Scenic Graphs II: Non-Traceable Graphs

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Abstract

A path of a graph is maximal if it is not a proper subpath of any other path of the graph. A graph is scenic if every maximal path of the graph is a maximum length path. In [4] we give a new proof of C. Thomassen’s result characterizing all scenic graphs with Hamiltonian path. Using similar methods here we determine all scenic graphs with no Hamiltonian path.

1 Introduction

We employ the following notation some of which is non-standard. A path in a graph is a sequence of distinct vertices in which consecutive vertices are adjacent. The length of a path is the number of edges in the path. Thus a path $Q = (x_0, x_1, \ldots, x_k)$ has length $k$. All graphs we consider here are undirected. Therefore, although sequences have an orientation or direction, here we shall not distinguish between the sequences $(x_0, x_1, \ldots, x_k)$ and $(x_k, x_{k-1}, \ldots, x_0)$ as paths. For the path $Q = (x_0, x_1, \ldots, x_k)$ we will also use the notation $(x_0, Q, x_k)$, and $(x_i, Q, x_j)$ is the corresponding subpath. If $(x, P, y)$ and $(u, Q, v)$ are disjoint paths with $y$ and $u$ adjacent, then their concatenation is a path we denote by either $((x, P, y), (u, Q, v))$, or $(x, \ldots, y, (u, Q, v))$, or $((x, P, y), u, \ldots, v)$, or $(x, \ldots, y, u, \ldots, v)$. A similar natural extension of this notation is used for concatenations of concatenated paths. A path $P$ is a subpath of $Q$ if the sequence corresponding to $P$ appears as a consecutive subsequence of $Q$. A subpath $P$ of a path $Q$ is proper if $P \neq Q$. If $P$ is a proper subpath of $Q$ then $P$ is proper if $P \neq Q$. If $P$ is a proper

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subpath of $Q$, then we shall say that $P$ extends to $Q$, or $Q$ extends $P$, or $Q$ is an extension of $P$. A path is maximal if it is not a proper subpath of any other path, or equivalently, if it has no extension. The path spectrum of a connected graph $G$ is the set of lengths of all maximal paths in $G$. The concept of path spectrum was introduced by Jacobson et al. [3]. We say that a connected graph is scenic if its path spectrum is a singleton. A graph with a Hamiltonian path is called traceable.

The Prism is the graph $K_6 - C_6$ obtained from $K_6$ by removing the edges of a six-cycle. The Cube is the graph $K_{4,4} - 4K_2$ obtained by removing four disjoint edges from the complete $4 \times 4$ bipartite graph. Except for paths $P_n$ ($n \geq 1$), cycles $C_n$ ($n \geq 3$), the Prism, and the Cube, scenic graphs emerge from cliques, $K_n$ ($n \geq 1$), and from the complete bipartite graphs $K_{p,p}$ and $K_{p,p+1}$ ($p \geq 1$). Traceable scenic graphs were determined by C. Thomassen [9] and a different proof can be found in [4]. To present the family we need some notation. The union of $t$ mutually disjoint edges (a matching) will be denoted by $tK_2$. The graph obtained from $K_n$ by removing the edges of a copy of $tK_2$ ($1 \leq t \leq n/2$) is denoted by $K_n - tK_2$. The complete $p \times p$ bipartite graph plus (resp. minus) an edge is denoted $K_{p,p} + K_2$ (resp. $K_{p,p} - K_2$). The graph obtained from the complete $p \times p$ bipartite graph by adding one edge into each partite set is denoted $K_{p,p} + 2K_2$. If $H \in \{K_3, 2K_2, K_{1,q}\}$, then $K_{p,p+1} + H$ denotes the graph obtained from the complete $p \times (p+1)$ bipartite graph by adding all the edges of $H$ to the largest partite set containing $p+1$ vertices. In [4] we give a new proof of the following theorem of C. Thomassen [9]:

**Theorem 1.1** A traceable graph is scenic if and only if it belongs to one of the following families:

\[
\begin{align*}
\Phi[K_n] & = \{K_n, \ K_n - tK_2 \ (1 \leq t \leq n/2)\}, \\
\Phi[K_{p,p}] & = \{K_{p,p}, K_{p,p} - K_2, K_{p,p} + K_2, K_{p,p} + 2K_2\}, \\
\Phi[K_{p,p+1}] & = \{K_{p,p+1}, K_{p,p+1} + K_3, K_{p,p+1} + 2K_2, K_{p,p+1} + K_{1,q} \ (1 \leq q \leq p)\}, \\
\Psi & = \{P_n, C_n, Prism, Cube\}.
\end{align*}
\]

In this paper we determine all non–traceable scenic graphs\(^1\). In Section 2 we prove that every non–traceable scenic graph is bipartite. Let $K^s_{1,r}$ ($r \geq 3$) be the equi-subdivided star obtained from a $K_{1,r}$ by subdividing each edge with $s \geq 0$ vertices. For $p \geq 2$ and $q \geq p + 2$, we call $K_{p,q} - F$ a $p \times q$ generic graph if it is obtained from $K_{p,q}$ by removing an arbitrary star forest $F$ with its star components centered in the $q$–element (i.e. largest) partite set of $K_{p,q}$. Note that a disconnected generic graph has the form $K_{p,q} - K_{p,1}$ (or equivalently $K_{p,q-1} + y$, where $y$ is an isolated vertex in the larger partite set). We show that besides a few exceptions, every non–traceable scenic graph is either an equi-subdivided star or a connected generic graph. The main result is formulated in the following theorem.

\(^1\)The same problem has been considered independently by M. Tarsi [8] (personal communication by editors of JGT and JCT B).
Theorem 1.2 A non–traceable graph is scenic if and only if it is one of the graphs $G_1, \ldots, G_6$ in Fig. 1, an equi-subdivided star, or a connected generic graph.

![Graphs G1 to G6](image)

Figure 1:

It is a routine to check that the six small graphs in Fig. 1 and the equi-subdivided stars are non–traceable and scenic. To prove the same for a connected $p \times q$ generic graph one may easily show that every maximal path covers the $p$–element partite set of $G$ and both of its endvertices must be in the $q$–element partite set. Therefore, all maximal paths in a connected generic graph have the same length, namely $2p \leq p + q - 2$. Hence connected generic graphs are scenic and non–traceable.

The next sections contain the proof of the ‘only if’ part of Theorem 1.2. The basic idea in the proof is the reduction of a non–traceable scenic graph $G$ by removing a copy of $K_{2,2}$ from $G$ together with all adjacent edges. To some extent the removal of a $K_{2,2}$ preserves the scenic property — the only exceptions are when the resulting graph is small or disconnected. Moreover, besides some exceptional cases discussed in Sections 4 and 5, both $G$ and $H$ must be generic graphs.

The problem of determining the maximum path length of a graph is NP–complete, and the same is true for computing the independence number (maximum number of mutually non–adjacent vertices), see [6]. R.S. Sankaranarayana and L.K. Stewart [7] have shown that deciding whether a graph is well-covered, i.e., deciding whether all maximal independent sets of a graph have the same cardinality, is a co-NP–complete problem. Concerning the analogous decision problem whether all maximal paths are maximum Theorems 1.1 and 1.2 imply that the property of being scenic can be tested in polynomial time.
Proposition 2.1 A tree is non–traceable and scenic if and only if it is an equi-
subdivided star $K_{s,r}^a$ ($r \geq 3$, $s \geq 0$).

Proof. Let $G$ be a non–traceable scenic tree, i.e., let it be different from a path.
For arbitrary $x, y \in V(G)$, we use $(x, G, y)$ to denote the (unique) path of $G$ with
endvertices $x$ and $y$. Let $P = (x, G, y)$ be a maximal path of $G$ and let $z \in V(P) \setminus$
\{x, y\} be a vertex of degree at least three. Clearly, both $x$ and $y$ are leaves of $G$, and
the subpaths $(x, G, z)$ and $(y, G, z)$ must have the same length. Therefore, $z$ is the
(unique) midvertex of $P$.

Assume that $G$ has two distinct vertices of degree at least three, $u$ and $v$. Consider
a maximal extension $P$ of $(u, G, v)$. By the observation above, both $u$ and $v$ are
midpoints of $P$, a contradiction. Therefore, $G$ has exactly one vertex of degree
$r \geq 3$ which is the midpoint of all paths between any two leaves. Thus $G$ is an
equi–subdivided star $K_{s,r}^a$, for some $s \geq 0$.

