The $k$th Upper Generalized Exponent Set for the Class of Non–symmetric Primitive Matrices

Yu-Bin Gao    Yan-Ling Shao

Department of Mathematics
North China Institute of Technology
Taiyuan, Shanxi 030051, P. R. China

Abstract

Let $QB_n$ be the set of $n \times n$ ($n > 8$) non–symmetric primitive matrices with at least one pair of nonzero symmetric entries. For each positive integer $2 \leq k \leq n - 2$, we give the $k$th upper generalized exponent set for $QB_n$ by using a graph theoretical method.

1 Introduction

An $n \times n$ nonnegative matrix $A$ is called primitive if there exist some positive integer $t$ such that $A^t > 0$. The least such positive integer $t$ is called the exponent of $A$, denoted by $\gamma(A)$.

In [1], Brualdi and Liu defined the $k$th upper generalized exponent $F(A, k)$ as follows.

Definition 1.1 ([1]) Let $A$ be a primitive matrix of order $n$ and $1 \leq k \leq n - 1$. Set

$$F(A, k) = \min\{p \mid \text{no set of } k \text{ rows of } A^p \text{ has a column of all zeros} \}.$$  

$F(A, k)$ is called the $k$th upper generalized exponent of $A$.

The $k$th upper generalized exponent is a generalization of the traditional concept of the exponent. Background can be found in [1].

It is well–known that for each nonnegative matrix $A$ there exists an associated digraph $D(A)$ whose adjacency matrix has the same zero entries as $A$. A digraph $D$ is primitive iff $D$ is strongly connected and $g.c.d(r_1, r_2, \cdots, r_{\lambda}) = 1$, where $\{r_1, r_2, \cdots, r_{\lambda}\} = L(D)$ is the set of distinct lengths of the directed cycles of $D$. $A$ is primitive iff $D(A)$ is primitive.

Definition 1.2 ([1]) Let $X$ be the vertex subset of a primitive digraph $D$. The exponent $\exp_D(X)$ is the smallest positive integer $p$ such that for each vertex $y$ of $D$, there exists a walk of length $p$ from at least one vertex in $X$ to $y$.

Definition 1.3 ([1]) Let $D$ be a primitive digraph of order $n$ and $1 \leq k \leq n - 1$. Set

$$F(D, k) = \max \{ \exp_D(X) \mid X \subseteq V(D), |X| = k \}. \quad (1.1)$$

$F(D, k)$ is called the $k$th upper generalized exponent of $D$.

It is obvious that

$$F(A, k) = F(D(A), k). \quad (1.2)$$

Definition 1.4 Let $a_1, \cdots, a_k$ be positive integers. The Frobenius set $S(a_1, \cdots, a_k)$ of the numbers $a_1, \cdots, a_k$ is defined as

$$S(a_1, \cdots, a_k) = \{ \sum_{i=1}^{k} x_i a_i \mid x_1, \cdots, x_k \text{ are nonnegative integers} \}.$$ 

It is well-known, by a lemma of Schur, that if $\text{g.c.d}(a_1, \cdots, a_k) = 1$, then $S(a_1, \cdots, a_k)$ contains all sufficiently large nonnegative integers. In this case we define the Frobenius number $\phi(a_1, \cdots, a_k)$ to be the least integer $\phi$ such that $m \in S(a_1, \cdots, a_k)$ for all integers $m \geq \phi$.

For the case $k = 2$, it is well-known that if $a$ and $b$ are relatively prime positive integers, then the Frobenius number is

$$\phi(a, b) = (a - 1)(b - 1). \quad (1.3)$$

It is easy to see the following result.

