exists a number $L$ such that if $j \geq L$ and $\ell \geq (1 + 2\delta)j$, then

$$(4.8) \quad r_\ell < r_j^{1+\delta}.$$}

**Proof:** Choose $L$ so that $L \geq N$, $c_j^{1/j} \tau < 1/2$, and $(c_j^{1/j} \tau)^\delta < 1/2$ for $j \geq L$. Then it follows from our assumptions that

$$c_{\ell+\nu} \tau^{\ell+\nu} \leq (c_{\ell} \tau^{\ell})^{(\ell+\nu)/\ell} \cdot (c_{\ell}^{1/2} \tau)^\nu \leq c_{\ell} \tau^{\ell} (1/2)^\nu$$

for $\nu = 0, 1, \ldots$. Hence $r_\ell \leq 2c_\ell \tau^\ell$.

On the other hand, it also follows from our assumptions that

$$c_\ell \tau^\ell \leq (c_j^{1/j} \tau)^\ell = (c_j^{1/j} \tau)^{\ell - \delta j} \cdot (c_j^{1/j} \tau)^{\delta j}$$

$$< (c_j^{1/j} \tau)^{(1+\delta)j} \cdot (1/2)^j = (c_j \tau^j)^{1+\delta} \cdot (1/2)^j < r_j^{1+\delta} \cdot (1/2)^j.$$
Hence, $r_\ell \leq 2c_\ell r^\ell < r_j^{1+\delta} \cdot (1/2)^{j-1} \leq r_j^{1+\delta}$, as required.

We remark that it can be shown that conclusion (4.8) also holds if condition (4.7) is replaced by either of the following conditions: there exist positive numbers $N$ and $H$ such that $\max\{c_h^{1/k} : k \leq h \leq 2k\} \leq Hc_k^{1/k}$ for all $k \geq N$ or $c_m > 0$ for all $m$ and $c_{k+1}/c_k$ decreases to 0 as $k \to \infty$.

5. Main result.

Let $D(n) = \max\{k : nr_k(\tau) \geq 1\}$. We now show that the distribution of $\Delta(T_n)$ is concentrated around $D(n)$ if the coefficients $c_m$ are reasonably well-behaved.

**Theorem 1.** Suppose the coefficients $c_m$ satisfy the conditions of Lemma 3 or of Lemma 4. Then for every $\epsilon > 0$

$$\Pr\{(1-\epsilon)D(n) < \Delta(T_n) < (1+\epsilon)D(n)\} \to 1$$

as $n \to \infty$.

**Proof:** We first show that

$$Pr\{\Delta(T_n) \geq (1+\epsilon)D(n)\} \to 0$$

as $n \to \infty$. Let $h = D(n)$, $\ell = [(1+\epsilon)h]$, and $\delta = \epsilon/3$. Since we are assuming that $\varphi(t)$ is not a polynomial, it follows that $h \to \infty$ as $n \to \infty$. Thus we may suppose that $n$ is large enough to ensure that $(1+2\delta)(h+1) < \ell$ and that, by Lemma 3 or Lemma 4, $r_\ell < (r_{h+1})^{1+\delta}$. But $r_{h+1} < n^{-1}$, by the definition of $h$, so $nr_\ell < n \cdot n^{-1-\delta} = n^{-\delta}$. Hence it follows from Lemma 2 that

$$Pr\{\Delta(T_n) \geq (1+\epsilon)h\} \leq Pr\{\Delta(T_n) \geq \ell\}$$

$$\leq Knr_\ell < Kn^{-\delta},$$

and this implies (5.1).

We now show that

$$Pr\{\Delta(T_n) \leq (1-\epsilon)D(n)\} \to 0$$
as \( n \to \infty \). As before, we let \( h = D(n) \) and \( \delta = \epsilon/3 \) but this time we let \( j = [(1 - \epsilon)h] + 1 \). We may suppose that \( n \) is large enough to ensure that \((1 + 2\delta)j \leq h\) and that, by Lemma 3 or Lemma 4, \( r_h < r_j^{1+\delta} \). But \( r_h \geq n^{-1} \), so
\[
nr_j > nr_h^{1/(1+\delta)} \geq n \cdot n^{-1/(1+\delta)} = n^{\epsilon/(1+\delta)}.
\]

Hence it follows from Lemma 1 that
\[
Pr\{\Delta(T_n) \leq (1 - \epsilon)h\} = Pr\{\Delta(T_n) < j\} < An^{1/2} \cdot e^{-Bnr_j} < An^{1/2} \cdot e^{-Bn^{\epsilon/(1+\delta)}},
\]
and this implies (5.2) and completes the proof of the theorem.

6. The behaviour of \( D(n) \).

We now consider the problem of estimating the function
\( D(n) = \max\{k : nr_k(\tau) \geq 1\} \). It turns out that the behaviour of \( D(n) \) depends on whether \( R \), the radius of convergence of \( \varphi(t) \), is finite or infinite.