Theorem 2.2 Let $G$ be a non–traceable scenic graph. If $G$ is different from a tree,
then it is a $p \times q$ bipartite graph with $p \geq 2$ and $q \geq p + 2$ vertices in the partite sets.
Furthermore, $G$ has a dominating cycle on $2p$ vertices and the maximum path length
in $G$ equals $2p$.

Proof. Let $C$ be a cycle of $G$ with maximum length $k = |V(C)|$. Observe
that $3 < k < |V(G)|$. Indeed, $C$ can not be a Hamiltonian cycle, because $G$ is non–
traceable. On the other hand, $k \neq 3$ holds by the following argument. Assuming that
$C = (x_1, x_2, x_3)$, at least two vertices of $C$ have degree greater than two (otherwise
$G$ would not be scenic). Let $x_1y_1, x_2y_2 \in E(G)$, for some vertices $y_1 \neq y_2$ and $y_1, y_2$
not in $C$. Because $C$ is a maximum cycle, every maximal extension $Q$ of the path
$(y_1, x_1, x_2, y_2)$ misses $x_3$. A maximal path longer than $Q$ can be found by including
$x_3$ into $Q$ between $x_1$ and $x_2$, contradicting that $G$ is scenic.

A path $T \subset G$ with $|V(T) \cap V(C)| = 1$ is called a tail of $C$. For a given vertex
$z \in C$, let $T(z)$ denote the longest tail of $C$ ending at $z$. Choose a maximum cycle
$C$ of $G$ having a tail $T$ of maximum possible length. Assume that $T = T(x)$ is a
maximum tail of $C$ at $x$, clearly it has length $t \geq 1$.

Let $y, x, y', x'$ be consecutive vertices on $C$ (they are distinct, since $k \geq 4$). Let
$T(y)$ and $T(y')$ be maximum length tails of $C$ at $y$ and $y'$, respectively. Because
$C$ is a maximum cycle, both $T(y)$ and $T(y')$ are disjoint from $T(x)$. Observe that
$(T(x), (x, C, y), T(y))$ and $(T(x), (x, C, y'), T(y'))$ are maximal paths of $G$, and be-
because $G$ is scenic, $T(y)$ and $T(y')$ have the same length $s$. Clearly, $1 \leq t$, $0 \leq s \leq t$,
and the maximum path length in $G$ is $s + t + (k - 1)$.

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First we show that there is no vertex \( z \in V(G) \setminus V(C) \) with \( yz, y'z \in E(G) \). Suppose that such a \( z \) exists. Because \( C \) has maximum length, \( z \) is not on \( T(x) \). Hence \( z \) could substitute for \( x \) in \( C \); that is \( (C - x) + z \) would be a maximum cycle with longer tail \( T(x) + y \) at \( y \), a contradiction.

The previous paragraph implies that \( T(y) \) and \( T(y') \) are disjoint. If \( s \neq 0 \) then \( (T(y), (y, C - x, y'), T(y')) \) is a maximal path of length \( 2s + (k - 2) \). From \( 2s + k - 2 = s + t + k - 1 \) we obtain \( s = t + 1 > t \), a contradiction. Consequently, \( s = 0 \). Next we show that \( t = 1 \). Note that this will imply that every vertex not in \( C \) is adjacent to some vertex of \( C \) (that is \( C \) is a dominating cycle in \( G \)), and the maximum path lengths equals \( k \).

Consider a maximum length tail \( T(x') \) at \( x' \). Because \( G \) is scenic, and \( ((y', C, x'), T(x')) \) is a maximal path, \( T(x') \) has \( t \) edges. If \( T(x) \) and \( T(x') \) are disjoint, then \( (T(x), (x, C - y', x'), T(x')) \) is a maximal path of length \( 2t + (k - 2) = t + k - 1 \) implying \( t = 1 \). If \( T(x) \) and \( T(x') \) are not disjoint, then they must share a vertex \( z \notin V(C) \) such that \( zx, zx' \in E(G) \). In this case \( (y', x, z, (x', C - \{x, y'\}, y)) \) is a maximal path of length \( k = t + k - 1 \), implying \( t = 1 \).

The argument above shows that the vertices of \( C \) have a two-coloring, namely \( z \in C \) is assigned color \( |T(z)| (= 0 \text{ or } 1) \). In particular, \( C \) is an even cycle of length \( k = 2p \), for some \( p \geq 2 \). Let us color all vertices off of \( C \) with 0. We claim that this is a proper two-coloring of \( G \), i.e., \( G \) is bipartite.

Any vertex off of \( C \) can only be adjacent to vertices of color 1 on \( C \), by the definition of our coloring and because \( G \) is connected. Now assume that \( uv \) is a chord of \( C \) between vertices of the same color \( \epsilon \). Let \( u' \) and \( v' \) be neighbors of \( u \) and \( v \), respectively, such that \( V(C) \) is partitioned into two subpaths of \( C \): \( C_1 \) going from \( u \) to \( v' \) and \( C_2 \) going from \( v \) to \( u' \). If \( \epsilon = 1 \), then both \( u' \) and \( v' \) have color 0, and the concatenation of \( C_1 \) and \( C_2 \) along the edge \( uv \) would result in a maximal path \( Q \) of length \( k - 1 \). Therefore, \( \epsilon = 0 \), and both \( u' \) and \( v' \) have color 1. This implies that \( u' \) and \( v' \) have a neighbor \( z \) and \( w \) not in \( C \), respectively. If \( z = w \), then the path \( Q \) above together with \( z \) would result in a cycle of length \( k + 1 \). Hence \( z \neq w \), and \( (z, (u', Q, v'), w) \) is a maximal path of length \( k + 1 \), a contradiction.

Therefore \( G \) is a (connected) bipartite graph with \( p \) vertices in one partite sets and \( q \geq p + 1 \) in the other one. If there was just one vertex not in \( C \) then \( G \) would be traceable. This shows that \( q \geq p + 2 \) and the maximum path length is \( 2p \). \( \square \)

### 3 Small Non-Traceable Scenic Graphs

For \( p \geq 2 \) and \( q \geq p + 2 \), denote by \( G_{p,q} \) the class of all \( p \times q \) bipartite graphs which are non-traceable scenic graphs different from trees. Notice that members of \( G_{p,q} \) have all properties described in Theorem 2.2. In this section we determine \( G_{p,q} \) for
Proposition 3.1 If $G \in \mathcal{G}_{2,q}$, then $G$ is a connected generic graph.

Proof. Let $\{x, y\}$ be the smallest partite set of $G$ and $Q = V(G) \setminus \{x, y\}$. Note that 4 is the maximum path length in $G$ (by Theorem 2.2). Because $G$ is connected, every vertex of $Q$ is adjacent to either $x$ or $y$. Assume that one of $x$ and $y$, say $y$, is non-adjacent to $u, v \in Q$. In this case the path $(u, x, v)$ would be maximal, a contradiction. This proves that $G \cong K_{2,q} - F$, where $F \cong K_2$ or $2K_2$. □

Proposition 3.2 If $G \in \mathcal{G}_{3,q}$ is not generic, then $G$ is either $G_1, G_2$ or $G_3$.

Proof. Suppose $P = \{x_1, x_2, x_3\}$ is the smallest partite set of $G$ and $Q = V(G) \setminus P$. Note that $G$ has a dominating 6-cycle $C$ and 6 is the maximum path length in $G$ (by Theorem 2.2). Because $G$ is connected, every vertex not in $C$ is adjacent to at least one vertex of $P$. For every $I \subseteq \{1, 2, 3\}$, define $Q(I) = \{z \in Q \setminus V(C) : zz_i \in E(G) \text{ iff } i \in I\}$. Obviously, $Q(I) \cap Q(J) = \emptyset$ holds for every $I \neq J$. Set $q(I) = |Q(I)|$. Observe that $\sum_{|I|=1} q(I) \leq 1$ and, for $|I| = 2, q(I) \leq 1$ must hold, because otherwise, one easily finds maximal paths of length 4 or 2. On the other hand, $q(I) \geq 2$, for some $I$, because $G$ is non-traceable.