Lemma 1.5 Let $X$ be a set of $k$ vertices of a primitive digraph $D$ of order $n$ and $1 \leq k \leq n - 1$. Let $R = \{ r_{i_1}, \cdots, r_{i_t} \} \subseteq L(D)$ such that $\text{g.c.d}(r_{i_1}, \cdots, r_{i_t}) = 1$. Let $d_R(i, j)$ be the length of the shortest walk from vertex $i$ to vertex $j$ in $D$ which meets at least one cycle of each length $r_{i_1}, \cdots, r_{i_t}$. Let $d_R(X) = \max_{j \in V(D)} \min_{i \in X} d_R(i, j)$ and $\phi_R = \phi(r_{i_1}, \cdots, r_{i_t})$. Then we have

$$\exp_D(X) \leq d_R(X) + \phi_R. \quad (1.4)$$

Let $QB_n$ be the set of $n \times n$ $(n > 8)$ non-symmetric primitive matrices with at least one pair of nonzero symmetric entries, $QB^+_n$ the set of matrices in $QB_n$ with nonzero trace and $QB^0_n$ the set of matrices in $QB_n$ with zero trace. For each positive integer $1 \leq k \leq n - 1$, let $E_{nk}$ be the set of $k$th upper generalized exponents of the matrices in $QB_n$, $E^+_n$ the set of $k$th upper generalized exponents of the matrices in $QB^+_n$ and $E^0_n$ the set of the $k$th upper generalized exponents of the matrices in $QB^0_n$. In this paper, we give the complete characterizations of $E^+_n$ and $E^0_n$, so that the $k$th upper generalized exponent set problem for $QB_n$ is settled.

Notice that if $k = 1$, then $F(A, k) = \gamma(A)$. In this case, the exponent sets $E^+_n$ and $E^0_n$ have already been determined in [3]. So we will only consider the cases $2 \leq k \leq n - 2$.

We will make use of the following notations. Let $D$ be an primitive digraph with $D = (V(D), E(D))$. Let $C_r$ be a cycle of length $r$ (called an $r$–cycle). We denote the distance from vertex $x$ to vertex $y$ of $D$ by $d(x, y)$. If $i, j \in V(D)$, then $(i, j)$ denotes an arc from vertex $i$ to vertex $j$ and $[i, j]$ denotes a edge between two vertices $i$ and $j$, i.e. a 2–cycle.
2 The generalized exponent set $E_{nk}^+$

In this section we will determine the generalized exponent set $E_{nk}^+$.

**Theorem 2.1** Let $n, k$ be positive integers with $2 \leq k \leq n - 2$ and $A \in QB_n^+$. Then

$$F(D(A), k) \leq 2n - k - 2. \quad (2.1)$$

**Proof.** Let $X$ be any $k$-vertex subset of $D(A)$, $w$ a loop of $D(A)$ and $[u, v]$ a edge of $D(A)$.

Case 1: $w \in X$. Then $\exp_{D(A)}(X) \leq \max_{y \in V(D(A))} d(w, y) \leq n - 1 \leq 2n - k - 2$.

Case 2: $\{u, v\} \subseteq X$. Then $\exp_{D(A)}(X) \leq \max_{y \in V(D(A))} \min\{d(u, y), d(v, y)\} \leq n - 2 < 2n - k - 2$.

Other cases: Let $l = \max_{y \in V(D(A))} d(w, y)$ and $h = \min_{x \in X} d(x, w)$. Then $l \leq n - 1$ and $h \leq n - k$.

1. $l \leq n - 2$ or $h \leq n - k - 1$. Then $\exp_{D(A)}(X) \leq h + l \leq 2n - k - 2$.
2. $l = n - 1$ and $h = n - k$. Then $\exp_{D(A)}(X) \leq n \leq 2n - k - 2$.

The proof of the theorem is completed. \hfill \Box

**Theorem 2.2** Let $n, k$ be positive integers with $2 \leq k \leq n - 2$. Then

$$\{k + 1, k + 2, \cdots, 2n - k - 2\} \subseteq E_{nk}^+. \quad (2.2)$$

**Proof.** Suppose $k + 1 \leq m \leq n - 1$. Firstly, we consider $D_1 = D(A)$ with vertex set $V(D_1) = \{1, 2, \cdots, n\}$ and arc set $E(D_1) = \{(1, 1), [1, 2], (2, 3), (3, 4), \cdots, (m - 1, m), (m, m + 1), (m, m + 2), \cdots, (m, n), (m + 1, 1), (m + 1, 2), \cdots, (n, 1)\}$.