**Theorem 2.** Suppose that \( R < \infty \) and that
\[
c_m^{1/m} \to R^{-1}
\]
as \( m \to \infty \). Then
\[
D(n) \sim \log n / \log(R/\tau)
\]
as \( n \to \infty \).

**Proof:** We saw in the proof of Lemma 3 that if \( \epsilon \) is any sufficiently small positive constant, then there exists an integer \( N \) such that if \( k \geq N \) then
\[
(\tau/(R + \epsilon))^k < r_k < U \cdot (\tau/(R - \epsilon))^k
\]
where \( U = 2R/(R - \tau) \). Now let \( h = D(n) \); we may suppose that \( n \) is large enough to ensure that \( h \geq N \). Then it follows from (6.1) and the definition of \( h \) that
\[
U^{-1} \cdot ((R - \epsilon)/\tau)^h < r_h^{-1} \leq n < r_{h+1}^{-1} < ((R + \epsilon)/\tau)^{h+1}.
\]

Consequently
\[
h \log((R - \epsilon)/\tau) - \log U < \log n < (h + 1) \log((R + \epsilon)/\tau),
\]

154
which implies the required result.

**Theorem 3.** Suppose that \( R = \infty \) so that

\[
c_m^{1/m} \to 0
\]

as \( m \to \infty \). Then

\[
D(n) = o(\log n)
\]

as \( n \to \infty \).

**Proof:** For any small positive \( \epsilon \), let \( \delta = \tau^{-1} \cdot \epsilon^{-1/\epsilon} \); we may suppose that \( \epsilon < 1/\log 2 \) so that \( \delta \tau < 1/2 \). There exists an integer \( N \) such that if \( m \geq N \) then \( c_m < \delta^m \); hence if \( k \geq N \), then

\[
r_k < \sum_k^{\infty} (\delta \tau)^m < 2(\delta \tau)^k.
\]

Now let \( h = D(n) \); we may suppose that \( n \) is large enough to ensure that \( h \geq N \). Then, since \( r_h \geq n^{-1} \), it follows that

\[
n \geq r_h^{-1} > \frac{1}{2}(\delta \tau)^{-h} = \frac{1}{2} \cdot \epsilon^{h/\epsilon}.
\]

Consequently, \( h < \epsilon \log(2n) \) and this implies the required result.

7. Special cases.

The plane trees and the rooted labelled trees illustrate the contrast in the behaviour of \( D(n) \) when \( R < \infty \) and when \( R = \infty \). For the plane trees \( \Phi(t) = (1 - t)^{-1} \), so \( R = 1 \), \( \tau = 1/2 \), and \( r_k(\tau) = (1/2)^{k-1} \); consequently, \( D(n) = 1 + [\log_2 n] \) in accordance with Theorem 2. For the rooted labelled trees \( \Phi(t) = e^t \) so \( R = \infty \) and \( \tau = 1 \); it is not difficult to see that for this case \( 1/k! < r_k(\tau) < (1 + k^{-1})/k! \) from which it follows that \( D(n) \sim \log n / \log \log n \) (see also [6]).

In general, when \( R = \infty \) the behaviour of \( D(n) \) depends very much on the rate at which the coefficients \( c_m \) of \( \Phi(t) \) approach zero. For example, let \( g(x) \) denote an increasing function of \( x \) such that \( g(x) \to \infty \) and \( g(x + 1) - g(x) \to \infty \) as \( x \to \infty \). Consider the function \( \Phi(t) = 1 + \sum_{m=1}^{\infty} c_m t^m \) where \( c_m = e^{-g(m)} \) for \( m \geq 3 \), \( c_2 = 1 - \sum_{m=3}^{\infty} (m-1)c_m \), and \( c_1 = 0 \). Then \( R = \infty \), \( \tau = 1 \), and \( r_k(\tau) \sim c_k \sim \)
$e^{-g^{(k)}}$ as $k \to \infty$; consequently, $D(n) \sim g^{-1}(\log n)$ for this family, where $g^{-1}$ denotes the inverse of the function $g$. In particular, if $g(x) = e^x$ then $D(n) \sim \log \log n$, if $g(x) = e^{e^x}$ then $D(n) \sim \log \log \log n$, and so on. And if, for example, $g(x) = x \log x$ then $D(n) \sim \log n / \log \log n$, and if $g(x) = x \log \log x$, then $D(n) \sim \log n / \log \log \log n$, and so on. Consequently, $D(n)$ can approach infinity arbitrarily slowly and Theorem 3 is, in a sense, best possible.

We remark in closing that it is not difficult to show that the expected value of $\Delta(T_n)$ is asymptotically equal to $D(n)$ as $n \to \infty$, assuming that $\Phi(t)$ satisfies the hypothesis of Lemma 3 or of Lemma 4.

The preparation of this paper was assisted by grants from the Natural Sciences and Engineering Research Council of Canada.

References