Case a: $C$ is an induced 6-cycle of $G$. If $q(\{i\}) = 1$ for some $i \in \{1, 2, 3\}$, then $q(I) = 0$ must hold for every $I$ containing $i$, because otherwise, one easily finds a maximal path of length 4. Therefore, $q(\{1, 2, 3\} \setminus \{i\}) = 1$ and $G \cong G_1$ follows. Assume now that $q(I) = 0$, for every $|I| = 1$. If $q(\{1, 2, 3\}) = 0$, then $\sum_{|I|=2} q(I) \geq 2$, because otherwise, $G$ would be traceable. Therefore $G$ is isomorphic to one of $G_2$ and $G_3$. If $q(\{1, 2, 3\}) > 0$, then $q(I) = 0$, for every $|I| = 2$. This implies that $G$ is generic.

Case b: $G$ has no induced 6-cycle. Assume first that $C$ has just one chord, say at $x_3$. In this case $q(\{1, 2\}) = 0$ (otherwise $G$ would contain a $C_6$). Furthermore, $q(\{1, 3\}) = q(\{2, 3\}) = 0$ and $q(I) = 0$, for every $|I| = 1$, because $G$ has no maximal paths of length less than 6. This proves that $G$ is generic. Assume now that every 6-cycle of $G$ induces at least two chords. A similar argument as above shows that $G$ must be generic. This proves the proposition. □

4 $K_{2,2}$-removal

In this section our goal is to prove that (to some extent) the removal of a $K_{2,2}$ preserves the scenic property — the only exceptions are when the resulting graph is small or disconnected.
Proposition 4.1 If \( p \geq 4 \) and \( G \in \mathcal{G}_{p,q} \) is different from \( G_4 \), then \( G \) contains a copy of \( K_{2,2} \).

Proof. By Theorem 2.2, \( G \) is bipartite and has a dominating cycle \( C = (x_1, y_1, x_2, y_2, \ldots, x_p, y_p) \) of length \( 2p \), where \( P = \{x_1, \ldots, x_p\} \) is one of the partite sets of \( G \). Furthermore, the proof of Theorem 2.2 implies that every vertex of \( P \) has a neighbor off of \( C \). Assume that \( G \) has no \( K_{2,2} \). For every \( i, 1 \leq i \leq p \), there exist vertices \( u, v \in V(G - C) \) with \( ux_i, vy_{x_i+2} \in E(G) \) and \( ux_{i+1}, vy_{x_{i+1}} \notin E(G) \) (because \( G \) is \( K_{2,2} \)-free). (Indices are reduced modulo \( p \) in this paragraph.) If \( u \neq v \), then the path \((u, (x_i, C - \{y_i, x_{i+1}, y_{i+1}\}, x_{i+2}), u)\) is maximal and has length \( 2p - 2 \). Hence \( u = v \) follows, moreover, \( u \) must be adjacent to all vertices \( x_{i+1}, x_{i+2}, \ldots, x_{i-2} \). The same argument shows that there exists a vertex \( w \in V(G - C) \) different from \( u \) and adjacent to all \( x_{i+1}, x_{i+3}, \ldots, x_{i-1} \). This implies that \( p \) must be even, in particular \( C \) has length \( 2p \geq 8 \).

If \( p \geq 6 \), then the path \((y_1, x_1, u, x_3, y_3, x_4, w, x_2, y_2)\) of length 8 cannot be maximal, hence it extends by an edge \( y_\epsilon x_i \), where \( \epsilon = 1 \) or 2, and \( 4 \leq i \leq p \). Now \( j = \epsilon \) or \( \epsilon + 1 \) has the same parity as \( i \), hence \( x_i \) and \( x_j \) are adjacent to the same vertex \( z = u \) or \( w \). Then \( \{y_\epsilon, x_i, z, x_j\} \) induces a \( K_{2,2} \), a contradiction. Thus we have \( p = 4 \). Because any additional vertices or any further edges included to \( C \cup \{u, w\} \) would complete a \( K_{2,2} \). \( G \cong G_4 \) follows.

For \( G' \subset G \), \( G - V(G') \) denotes the graph obtained from \( G \) by removing the vertices of \( G' \) together with all incident edges.

Theorem 4.2 For \( p \geq 4 \), let \( G \in \mathcal{G}_{p,q} \) \((q \geq p + 2)\), and let \( K \cong K_{2,2} \) be a subgraph of \( G \). If \( G \) is different from \( G_5 \), then either \( G - V(K) \in \mathcal{G}_{p-2,q-2} \) or \( G - V(K) \) is a scenic graph (traccable or non–traceable) plus an isolated vertex.

Proof. Let \( a_1, a_2, b_1, \) and \( b_2 \) be the vertices of \( K \). Let \( H = G - V(K) \), let \( P \) and \( Q \) be the partite sets of \( H \) with \(|P| = p - 2\) and \(|Q| = q - 2\), furthermore, let \( \{a_1, a_2\} \cup P \) and \( \{b_1, b_2\} \cup Q \) be the partite sets of \( G \). We know from Theorem 2.2 that

\[(*)\text{ every maximal path of } G \text{ has both end vertices in the larger partite set, } \{b_1, b_2\} \cup Q, \text{ and contains all vertices from the smaller one, } \{a_1, a_2\} \cup P.\]

Our goal is to show that similar properties are satisfied by the maximal paths of \( H \), as well. Let \( M = (u, \ldots, v) \) be a maximal path in \( H \) (we may assume \( u \neq v \)). We shall prove that \( u, v \in Q \), moreover, \( M \) contains all vertices of \( P \). Because \( M \) has an extension in \( G \) containing \( a_1 \) and \( a_2 \), we may assume that there is an edge, say \( uz \in E(G) \), for some \( z \in V(K) \). Thus \( M \) can be extended in \( G \) from its endvertex \( u \) to include the four vertices of \( K \). This new path has no extension in \( G \) from the other endvertex \( v \) (because \( (u, M, v) \) is maximal in \( H \)). Hence (*) implies \( v \in Q \).

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Due to the argument above we may assume that if $M = (u, \ldots, v)$ is a maximal path of $H$, then $v \in Q$ and $u$ sends an edge to $K$. The proof of the theorem consists of two claims, each will be verified in several numbered steps.

**Claim I: Every maximal path of $H$ has both end vertices in $Q$.**

*Proof.* We assume that $M = (u, \ldots, v)$ is a maximal path of $H$ with $v \in Q$. Suppose to the contrary that $u \in P$, and let $ub_1 \in E(G)$. Let $M = (u, u', \ldots, v', v)$, and let $Y = V(H) \setminus V(M)$. Observe that $|Y| \geq 2$ holds, because $q \geq p + 2$.

Assume that $va_i \in E(G)$. The path $(u, M, v, a_i, K, b_j)$ extends in $G$ from $u$, by property (*). This contradicts the maximality of $M$ in $H$. Similar argument shows that $va_i, v'a_i \notin E(G)$, for each $i = 1$ and 2. Note also that, by property (*), path $(v, M, u, b_1, K, a_i)$ has an extension in $G$ from $a_i$. This implies that, for each $i = 1$ and 2, there exists $y_i \in Y$ with $a_iy_i \in E(G)$.

(1) **There is an edge from $Y$ to $M$.**

Suppose $b_2x \in E(G)$, for some $x \in Y \cap P$. The path $(x, b_2, K, a_i, y_i)$ covering $K$ extends in $G$ to include all vertices of $P$, as required by property (*). In this case there must be an edge from $Y$ to $M$.

Next we suppose that $\{b_1, b_2\}$ has no neighbor in $Y \cap P$. If $y_1 \neq y_2$, the path $(y_1, a_1, b_1, a_2, y_2)$ extends to include $P$ which requires of using some edge going from $Y$ to $M$. Thus we may also assume that $y_1$ is the unique neighbor of $\{a_1, a_2\}$ in $Y$. Because $q \geq p + 2$, there is some $y' \in Y \cap Q$ different from $y_1$. By the connectivity of $G$, there is an edge $zy' \in E(G)$. If $z \in Y \cap P$, then the path $((v, M, u), (b_1, a_1, y_1, a_2, b_2))$ has no extension to include $z$, thus $z \in V(M)$ follows.

Note that (1) implies that $M$ has at least 4 vertices. In particular, $u' \neq v$, and $u \neq v'$.

(2) **There is a vertex $y \in Y \cap Q$ such that $yy' \in E(G)$.**

Let $x \in V(M)$ be the closest vertex to $v$ such that $xy \in E(G)$, for some $y \in Y$. By (1), such $x$ exists, we shall show that $x = v'$. Suppose to the contrary that $x \neq v'$.