It is obvious that $A \in QB_n^+$. Take $X_0 = \{3, 4, \cdots, k + 2\}$. It is not difficult to verify that there is no walk of length $2m - k - 1$ from any vertex of $X_0$ to the vertex $m + 1$. So we have

$$F(D_1, k) \geq \exp_{D_1}(X_0) \geq 2m - k. \quad (2.3)$$

On the other hand, let $X$ be any $k$-vertex subset of $D_1$. If $\{1, 2\} \cap X \neq \emptyset$, then

$$\exp_{D_1}(X) \leq m + 1 \leq 2m - k. \quad (2.4)$$

If $\{1, 2\} \cap X = \emptyset$, letting $i$ be the vertex of $X$ which is closest to 1, then $d(i, 1) \leq m + 1 - k - 2 + 1 = m - k$ and so

$$\exp_{D_1}(X) \leq m - k + m = 2m - k. \quad (2.5)$$

Combining (2.3), (2.4) and (2.5) we have

$$F(D_1, k) = 2m - k. \quad (2.6)$$

Next, we consider $D_2 = D(A)$ with vertex set $V(D_2) = \{1, 2, \cdots, n\}$ and arc set $E(D_2) = \{(1, 1), (2, 2), [1, 2], (2, 3), (3, 4), \cdots, (m - 1, m), (m, m + 1), (m, m + 2), \cdots, (m, n), (m + 1, 1), (m + 1, 2), \cdots, (n, 1), (m + 1, 2), (m + 2, 2), \cdots, (n, 2)\}$. 

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It is obvious that $A \in QB_n^+$. Take $X_0 = \{3, 4, \cdots, k + 2\}$. It is not difficult to verify that there is no walk of length $2m - k - 2$ from any vertex of $X_0$ to the vertex $m + 1$. Then $F(D_2, k) \geq \exp_{D_2}(X_0) \geq 2m - k - 1$.

On the other hand, let $X$ be any $k$-vertex subset of $D_2$. If $\{1, 2\} \cap X \neq \emptyset$, then $\exp_{D_2}(X) \leq m \leq 2m - k - 1$. If $\{1, 2\} \cap X = \emptyset$, letting $j$ be the vertex of $X$ which is closest to 2, then $d(j, 2) \leq m + 1 - k - 2 + 1 = m - k$ and $\exp_{D_2}(X) \leq m - k + m - 1 = 2m - k - 1$.

So we have

$$F(D_2, k) = 2m - k - 1. \quad (2.7)$$

Notice that $k + 1 \leq m \leq n - 1$. Combining (2.6) and (2.7) we obtain (2.2). \[ \blacksquare \]

**Theorem 2.3** Let $n, k$ be positive integers with $2 \leq k \leq n - 2$. Then

$$\{2, 3, \cdots, k\} \subseteq E_{nk}^+. \quad (2.8)$$

**Proof.** Suppose $2 \leq m \leq k$. We consider $D_2 = D(A)$ in theorem 2.2.

Take $X_0 = \{n, n - 1, \cdots, n - k + 1\}$. Then $|X_0| = k$. Since $n - k + 1 \geq 3$, it is not difficult to verify that there is no walk of length $m - 1$ from any vertex of $X_0$ to the vertex $m + 1$. Then $F(D_2, k) \geq \exp_{D_2}(X_0) \geq m$.

On the other hand, let $X$ be any $k$-vertex subset of $D_2$. If $1 \in X$, then $\exp_{D_2}(X) \leq m$. If $1 \notin X$, then $X \cap \{m + 1, m + 2, \cdots, n\} \neq \emptyset$ and so $\exp_{D_2}(X) \leq m$.

