Asymptotics of the total chromatic number for multigraphs*

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Abstract

For loopless multigraphs, the total chromatic number is asymptotically its fractional counterpart as the latter invariant tends to infinity. The proof of this is based on a recent theorem of Kahn establishing the analogous asymptotic behaviour of the list-chromatic index for multigraphs.

The total colouring conjecture, proposed independently by Behzad [1] and Vizing [11], asserts that the total chromatic number $\chi_t$ of a simple graph exceeds the maximum degree $\Delta$ by at most two. The most recent increment (better: giant leap) toward a proof of this conjecture was made by Molloy and Reed [8], who established by probabilistic means that the difference between $\chi_t$ and $\Delta$ is at most a constant (say $c$). An immediate consequence of their result is that for simple graphs, $\chi_t$ is asymptotically its fractional analogue $\chi_t^*$ as the latter tends to infinity: for this follows from $\Delta + 1 \leq \chi_t^* \leq \chi_t \leq \Delta + c$. This leads naturally to the following question: does $\chi_t$ enjoy the same asymptotic connection with $\chi_t^*$ for loopless multigraphs (henceforth multigraphs)? That this question has an affirmative answer was conjectured in [6].

The purpose of this note is to verify that conjecture:

**Theorem 1** For multigraphs,

$$\chi_t \sim \chi_t^* \quad \text{as} \quad \chi_t^* \to \infty. \quad \text{(1)}$$

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That is, for each \( \varepsilon > 0 \) there exists \( D = D(\varepsilon) \) such that every multigraph \( G \) with \( \chi^*_t(G) > D \) satisfies

\[
(1 + \varepsilon)^{-1} \frac{\chi_t(G)}{\chi^*_t(G)} < 1 + \varepsilon.
\] (2)

This adds \( \chi_t \) to a growing list of (hyper)graph colouring invariants exhibiting "asymptotically good" behaviour, in the sense elucidated, e.g., in [3] or [6].

Pausing briefly to fix notation, we point the reader to [5, 6] for background and further motivation, and to [2] for omitted definitions. In addition to \( \chi_t \), the colouring invariants that come into play here are the chromatic index \( \chi' \) and the list-chromatic index \( \chi'_t \). Regarding these as solutions to integer programming problems leads to their fractional variants \( \chi^*_t, \chi'^*, \chi'^*_t \), namely the optimal values of the linear relaxations of the respective IP's (see [10] for omitted LP/IP terminology). We can (and will) restrict our attention to \( \chi^*_t \) and \( \chi'^* \) since \( \chi'^* = \chi'^*_t \); see [9].

The key ingredient in the proof of Theorem 1 is the following result of Kahn [4]:

**Theorem 2** For multigraphs,

\[
\chi'_t \sim \chi'^* \quad \text{as} \quad \chi'^* \to \infty.
\]

The convergence here is in the same sense as that in (1), but we again spell out the quantifiers for later reference: for each \( \gamma > 0 \) there exists \( C = C(\gamma) \) such that every multigraph \( G \) with \( \chi'^*(G) > C \) satisfies \( \chi'_t(G) < (1 + \gamma)\chi'^*(G) \).

Our proof also employs the following elementary inequalities (in (4), \( k \) is a positive constant and the multigraph needs to be non-empty):

\[
\begin{align*}
\chi^*_t & \leq \chi_t; \quad (3) \\
\chi'^*_t & \leq k\chi'^*; \quad (4) \\
\chi_t & \leq \chi'_t + 2; \quad (5) \\
\chi'^* & \leq \chi'^*_t. \quad (6)
\end{align*}
\]

**Proof of (3).** The left side is the optimal value of the linear relaxation of the IP defining the right. ■

**Proof of (4).** Kostochka proved (see, e.g., [2, p. 86]) that \( \chi_t \leq [3\Delta/2] \), but, for our needs, this is using a sledge for a finishing nail; greedy colouring yields \( \chi_t \leq 2\Delta + 1 \). Either of these bounds together with (3) and the obvious \( \Delta \leq \chi'^* \) gives (4). ■

**Proof of (5).** See, e.g., [2, p. 87]. ■

**Proof of (6).** Straightforward; see [7]. ■

In light of (3), to complete the proof of Theorem 1 it remains only to establish the right-hand inequality in (2) for arbitrary \( \varepsilon > 0 \) and sufficiently large \( \chi^*_t \). Given \( \varepsilon > 0 \), let \( \gamma = \varepsilon/2 \), and choose \( C \) so large (according to Theorem 2) that

\[
\chi'^* > C \quad \text{implies} \quad \chi'_t < (1 + \gamma)\chi'^*.
\] (7)
Let $k$ be as in (4). If $\chi_t^* > D := \max\{kC, 4k/\varepsilon\}$, then, since $\chi'^* \geq \chi^*_t/k$ (by (4)), we see that $\chi'^*$ exceeds both $C$ and $4/\varepsilon = 2/\gamma$. Thus, provided $\chi_t^* > D$, we have

$$\chi_t \leq \chi'_t + 2 < (1 + \gamma)\chi'^* + \gamma\chi'^* = (1 + \varepsilon)\chi'^* \leq (1 + \varepsilon)\chi^*_t$$

(justifying the inequalities, respectively, by: (5); the preceding sentence and (7); and (6)), as desired.

References


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