If $x \in Q$, then no extension of the path $S = (y, (x, M, u), (b_1, K, a_i), y_i)$ ($i = 1$ or 2) can include $v'$, by the choice of $x$. This contradicts (*), thus $x \in P$ follows. Note also that the path $S$ above cannot exist, consequently, we have $y = y_i$, for $i = 1, 2$. Therefore, $y$ is the only vertex of $Y \cap Q$ which is adjacent to $a_1$ and $a_2$. The path $(y, (x, M, u), (b_1, K, a_2))$ extends in $G$ with some $a_2t \in E(G)$, where $t \in Q$. By the assumption on $y$, we know that $t \notin Y \cap Q$, that is $t$ is a vertex of $(x, M, v)$ different from $v$.

If $Y \cap P = \emptyset$, then every vertex of $Y$ sends an edge to $M$, because $G$ is connected. Define $x' \in V(M)$ as the first vertex along the subpath $(x, M, u)$ having some neighbor $y' \in Y \setminus \{y\}$. Because $xy' \notin E(G)$, we have $x' \neq x$. Let
$x^*$ be the last vertex on $(x, M, x')$ adjacent to $y$ (possibly $x^* = x$). The path $(y', (x', M, u), (b_1, K, a_2), (t, M, x^*), y)$ is maximal and misses $v'$, a contradiction.

If $Y \cap P \neq \emptyset$, then the path $((v, M, x), y, (a_2, K, b_1), (u, M, u''))$ is not maximal in $G$. Therefore, there exists a vertex $z \in Y \cap P$ with $u'z \in E(G)$. No extension of the path $(z, (u'', M, u), (b_1, K, a_2), (t, M, x), y)$ may contain $v'$, a contradiction. This proves (2).

(3) There is a vertex $w \in Y \cap P$ such that $wu' \in E(G)$.

By (2), there is a vertex $y \in Y \cap Q$ such that $yv' \in E(G)$. Let $C$ be the connected component of the subgraph of $H$ induced by $Y$ and containing $y$.

Assume that $uv \notin E(G)$. First we verify that in this case $C$ does not send any edge to $K$. Otherwise, let $S = (y, \ldots, y', z)$ be a shortest path from $y$ to $K$ (with $z \in V(K)$). Any extension of the path $((u, M, v'), (y, S, y'), (z, K, z'))$ (with $z$ and $z'$ in opposite partite sets) has endvertex at $u \in P$, which contradicts property (*). Hence $C \cup \{v\}$ has no neighbor in $K$. Let $t$ be the last vertex on $(v', M, u)$ that sends an edge to some $w \in C \cup \{v\}$. Either the path $(v, (t, M, v'), y)$ or the path $((v, M, t), w)$ leads to a contradiction, since no extension of these paths may include $a_i$ ($i = 1$ or 2).

So we may assume that $uv \in E(G)$. Recall that $u'a_i \notin E(G)$, for $i = 1$ and 2. The path $(v, (u, M, v'), y)$ extends to include $a_i$. Let $S = (y, \ldots, z)$ be a shortest path from $y$ to $K$ (with $z \in V(K)$). Consider the path $J$ obtained from the paths $((u', M, v'), (y, S, z))$ and $(b_1, u, v)$ by joining them in $K$ with a shortest path between $b_1$ and $z$. Because $J$ misses either $a_1$ or $a_2$, there exists a vertex $w \in Y \cap P$ such that $wu' \in E(G)$. This proves (3).

(4) To conclude the proof of Claim I we show that the existence of the vertices $y, w \in Y$ obtained in (2) and (3) leads to a contradiction.

Let $S = (y, \ldots, y', z)$ be a shortest path from $y$ to $K$ as introduced in (3) above. If $z = b_i$ ($i = 1$ or 2), then any extension of $(w, (u', M, v'), (y, S, y'), (b_i, K, a_2))$ misses $u$. Hence we may assume that $z = a_i$ ($i = 1$ or 2). Furthermore, the path $(w, (u', M, v'), (y, S, y'), (a_i, K, b_2))$ extends with $b_2u \in E(G)$.

Let $R = (w, \ldots, w', r)$ be some path that we start adding when the path $(v, u, b_1, (a_i, S, y), (v', M, u'), w)$ is extended to include all vertices of $P \cup \{a_1, a_2\}$. In particular, the extension will include $a_{3-i} \in K$, thus $R$ should enter $K$. Actually we assume that $r$ is the first vertex from $K$ along $R$. If $r = b_2$, then any extension of $(y, (v', M, u'), (w, R, w'), (b_2, K, a_2))$ would miss $u$. Hence $r = a_{3-i}$. From this we obtain that the path $(w, (u', M, v'), (y, S, a_i), b_2, (a_{3-i}, R, w'))$ must extend with $wb_1 \in E(G)$ to include $u$. Now any extension of $(y, (v', M, u'), w, (b_1, K, a_2))$ misses $u$, a contradiction. This concludes the proof of Claim I. \[\square\]
Claim II: Every maximal path of $H$ with distinct endvertices contains all vertices of $P$.

Proof. Suppose to the contrary that there exists a maximal path $M = (u, \ldots, v', v)$ of $H$ such that $P \setminus V(M) \neq \emptyset$. Assume that $M$ is the longest such path. By Claim I, we have $u, v \in Q$. Because $M$ extends in $G$, and by the symmetry of the endvertices, we may assume that $ua_1 \in E(G)$. Let $Y = V(H) \setminus V(M)$. For $i = 1$ or $2$, the path $((v, Mu), (a_1, K, b_i))$ extends in $G$. Hence, for every $i = 1$ and $2$, there exists a vertex $y_i \in Y \cap P$ with $y_i \in E(G)$.

(1) There is an edge from $Y$ to $M$.

First assume that $a_jz \in E(G)$, for some $z \in Y \cap Q$ and $j = 1$ or $2$. Any maximal extension of the path $(y_1, (b_1, K, a_j), z)$ has to cover vertices of $M$, thus there exists an edge between $Y$ and $M$. Assume now that there is no edge from $\{a_1, a_2\}$ to $Y$. If $y_1 \neq y_2$, then the path $(y_1, b_1, a_1, b_2, y_2)$ extends to include $a_2$, hence there is an edge from $Y$ to $M$. So we may suppose that $y_1 = y_2$ is the only neighbor of $b_1$ and $b_2$ in $Y$. In this case the path $((v, Mu), a_1, b_1, y_1, b_2, a_2)$ is maximal in $G$. This contradicts property (*) and concludes the proof of (1).

(2) There is a vertex $y \in Y \cap Q$ such that $yv' \in E(G)$.

By (1), there is an edge between $M$ and $Y$. Let $x$ be the first vertex along the path $(v, Mu)$ which has a neighbor from $Y$, say $xy \in E(G)$, for some $y \in Y$. Suppose to the contrary that $x \neq v'$.

If $x \in P$ then the path $(y, (x, Mu), (a_1, K, b_i), y_i)$, where $i = 1$ or $2$, has no extension including $v'$, by the choice of $x$. Hence $x \in Q$. Moreover, as the path above can not exist, $y = y_1 = y_2$ is the only vertex of $Y \cap P$ adjacent to $b_i$ ($i = 1, 2$) and $x$.

The path $((v, Mu), a_1, b_1, y, b_2, a_2)$ extends with $a_2w \in E(G)$, for some $w \in Y \cap Q$. The path $((v, Mu), (a_1, K, b_i), y)$ extends at $y$, thus $yz \in E(G)$, for some $z \in Y \cap Q$. If $z \neq w$, then the path $(z, y, (x, Mu), a_1, b_1, a_2, w)$ misses $v'$. Thus we conclude that $z = w$ is the only neighbor of $y$ from $Y \cap Q$.

For $i = 1$ or $2$, the path $(w, a_2, b_{3-i}, y, (x, Mu), b_i)$ must extend at $b_i$ to include $v'$. Thus there is an edge $b_it \in E(G)$, where $t \in P$ is a vertex of $(v, Mu, x)$. The path $(w, a_2, b_i, (t, Mu), a_1, b_{3-i}, y)$ misses $v'$ unless $t = v'$. Therefore, we may assume that $b_itv' \in E(G)$ for $i = 1$ and $2$. The path $(y, (x, Mu), a_1, b_1, v', b_2, a_2, w)$ has no extension at $y$. This contradicts property (*). Therefore, $yv' \in E(G)$ follows.

(3) For $j = 1$ or $2$, there exists a path $S = (y, \ldots, x, b_j)$ such that $V(S) \setminus \{b_j\} \subset Y$.