So we have $F(D_2, k) = m$. Noticing that $2 \leq m \leq k$, we obtain (2.8). \[ \blacksquare \]

**Theorem 2.4** Let $n, k$ be positive integers with $2 \leq k \leq n - 2$. Then

$$E_{nk}^+ = \{1, 2, 3, \cdots, 2n - k - 2\}. \quad (2.9)$$

**Proof.** We consider $D = D(A)$ with vertex set $V(D) = \{1, 2, \cdots, n\}$ and arc set $E(D) = \{(i, j) \mid i, j = 1, 2, \cdots, n\} \setminus \{(2, 1)\}$.

It is obvious that $A \in QB_n^+$ and $F(D, k) = 1$. So $1 \in E_{nk}^+$.

Combining (2.1), (2.2) and (2.8) we obtain (2.9). \[ \blacksquare \]

## 3 The generalized exponent set $E_{nk}^0$

In this section we will determine the generalized exponent set $E_{nk}^0$.

**Lemma 3.1** ([2]) Suppose $\Gamma$ is primitive digraph of order $n$ and $s$ is the length of the shortest directed cycles of $\Gamma$. Then

$$F(\Gamma, k) \leq (n - k)s + (n - s), \quad (1 \leq k \leq n - 1). \quad (3.1)$$

**Theorem 3.2** Let $n, k$ be positive integers with $2 \leq k \leq n - 2$.

1. If $n$ is even, then

$$\{11, 12, \cdots, 3n - 2k - 3\} \subseteq E_{nk}^0. \quad (3.2)$$

2. If $n$ is odd, then

$$\{11, 12, \cdots, 3n - 2k - 5, 3n - 2k - 4, 3n - 2k - 2\} \subseteq E_{nk}^0. \quad (3.3)$$
\textbf{Proof.} Firstly, let \( 4 \leq s \leq m \leq n - 1 \) and \( m - s = 0 \pmod{2} \). We consider 
\( D_1(m) = D(A) \) with vertex set \( V(D_1(m)) = \{1, 2, \cdots, n\} \) and arc set \( E(D_1(m)) = \{[1, 2], (2, 3), (2, 4), \cdots, (2, s-1), (3, s), (4, s), \cdots, (s-1, s), (s, s+1), (s+1, s+2), \cdots, \\
(m, m+1), (m, m+2), \cdots, (m, n), (m+1, 1), (m+2, 1), \cdots, (n, 1)\} \).

It is obvious that \( A \in QB_n^0 \). Let \( R = \{2, m-s+5\} \). We consider two cases.

Case 1: \( k \leq n - 4 \) and \( \max\{4, 2k-m+4\} \leq s \leq k+3 \leq m \leq n - 1 \). In this case, we will prove that

\[ F(D_1(m), k) = 3m - 2k - s + 5. \tag{3.4} \]

Take \( X_0 = \{3, 4, \cdots, s-1, s+1, s+3, \cdots, 2k-s+5\} \). Then \(|X_0| = k\) and \(2k-s+5 \leq m+1\). It is not difficult to verify that there is no walk of even length \(3m-2k-s+4\) from any vertex of \( X_0 \) to the vertex \( m+1 \). So we have
\[ F(D_1(m), k) \geq \exp_{D_1(m)}(X_0) \geq 3m - 2k - s + 5. \]

On the other hand, let \( X \) be any \( k \)-vertex subset of \( D_1(m) \). If \( \{1, 2\} \cap X \neq \emptyset \), then by (1.4) we have \( \exp_{D_1(m)}(X) \leq d(1,m+1) + \phi(2,m-s+5) \leq 3m - 2k - s + 5 \). If there are vertices \( i,j \in X \) such that \( (i,j) \in E(D_1(m)) \), then \( \exp_{D_1(m)}(X) \leq \max_{y \in V(D_1(m))} d(j,y) \leq 3m - 2k - s + 5 \). In addition, letting \( l \) be the vertex of \( X \) which is closest to 1, we have \( 1 \leq d(l,1) \leq m+1-2k+s-5+1 = m+s-2k-3 \) and \( \exp_{D_1(m)}(X) \leq d(l,1) + m-s+4 + \phi(2,m-s+5) \leq 3m - 2k - s + 5 \). So we obtain (3.4). By hypotheses we also have the following.