Let $C$ be the connected component of the subgraph of $G$ induced by $Y$ and containing $y$. First we show that there is a vertex $x \in C$ that is adjacent to some vertex of $K$. Suppose this is false. In particular, we may assume that the neighbor $y_1 \in Y \cap P$ of $b_1$ is not in $C$. 296
If $a_2 v \in E(G)$, then no extension of $(y, (v', M, u), a_1, b_1, a_2, v)$ contains $y_1$. Hence $a_2 v \notin E(G)$. Similarly, if $a_1 v \notin E(G)$, then no extension of $(y, (v', M, u), a_1, v)$ contains $y_1$. Hence $a_1 v \notin E(G)$. Let $t \in V(M)$ be the last vertex on $(v', M, u)$ adjacent to $v$ or to some vertex $x \in C$. One of the paths $((v, M, t), x)$ and $(y, (v', M, t), v)$ exists and misses $y_1$, a contradiction. Thus we obtain that some $x \in C$ is adjacent to some vertex of $K$.

The existence of $x$ implies that there is a path $S = (y, \ldots, x, z)$, for some $z \notin V(K)$, such that $V(S) \setminus \{z\} \subset Y$. Now suppose that in every such path $S$ we have $z = a_i$ ($i = 1$ or $2$). In particular, no vertex of $C$ is adjacent to $b_1$ or $b_2$. If $z = a_2$, then any extension of $((v, M, u), a_1, b_1, (z, S, y))$ would miss $y_1$. Hence $z = a_1$, for every path $S$, and $a_2$ has no neighbor in $C$. The path $((v, M, u), (z, S, y))$ has no extension that includes $a_2$, a contradiction. This proves (3).

(4) For $k = 1$ or $2$, $ua_k, va_{3-k} \in E(G)$.

Assume that $S = (y, \ldots, x, b_1)$ is a path guaranteed by (3). Let $R = (r, \ldots, y')$ be a path (possibly empty) such that $((v, M, u), (a_1, K, b_1), (x, S, y), (r, R, y'))$ is maximal in $G$. The path $((v, M, u), a_1, b_1, (x, S, y), (r, R, y'))$ has an extension to include $a_2$. Thus either $va_2 \in E(G)$ which proves (4), or we have $y'u \in E(G)$.

Assume that $va_2 \notin E(G)$. Let $v''$ be the neighbor of $v'$ in $M$ different from $v$. The path $(v, v', y, (r, R, y''), a_2, b_1, a_1, (u, M, v''))$ extends with $v''w \in E(G)$, for some $w \in V(S) \cap P$. Thus we obtain a path $M' = ((u, M, v''), (w, S, y), v', v)$ which is maximal in $H$ and longer than $M$. By the choice of $M$, we have $P \subset M'$, and $w = x$. This implies that $R$ is empty ($y = y''$), furthermore, $ya_2, v''x \in E(G)$, and $b_1x, b_2x \in E(G)$. Observe that the path $((u, M, v''), x, (b_1, K, a_2), y, v, v')$ is maximal in $G$, hence we have $S = (y, x, b_1)$.

The path $(b_2, a_1, (u, M, v''), x, y, a_2, b_1)$ extends to include $v'$, the only uncovered vertex of $P$; therefore, $b_jv' \in E(G)$, for $j = 1$ or $2$. The path $(v, v', b_j, a_1, b_{3-j}, x, (v'', M, u))$ extends to include $a_2$. Thus we have $ua_2 \in E(G)$ (recall that, by assumption, $va_2 \notin E(G)$). If $va_1 \in E(G)$, then we are done. Assuming that $va_1 \notin E(G)$, we obtain that $ya_1, E(G)$, by the symmetry of $a_1$ and $a_2$. For $i = 1$ and $2$, the path $((v, v', y, x, b_1, a_1, (u, M, v''))$ extends with $v'a_{3-i} \in E(G)$. The path $((u, M, v''), (a_1, K, b_j), v', v)$ is maximal in $G$ and misses $x$, a contradiction. This concludes the proof of (4).

In the next step we use $S = (y, \ldots, x, b_j)$, $j = 1$ or $2$, a path guaranteed by (3), together with further paths similar to those in the proof of (4).

(5) $P \setminus V(M) = \{x\}$, $xb_i \in E(G)$, for $i = 1, 2$, and there exists $z \in Y \cap Q$ such that $za_1, zx \in E(G)$.

By (4), and by the symmetry of $a_1$ and $a_2$, we may assume that $va_2 \in E(G)$. Also assume that $S = (y, \ldots, x, b_2)$. The path $N = (v, (a_2, K, b_2), (x, S, y), (v', M, u))$ is maximal, hence $(P \setminus V(M)) \subset V(S)$. Observe that $N$ has no chord induced by two
non-consecutive vertices of $S$; for otherwise, a shorter maximal path of $G$ would result by using that chord to skip over some vertex of $V(S) \cap P$. The same argument shows that if $b_1 y_1 \in E(G)$, for some $y_1 \in Y \cap P$, then $y_1 = x$ follows. Thus we have $b_1 x \in E(G)$.

The path $(a_1, b_1, x, b_2, a_2, (v, M, u))$ extends with $a_1 z \in E(G')$, for some $z \in Y \cap Q$. Note that $z \notin V(S)$, because otherwise, the maximal path $((u, M, v'), (y, S, z), a_1, b_1, a_2, v')$ would miss $x$. We show next that $xz \in E(G)$. Every extension of $(z, a_1, b_1, a_2, (v, M, u))$ contains $x$, thus $z \in C$, where $C$ is the connected component containing $y$ in the subgraph of $H$ induced by $Y$. This implies that $zz' \in E(G)$, for some $z' \in V(S) \cap P$. The maximal path $((u, M, v'), (y, S, z'), z, a_1, b_1, a_2, v')$ contains $x$, thus $z' = x$. Observe that the path $((u, M, v), a_2, b_1, a_1, z, x, b_2)$ must contain $V(S) \cap P$, on the other hand $S$ has no chord from $b_2$. Therefore, $S = (y, x, b_2)$ which concludes the proof of (5).

$$P = \{v', x\}, \ Q = \{u, v, y, z\}, \ and \ v'z \notin E(G).$$

The path $(z, x, b_1, a_1, (u, M, v'), y)$ extends to include $a_2$. Hence we have either $ya_2 \in E(G)$ or $za_2 \in E(G)$. Suppose first that $ya_2 \in E(G)$. The path $(v, v', y, a_2, b_1, a_1, (u, M, v''))$ extends to include $x$, thus $v''x \in E(G)$. The path $(z, a_1, (u, M, v''), x, b_1, a_2, b_2)$ extends to include $v'$. Hence we have either $v'b_2 \in E(G)$ or $v'z \in E(G)$. None of them is possible, because in the first case $((u, M, v'), (b_2, K, a_2), v)$, and in the second case $((u, M, v'), z, a_1, b_2, a_2, v)$ is a maximal path of $G$ missing $x$. Therefore, we may assume that $ya_2 \notin E(G)$ and $za_2 \in E(G)$, that is $y$ and $z$ are not interchangeable. If $zv' \in E(G)$, then $y$ and $z$ are interchangeable with respect to $v'$. Thus we may also assume that $zv' \notin E(G)$.

We show that $v'' = u$. Suppose that this is false, that is $u' \neq v'$, where $u'$ is the neighbor of $u$ in $M$. The path $(y, v', v, (a_2, K, b_2), x, z)$ extends to include uncovered vertices of $V(M) \cap P$. Let $w$ be the last vertex on $(v'', M, u)$ adjacent to $y$ or $z$. In the first case $(y, (w, M, v), (a_2, K, b_2), x, z)$ and in the second case $(z, (w, M, v), (a_2, K, b_2), x, y)$ is a maximal path, therefore, $w = u'$ must hold. Observe that $u'z \notin E(G)$, for otherwise, the maximal path $((v, M, u'), z, a_2, b_1, a_1, u)$ in $G$ would miss $x$. Hence we have $u'y \in E(G)$.

The path $((v'', M, u'), y, v', y, a_2, b_1, a_1, u)$ extends with $v''x \in E(G)$. The path $(z, a_1, (u, M, v''), x, b_1, a_2, b_2)$ extends with $b_2v' \in E(G)$. Thus we obtain that $((u, M, v'), (b_2, K, a_2), v)$ is a maximal path of $G$ missing $x$, a contradiction. Therefore, $u' = v'$ and (6) follows.

To conclude the proof of Claim II we show that $G \cong G_5$. By (5) and (6), $G$ is a $4 \times 6$ bipartite graph such that its edges determined so far (explicitly or by symmetry) induce a $G_5$. It is easy to check that including any of the four edges $ua_2, va_1$, or $v' bi_i$, $i = 1, 2$, would result in a non-scenic graph containing a maximal path of length less than 8. Therefore, $G \cong G_5$ follows, contradicting the assumption of the theorem. □
Claim II implies that $H$ has at most one non-trivial connected component, and this component is scenic. If $H$ is connected, then it is non-traceable, because $q \geq p + 2$. If $H$ is disconnected, then it has exactly one trivial component (i.e., isolated vertex). Indeed, in case of two isolated vertices $u, u' \in V(H)$, one would easily find a path $M \subset K + \{u, u'\}$ which is maximal in $G$ and misses all vertices in the non-trivial component of $H$. This contradicts (*) and concludes the proof of Theorem 4.2. □

5 $K_{2,2}$-extension

In this section we consider ways that a $K_{2,2}$ can be "added" to non-traceable scenic graphs so that the property of being scenic is preserved. If $G$ is a non-traceable scenic graph containing a copy $K \cong K_{2,2}$, then we say that $G$ is a scenic $K_{2,2}$-extension of $H = G - V(K)$.

We use the following notations throughout this section. We assume that $G$ is scenic non-traceable $K_{2,2}$-extension of $H = G - V(K)$. The vertices of $K$ are $a_1, a_2, b_1,$ and $b_2$, the partite sets of $H$ are $P$ and $Q$ with $|P| \leq |Q| - 2$, and the partite sets of $G$ are $P \cup \{a_1, a_2\}$ and $Q \cup \{b_1, b_2\}$. In the figures accompanying the proofs, black circles indicate vertices in the smaller partite set of $G$. Let $(a_i, K, b_j)$ denote the Hamiltonian path of $K$ from $a_i$ to $b_j$ ($1 \leq i, j \leq 2$). For $H' \subseteq H$ and $u, v \in V(H')$, we denote by $(u, H', v)$ a path of $H'$ between $u$ and $v$ spanning as many vertices of $V(H') \cap P$ as possible.

By Theorem 4.2, one may assume that $H$ is either a non-traceable scenic graph or a (traceable or non-traceable) scenic graph plus an isolated vertex. We need the following easy corollaries of Theorem 2.2.

**Lemma 5.1** Let $G$ be a scenic $K_{2,2}$-extension of $H$.

(i) If there is a maximal path of $H$ between $y, y' \in Q$, then there is an edge from $(y, y')$ to $(a_1, a_2)$.

(ii) If at least two vertices of $Q$ are adjacent to $(a_1, a_2)$, then there exist two independent edges $y_1a_1, y_2a_2 \in E(G)$, for some $y_1, y_2 \in Q$.

**Proof.** Because $G$ is scenic, every maximal extension of the path between $y$ and $y'$ contains $a_1$ and $a_2$ which proves (i). The maximum path length in $G$ is $2|P| + 2$, thus no maximal extensions of the path $(a_1, K, b_2)$ or $(a_2, K, b_2)$ may start at $a_1$ or at $a_2$. Therefore, both $a_1$ and $a_2$ are adjacent to $Q$. This observation together with the condition in (ii) imply that the edges between $Q$ and $(a_1, a_2)$ can not be covered with one vertex. Hence there exist two independent edges, and (ii) follows. □

**Proposition 5.2** The equi-subdivided star $K^s_{1,r}$ ($r \geq 3$, $s \geq 1$) and the graphs $G_1, \ldots, G_6$ have no scenic $K_{2,2}$-extensions.
Proof. Suppose on the contrary that $G$ is a scenic $K_{2,2}-$extension of $H$, where $H$ is one of the seven graphs in the proposition.

Case 1: $H = K_{1,r}^s$ ($r \geq 3, s \geq 1$). Because $|P| \leq |Q| - 2$, the center of $H$ is a vertex $x_0 \in P$, and all leaves of $H$ are in $Q$. Let $y_1, y_2, y_3 \in Q$ be distinct leaves of $H$. By Lemma 5.1 (i), one may assume that $y_1a_1 \in E(G)$. The path $((y_2, H, y_1), (a_1, K, b_1))$ is not maximal, thus $b_1x_1 \in E(G)$ holds, for some $x_1 \in (x_0, H, y_3)$ (see Fig. 2). If $y_4 \in Q$ is an arbitrary vertex on $(x_0, H, x_1)$, then no extension of the path $((y_4, H, y_1), (a_1, K, b_1), (x_1, H, y_3))$ contains the vertices of $P$ on the path $(x_0, H, y_2)$, a contradiction.

Case 2: $H = G_1, G_2$ or $G_3$. Let $y, y' \in Q$ be any pair of vertices such that their removal does not disconnect $H$ (note that all pairs satisfy this in $H = G_2$ or $G_3$, and just one pair fails it in $H = G_1$). It is easy to check that between $y$ and $y'$ there exists a maximal path in $H$ (actually, covering all vertices in $P$). Hence, by Lemma 5.1 (i) and (ii), there exist $y_1a_1, y_2a_2 \in E(G)$, with distinct $y_1, y_2 \in Q$. Consider a maximal path $(b_1a_1, (y_1, H - x_1, y_2), a_2b_2)$ in $H$ which does not cover a vertex $x_1 \in P$. This path has an extension $b_1x_1 \in E(G)$ to include $x_1$. Fig. 3 (a) shows a particular case, where $H = G_1$. The argument works for any other choice of $H$, and for other positions of $y_1$ and $y_2$, as well. Thus we always have $x_1b_1 \in E(G)$, for some vertex $x_1 \in P$.

Let $x_2$ and $x_3$ be the other two vertices in $P$. If $x_2$ and $x_3$ have two common neighbors in $H$, then, by Lemma 5.1 (i), one of them is adjacent to $K$, say $y_2a_2 \in E(G)$. The maximal path $(y_1, x_1, (b_1, K, a_2), y_2, x_3, y_3)$ shown in Fig. 3 (b) misses $x_2$, a contradiction. Assume now that the previous argument does not apply (even if we relabel the vertices of $P$), because there is no edge from $\{x_2, x_3, y_2\}$ to $K$. In this case any path of $H$ between $x_2$ and $y_2$ not containing edge $x_2y_2$ is maximal in $G$ and misses $K$, a contradiction.

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Case 3: $H = G_4$. Since $G$ is connected, either $x_1b_\epsilon \in E(G)$ or $y_1a_\epsilon \in E(G)$ holds, for some $x_1 \in P$ or $y_1 \in Q$, and $\epsilon = 1$ or $2$. Assume that $y_1a_1 \in E(G)$ and let $x_1$ be a neighbor of $y_1$. The path $((b_1, K, a_1), (y_1, H - x_1, y_3))$ extends to include $x_1$ (see Fig. 4 (a)). Thus $x_1b_1 \in E(G)$ follows. Because there is a path of $H$ between $y_2$ and $y_3$ that covers all vertices of $P$, say $y_2a_2 \in E(G)$. The path $(y_1, x_1, (b_1, K, a_2), (y_2, H - \{x_1, x_4\}, y_4))$ in Fig. 4 (b) is maximal and misses $x_4$, a contradiction.

Case 4: $H = G_5$. It is easy to verify that between any pair $y, y' \in Q$ there exists a maximal path in $H$. Hence by Lemma 5.1, $y_1a_1, y_2a_2 \in E(G)$, for some $y_1, y_2 \in Q$. The path $(b_1, a_1, (y_1, H - x_1, y_2), a_2, b_2)$ as shown in Fig. 5 (a) extends to include $x_1$. Thus one may assume that $x_1b_1 \in E(G)$, so $(y_1, x_1, (b_1, K, a_2), (y_2, H - \{x_1, x_4\}, y_4))$ in Fig. 5 (b) is a maximal path missing $x_4$, a contradiction.
Case 5: \( H = G_6 \). Label the vertices of \( H \) as shown in Fig. 6. An easy argument using Lemma 5.1 shows the existence of \( x_1 b_1, y_2 a_2 \in E(G) \). The maximal path \( (y_3, x_3, y_1, x_1, (b_1, K, a_2), y_2, x_2, y_4) \) misses \( x_4 \), a contradiction.