(i) If \( 3 \leq k \leq \frac{n-1}{2} \), then
\[ \{i \mid i \text{ is odd and } 3m-3k+2 \leq i \leq 4m-4k+1\} \subseteq E_{nk}^0, \quad (k+3 \leq m \leq 2k). \tag{3.5} \]

(ii) If \( \frac{n-1}{2} \leq k \leq n-4 \), then
\[ \{i \mid i \text{ is odd and } 3m-3k+2 \leq i \leq 4m-4k+1\} \subseteq E_{nk}^0, \quad (k+3 \leq m \leq n-1). \tag{3.6} \]

(iii) If \( 2 \leq k \leq \frac{n-1}{2} \), then
\[ \{i \mid i \text{ is odd and } 3m-3k+2 \leq i \leq 3m-2k+1\} \subseteq E_{nk}^0, \quad (2k \leq m \leq n-1). \tag{3.7} \]

Case 2: \( m = n - 1 \), \( \frac{n+1}{2} \leq k \leq n-2 \) and \( 4 \leq s \leq 2k-n+3 \). In this case, we will prove that

\[ F(D_1(n-1), k) = 3n - 2k - s + 2. \tag{3.8} \]

Take \( X_0 = \{2, 3, 4, \cdots, 2k-n+1, 2k-n+2, 2k-n+4, \cdots, n\} \). Then \(|X_0| = k\) and it is not difficult to verify that there is no walk of even length \(3n-2k-s+1\) from any vertex of \( X_0 \) to the vertex \( n \). So we have \( F(D_1(n-1), k) \geq \exp_{D_1(n-1)}(X_0) \geq 3n-2k-s+2 \).

On the other hand, let \( X \) be any \( k \)-vertex subset of \( D_1(n-1) \). There are adjacent vertices of \( D_1(n-1) \) in \( X \). Let \( l = \min\{d(j,1) \mid j \in X \text{ and there exist } i \in X \text{ such that } (i,j) \in E(D_1(n-1))\} \), which implies that \( l \leq 2n-2k-1 \). Then \( \exp_{D_1(n-1)}(X) \leq l + n - s + 3 \leq 3n - 2k - s + 2 \).
We obtain (3.8). Noticing that \(4 \leq s \leq 2k - n + 3\), we also have

\[
\{ i \ | \ i \text{ is odd and } 4n - 4k - 1 \leq i \leq 3n - 2k - 2 \} \subseteq E_{nk}^0, \quad \left(\frac{n+1}{2} \leq k \leq n - 2\right). \tag{3.9}
\]

Next, let \(4 \leq s < m \leq n - 1\) and \(m - s = 1 \pmod{2}\). We consider \(D_2(m) = D(A)\) with vertex set \(V(D_2(m)) = \{1, 2, \cdots, n\}\) and arc set \(E(D_2(m)) = \{[1, 2], (2, 3), (2, 4), \cdots, (2, s - 1), (3, s), (4, s), \cdots, (s - 1, s), (s, s + 1), (s + 1, s + 2), \cdots, (m - 1, m), (m, m + 1), (m, m + 2), \cdots, (m, n), (m + 1, 2), (m + 2, 2), \cdots, (n, 2), (m, 1)\}\).

It is obvious that \(A \in QB_n^0\). Let \(R = \{2, m - s + 4\}\). We consider two cases.

Case 1: \(k \leq n - 5\) and \(\max\{4, 2k - m + 5\} \leq s \leq k + 3 < m \leq n - 1\). In this case, we will prove that

\[
F(D_2(m), k) = 3m - 2k - s + 3. \tag{3.10}
\]

Take \(X_0 = \{3, 4, \cdots, s - 1, s + 1, s + 3, \cdots, 2k - s + 5\}\). Then \(|X_0| = k\) and \(2k - s + 5 \leq m\). It is not difficult to verify that there is no walk of odd length \(3m - 2k - s + 2\) from any vertex of \(X_0\) to the vertex \(m + 1\). So we have \(F(D_2(m), k) \geq \exp_{D_2(m)}(X_0) \geq 3m - 2k - s + 3\).