This concludes the proof of the proposition. \( \Box \)

Figure 5:

The following technical lemma will be used when proving that a \( K_{2,2} \)-extension of a generic graph is generic. We note in advance that the only exception will be the generic graph \( K_{2,4} - 2K_2 \) which has a non-generic \( K_{2,2} \)-extension, namely \( G_6 \). Recall that a \( p \times q \) generic graph has the form \( K_{p,q} - F \), where the partite sets \( P \) and \( Q \) contain \( p \geq 2 \) and \( q \geq p + 2 \) vertices, respectively, and \( F \) is a star forest with its star components centered in \( Q \).

Figure 6:
Lemma 5.3 Let $H$ be a $p \times q$ generic graph with partite sets $P$ and $Q$. If $H \neq K_{2,4} - 2K_2$, then

(A) $H$ has a maximal path between any two non-isolated vertices $y, y' \in Q$;

(B) for every $x \in P$ and for every $y, y' \in Q$ which are distinct non-isolated vertices of $H - x$, there is a path in $H - x$ between $y$ and $y'$ that contains all vertices in $P \setminus \{x\}$.

Proof. (A) Let $M = (y, x, \ldots, x', y')$ be a maximum length path of $H$ from $y$ to $y'$. We shall prove that $M$ contains $P$. Suppose on the contrary that $x_1 \in P \setminus V(M)$. First assume that there are vertices $y_1, y_2, y_3 \in Q \setminus V(M)$. Because $H$ is generic, $x_1$ is adjacent to $y$ or $y'$, say $x_1y' \in E(H)$. Moreover, by the pigeon hole principle, some $y_i$ is adjacent to both $x'$ and $x_1$, for $i = 1, 2$ or 3. The path $((y, M, x'), y_i, x_1, y')$ would be longer than $M$, a contradiction.

Thus we may assume that $Q \setminus V(M) = \{y_1, y_2\}$, $P \setminus V(M) = \{x_1\}$. Furthermore, $x_1$ is non-adjacent to one of $y_1$ and $y_2$, say $x_1y_2 \notin E(G)$. We have $x_1y_1, x_1y', x_1y \in E(G)$, and by the argument above, $xy_1, x'y_1 \notin E(G)$. Hence $xy_2, x'y_2 \in E(G)$. Also $H \neq K_{2,4} - 2K_2$, thus $p \geq 3$. In particular, $x \neq x'$, and $M = (y, x, \ldots, y'', x', y')$. We shall prove by induction on $p$ that in the particular generic graph $H$ described above there exists a path from $y$ to $y'$ that covers $P$. This will contradict our assumption and will prove (A).

For $p = 3$, the path $(y, x, y_2, x', y'', x_1, y')$ covers $P$. Thus (A) is true for $p = 3$. Assume that $p \geq 4$ and (A) is true for $p - 1$. Because our graph $H$ is generic, $x$ is adjacent to every vertex of $Q \setminus \{y_1\}$ and $x_1$ is adjacent to every vertex of $Q \setminus \{y_2\}$. Hence $y$ and $y''$ are not isolated vertices in $H' = H - \{x', y'\}$. By the induction hypothesis, $H'$ has a path $M' = (y, \ldots, y'')$ that contains $P \setminus \{x'\}$. The path $((y, M', y''), x', y')$ covers $P$, a contradiction. Thus (A) follows.

(B) If $H$ or $H' = H - x$ has an isolated vertex $u \in Q$, then $H - \{x, u\}$ is a complete bipartite graph and (B) obviously holds. Assume that $H$ and $H'$ are both connected, in particular, $H' \neq K_{2,4} - 2K_2$. Now (B) follows by applying (A) for the generic graph $H'$.

Proposition 5.4 If $H$ is the union of an isolated vertex and one of the following graphs: $G_1, \ldots, G_6$, an equi-subdivided star $K^*_1, r$ ($r \geq 2, s \geq 1$), or a connected $p \times q$ generic graph ($p \geq 2, q \geq p + 2$) different from a complete bipartite graph, then $H$ has no scenic $K_{2,2}$-extension.

Proof. Let $u$ be the isolated vertex of $H$ and let $H' = H - u$ be one of the graphs in the proposition. Suppose on the contrary that $G$ is a scenic $K_{2,2}$-extension of $H$. Observe that $u \in Q$, for otherwise, $G$ would have a path $(u, (b_1, K, a_1))$ and a maximal extension of it with a black end vertex $u \in P$. One may assume that $ua_1 \in E(G)$. The path $S = (u, (a_1, K, b_2))$ extends in $G$ with an edge $b_2z$, for some
Now suppose that $H'$ is a connected generic graph different from a complete bipartite graph. The previous argument shows that $H' \neq K_{2,4} - 2K_2$. Let $xy \notin E(H')$, for some $x \in P$ and $y \in Q \setminus \{u\}$. By the connectivity of $H'$, $y$ is non-isolated in $H' - x$. In addition, because $H' \neq K_{2,4} - 2K_2$, we may choose $x$ and $y$ such that every $y' \in Q \setminus \{u\}$ is a non-isolated vertex of $H' - x$.

By Lemma 5.3 (A), there is a maximal path $S_1$ of $H'$ between any two distinct vertices $y', y'' \in Q \setminus \{u, y\}$. This path covers $P$ and extends in $G$, say from end vertex $y'$ with an edge to $\{a_1, a_2\}$. If $y'a_2 \in E(G)$, then $M_1 = ((y'', S_1, y'), a_2, b_1, a_1, u)$ is a maximal path of $G$. If $y'a_2, y''a_2 \notin E(G)$, then one may assume that $y'a_1, ua_2 \in E(G)$, and hence $M_2 = ((y'', S_1, y'), a_1, b_1, a_2, u)$ is a maximal path of $G$.

By Lemma 5.3 (B), $H' - x$ has a path $S_2$ between $y'$ and $y$ covering all vertices in $P \setminus \{x\}$. By a similar argument as above, we obtain that either $M_1' = ((y, S_2, y'), a_2, b_1, a_1, u)$ or $M_2' = ((y, S_2, y'), a_1, b_1, a_2, u)$ exists and is a maximal path of $G$. The lengths of the maximal paths $M_i$ and $M_i'$ are different, for $i = 1$ or 2, hence $G$ is not scenic. This contradiction concludes the proof of the proposition.

**Proposition 5.5** If $G$ is a scenic $K_{2,2} - \text{extension}$ of a $p \times q$ generic graph $H$, then either $G \cong G_6$ or $G$ is generic.

**Proof.** By definition, $G$ is generic if and only if at most one edge is missing at any vertex of $P \cup \{a_1, a_2\}$.

**Case 1:** $H$ is connected and different from $K_{2,4} - 2K_2$. Suppose that $xy \notin E(G)$, for some $x \in P$ and $y \in Q$. By Lemma 5.3 (A), there is a maximal path of $H$ between any two distinct vertices $y_1, y_2 \in Q \setminus \{y\}$. This path extends in $G$, say $y_1a_i \in E(G)$ ($i = 1$ or 2). Obviously, $y$ and $y_1$ are non-isolated vertices in $H - x$, thus by Lemma 5.3 (B), there is a path $S = (y, \ldots, y_1)$ in $H - x$ containing $P \setminus \{x\}$. The path $((y, S, y_1), (a_i, K, b_j))$ extends, hence $b_jx \in E(G)$ holds, for $j = 1$ and 2.

Let $x \in P$ be a vertex such that $xb_i \notin E(G)$, $i = 1$ or 2. We shall prove that $xb_{3-i} \in E(G)$. By the argument above, $xy \in E(G)$, for every $y \in Q$. Lemmas 5.3 (A) and 5.1 imply the existence of independent edges $ya_1, y'a_2 \in E(G)$, $y, y' \in Q$. If $y$ and $y'$ are non-isolated in $H - x$, then by Lemma 5.3 (B), $H - x$ has a path $S$ between $y$ and $y'$ which contains $P \setminus \{x\}$. The path $(b_i, a_1, (y, S, y'), a_2, b_{3-i})$ extends with $b_{3-i}x \in E(G)$.