On the other hand, let \(X\) be any \(k\)-vertex subset of \(D_2(m)\). If \(\{1, 2\} \cap X \neq \emptyset\), then by (1.4) we have \(\exp_{D_2(m)}(X) \leq d(1, m + 1) + \phi(2, m - s + 5) \leq 3m - 2k - s + 3\). If there are vertices \(i, j \in X\) such that \((i, j) \in E(D_2(m))\), then \(\exp_{D_2(m)}(X) \leq \max_{y \in V(D_2(m))} d(j, y) < 3m - 2k - s + 3\). In addition, letting \(l\) be the vertex of \(X\) which is closest to 2, we have \(1 \leq d(l, 2) \leq m + 1 - 2k + s - 5 + 1 = m + s - 2k - 3\) and \(\exp_{D_2(m)}(X) \leq d(l, 2) + m - s + 3 + \phi(2, m - s + 4) \leq 3m - 2k - s + 3\).

So we obtain (3.10). By hypotheses we also have the following.

(i) If \(3 \leq k \leq \frac{n-2}{2}\), then

\[
\{ i \ | \ i \text{ is even and } 3m - 3k \leq i \leq 4m - 4k - 2 \} \subseteq E_{nk}^0, \quad (k + 4 \leq m \leq 2k + 1). \tag{3.11}
\]

(ii) If \(\frac{n-2}{2} \leq k \leq n - 5\), then

\[
\{ i \ | \ i \text{ is even and } 3m - 3k \leq i \leq 4m - 4k - 2 \} \subseteq E_{nk}^0, \quad (k + 4 \leq m \leq n - 1). \tag{3.12}
\]

(iii) If \(2 \leq k \leq \frac{n-2}{2}\), then

\[
\{ i \ | \ i \text{ is even and } 3m - 3k \leq i \leq 3m - 2k - 1 \} \subseteq E_{nk}^0, \quad (2k + 1 \leq m \leq n - 1). \tag{3.13}
\]

Case 2: \(m = n - 1, \frac{n}{2} \leq k \leq n - 2\) and \(4 \leq s \leq 2k - n + 4\). In this case, we will prove that

\[
F(D_2(n - 1), k) = 3n - 2k - s. \tag{3.14}
\]

Take \(X_0 = \{2, 3, 4, \cdots, 2k - n + 2, 2k - n + 3, 2k - n + 5, \cdots, n - 1\}\). Then \(|X_0| = k\) and it is not difficult to verify that there is no walk of odd length \(3n - 2k - s - 1\) from any vertex of \(X_0\) to the vertex \(n\). So we have \(F(D_2(n - 1), k) \geq \exp_{D_2(n-1)}(X_0) \geq 3n - 2k - s\).

On the other hand, let \(X\) be any \(k\)-vertex subset of \(D_2(n - 1)\). There are adjacent vertices of \(D_2(n - 1)\) in \(X\). Let \(l = \min\{d(j, 2) \mid j \in X \text{ and there exist } i \in\)

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X such that \((i, j) \in E(D_2(n - 1))\)}, which implies that \(l \leq 2n - 2k - 2\). Then \(\exp_{D_2(n-1)}(X) \leq l + n - s + 2 \leq 3n - 2k - s\).

So we obtain (3.14). Noticing that \(4 \leq s \leq 2k - n + 4\) we also have

\[
\{i \mid i \text{ is even and } 4n - 4k - 4 \leq i \leq 3n - 2k - 4\} \subseteq E_{nk}^0, \quad \left(\frac{n}{2} \leq k \leq n - 2\right). \tag{3.15}
\]

The theorem now follows from (3.5)–(3.7), (3.9) and (3.11)–(3.13), (3.15).