We show that the previous argument applies even if one of $y$ and $y'$, say $y'$, is an isolated vertex of $H - x$. Note that no $y'' \in Q \setminus \{y', y\}$ is isolated in $H - x$. Thus if we can not replace $y'$ with some $y'' \in Q \setminus \{y, y\}$, and proceed as above, this is
because $y'' a_2 \notin E(G)$, for every $y'' \in Q \setminus \{y\}$. We prove that this can not happen. Because $y' x' \notin E(G)$ holds for each $x' \in P \setminus \{x\}$, there exists $u x' \in E(G)$ with $x' \in P \setminus \{x\}$ and $u \in Q \setminus \{y'\}$ such that $H - \{x', u\}$ is a connected generic graph. By Lemma 5.3 (A), the generic graph $H - \{x', u\}$ has a path $S$ between any two vertices $y_1, y_2 \in Q \setminus \{u, y'\}$ which contains $P \setminus \{x'\}$. We know that there is an edge between $\{y_1, y_2\}$ and $\{a_1, a_2\}$. By our assumption, $y_1$ or $y_2$ is adjacent to $a_1$, say $a_1 y_1 \in E(G)$. Thus the path $(u, x', b_1, a_1, (y_1, S, y_2))$ misses $a_2$, a contradiction. We conclude that at every $x \in P$ at most one edge is missing in $G$.

Next assume that $a_i y_1, a_i y_2 \notin E(G)$, for some $y_1, y_2 \in Q$ and $i = 1$ or 2. By Lemma 5.3 (A), $H$ has a path $S = (y_1, x, \ldots, y_2)$ containing $P$. Furthermore, we know that one of $y_1$ and $y_2$ sends an edge to $K$, say $y_1 a_3 - i \in E(G)$. The path $(y_1, a_3 - i, b_1, (x, S, y_2))$ is maximal in $G$ and misses $a_i$, a contradiction. Therefore $G$ is scenic.

Case 2: $H$ is disconnected. By Proposition 5.4, $H = H' + u$, where $u \in Q$ is an isolated vertex of $H$, and $H'$ is a complete bipartite graph. We may assume that $u a_j \in E(G)$ ($j = 1$ or 2). For every $i = 1, 2$, the path $(u, (a_j, K, b_i))$ extends with an edge, say $b_i x_i \in E(G)$, where $x_i \in P$.

First we show that $y a_{3 - j} \in E(G)$, for some $y \in Q \setminus \{u\}$. This is obvious if $u a_{3 - j} \notin E(G)$, because the path $(u, a_j, b_1, (x_1, H', y))$ extends with $y a_{3 - j} \in E(G)$. If $u a_{3 - j} \in E(G)$, then any maximal path $(y, \ldots, y')$ of $H'$ extends with an edge, say $y a_k \in E(G)$. Now the claim follows by choosing $j = 3 - k$, because $u a_j \in E(G)$ holds for every $i = 1, 2$, by assumption.

Our next claim is that $b_i x \in E(G)$, for every $i = 1, 2$ and $x \in P$. For any $x \in P$, $H' - x$ is a complete bipartite graph, hence it has a path $S = (x_i, \ldots, y)$ containing all vertices in $P \setminus \{x\}$. The path $(u, a_j, b_3 - i, a_{3 - j}, (y, S, x_i), b_i)$ extends with $b_i x \in E(G)$.

Suppose now that $u a_i, y a_i \notin E(G)$, for some $y \in Q \setminus \{u\}$ and $i = 1$ or 2. Let $S = (x_1, \ldots, y)$ be a path of $H'$ containing $P$. The path $(u, a_3 - i, b_1, (x_1, S, y))$ is maximal in $G$ and misses $a_i$, a contradiction. Therefore, it remains to show that if $y a_i \notin E(G)$, for some $y \in Q \setminus \{u\}$ and $i = 1$ or 2, then $y' a_i \in E(G)$, for every $y' \in Q \setminus \{u, y\}$. Let $x, x' \in P$, and let $S = (x', \ldots, y')$ be a path of $H' - \{x, y\}$ covering all vertices in $P \setminus \{x\}$. The path $(y, x, b_1, a_{3 - i}, b_2, (x', S, y'))$ extends with $y' a_i \in E(G)$. This proves that $G$ is generic and concludes the proof of the proposition.

Case 3: $H = K_{2,4} - 2K_2$. We show that if $G$ is not generic, then $G \cong G_6$.

Let $P = \{x_1, x_2\}$, $Q = \{y_1, \ldots, y_4\}$, and assume that the two missing edges are $x_1 y_3, x_2 y_4 \notin E(H)$. Suppose that $G \cong K_{4, q} - F$, and $G$ is not generic.

First we assume that one of $x_1$ or $x_2$ has degree more than 1 in $F$, say $x_2 b_2 \notin E(G)$. If $y_i a_j \in E(G)$ holds, for some $1 \leq i, j \leq 2$, then the maximal path $((b_2, K, a_j), y_1, x_1, y_4)$ would miss $x_2$. Hence there are no edges between the sets $\{a_1, a_2\}$ and $\{y_1, y_2\}$. This observation together with Lemma 5.1 imply the existence of two independent edges between the sets $\{a_1, a_2\}$ and $\{y_3, y_4\}$. Assume that
\[ a_1y_3, a_2y_4 \in E(G). \] We shall verify that there are no further edges between \( H \) and \( K \).

If \( x_2b_1 \in E(G) \), then the maximal path \( (y_1, x_1, y_4, a_2, b_1, x_2, y_2) \) misses \( a_1 \), a contradiction. If \( x_1b_j \in E(G) \) \( (j = 1 \text{ or } 2) \), then the maximal path \( (y_1, x_1, b_j, a_1, y_3, x_2, y_2) \) misses \( a_2 \), a contradiction. Assume now that one of \( a_1y_4 \) and \( a_2y_3 \) is an edge, say \( a_1y_4 \in E(G) \). The maximal path \( (y_1, x_1, y_4, a_1, y_3, x_2, y_2) \) misses \( a_2 \), a contradiction. Thus we obtain that \( G \cong G_6 \).

Second we assume that one of \( a_1 \) and \( a_2 \) has degree more than one in \( F \). Because \( G \) is not generic, and \( \{ x_1, y_1, x_2, y_2 \} \) induces a \( K_{2,2} \) in \( G \), we have \( G - \{ x_1, x_2, y_1, y_2 \} \cong G - \{ a_1, a_2, b_1, b_2 \} \cong K_{2,4} - 2K_2 \). By the symmetry of the sets \( \{ x_1, x_2 \} \) and \( \{ a_1, a_2 \} \) in \( G \), the previous argument applies, and \( G \cong G_6 \) follows.

\[ \Box \]

**Proof of Theorem 1.2.** Let \( G \) be a scenic non-traceable graph. If \( G \) has no cycle, then it is an equi-subdivided star by Proposition 2.1. Otherwise, by Theorem 2.2, \( G \) is a \( p \times q \) bipartite graph with \( p \geq 2 \) and \( q \geq p + 2 \). If \( p = 2 \) or \( 3 \) then, by Propositions 3.2 and 3.1, \( G \) is either \( G_1, G_2, G_3 \), or a connected generic graph.

From now on assume that \( p \geq 4 \). If \( G \neq G_4 \) then, by Proposition 4.1, there exists a subgraph \( K \cong K_{2,2} \) of \( G \), so that \( G \) is a scenic \( K_{2,2} \)-extension of \( H = G - V(K) \). If \( G \neq G_5 \), then by Theorem 4.2, either \( H \) is a scenic non-traceable graph or \( H \) is disconnected.

If \( H \) is a scenic non-traceable graph, then \( H \) must be generic. This follows by Proposition 3.1, for \( p = 4 \), and by Proposition 5.2, for \( p > 4 \). If \( H \) is disconnected, then by Theorem 4.2, \( H = H' + u \), where \( H' \) is scenic and \( u \) is an isolated vertex. If \( H' \) is traceable, then \( H' \cong K_{p,p+1} \), by Theorem 1.1. If \( H' \) is non-traceable, then by definition, \( H' \in \mathcal{G}_{p-2,q-3} \). By Proposition 5.4, \( H' + u \) might have a scenic \( K_{2,2} \)-extension only if \( H' \) is a complete bipartite graph. In these cases \( H \) is a disconnected \((p - 2) \times (q - 2) \) generic graph.

The previous paragraph shows that, whether or not \( H \) is connected, it must be generic. Proposition 5.5 implies that \( G \) is a connected generic graph or \( G \cong G_6 \). Consequently, every \( G \in \mathcal{G}_{4,q} \) is either \( G_4, G_5, G_6 \), or a connected generic graph. Furthermore, each graph in \( \mathcal{G}_{5,q} \) and \( \mathcal{G}_{6,q} \) is generic. Proposition 5.5 implies that the same is true for every \( \mathcal{G}_{p,q} \), \( p \geq 7 \). This concludes the proof of Theorem 1.2.

**References**


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