**Theorem 3.3** Let \(n\) be odd and \(2 \leq k \leq n - 2\). Then

\[
3n - 2k - 3 \in E_{nk}^0. \tag{3.16}
\]

**Proof.** We consider \(D = D(A)\) with vertex set \(V(D) = \{1, 2, \ldots, n\}\) and arc set \(E(D) = \{(1, 2), (2, 3), (3, 4), (4, 5), \ldots, (n - 1, n), \ldots, (n, 1)\}\).

It is obvious that \(A \in Q B_n^0\). Let \(R = \{2, n\}\). We will prove that

\[
F(D, k) = 3n - 2k - 3. \tag{3.17}
\]

Case 1: \(2 \leq k \leq \frac{n - 1}{2}\). Take \(X_0 = \{4, 6, \ldots, 2k + 2\}\) (if \(k = \frac{n - 1}{2}\), then \(X_0 = \{4, 6, \ldots, n - 1, 1\}\}). Then \(|X_0| = k\) and there is no walk of odd length \(3n - 2k - 4\) from any vertex of \(X_0\) to the vertex \(n\). So \(F(D, k) \geq 3n - 2k - 3\).

On the other hand, let \(X\) be any \(k\)-vertex subset of \(D\). If \(\{1, 2, 3\} \cap X \neq \emptyset\), then by (1.4) we have \(\exp_D(X) \leq n - 1 + n - 1 \leq 3n - 2k - 3\). If \(\{1, 2, 3\} \cap X = \emptyset\) and there are adjacent vertices of \(D\) in \(X\), then \(\exp_D(X) \leq n - 5 + n < 3n - 2k - 3\). If \(\{1, 2, 3\} \cap X = \emptyset\) and there are not adjacent vertices of \(D\) in \(X\), then \(k \leq \frac{n - 3}{2}\). By (1.4) we have \(\exp_D(X) \leq n - 2k - 2 + n + n - 1 = 3n - 2k - 3\).

So we obtain (3.17) for \(2 \leq k \leq \frac{n - 1}{2}\).

Case 2: \(\frac{n + 1}{2} \leq k \leq n - 2\). Take \(X_0 = \{1, 3, 4, 5, \ldots, 2k - n + 3, 2k - n + 5, \ldots, n - 1\}\). Then \(|X_0| = k\) and there is no walk of odd length \(3n - 2k - 4\) from any vertex of \(X_0\) to the vertex \(n\). So \(F(D, k) \geq 3n - 2k - 3\).

On the other hand, let \(X\) be any \(k\)-vertex subset of \(D\). There are adjacent vertices of \(D\) in \(X\). Let \(l = \min\{d(j, 1) \mid j \in X\}\) and there exist \(i \in X\) such that \((i, j) \in E(D)\), which implies that \(l \leq 2n - 2k - 2\). Then \(\exp_D(X) \leq l + n - 1 \leq 3n - 2k - s\).

So we obtain (3.17) for \(\frac{n + 1}{2} \leq k \leq n - 2\).

Now it is straightforward to obtain (3.16) from Case 1 and Case 2.

**Lemma 3.4** (4) Let digraph \(D^t\) be the digraph with the same vertex set as \(D\) in which there is an arc from \(x\) to \(y\) iff there is a walk of length \(t\) from \(x\) to \(y\) in \(D\). If \(D\) is a primitive digraph, then for any positive integer \(t\), \(D^t\) is a primitive digraph.

**Theorem 3.5** Let \(n\) be even.

1. If \(\frac{n + 4}{2} \leq k \leq n - 2\), then
   \[
   3n - 2k - 2 \in E_{nk}^0. \tag{3.18}
   \]

2. If \(2 \leq k \leq \frac{n + 2}{2}\) and \(A \in Q B_n^0\), then
   \[
   F(A, k) \leq 3n - 2k - 3. \tag{3.19}
   \]

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Proof. (1) $\frac{n+4}{2} \leq k \leq n-2$. We consider $D = D(A)$ with vertex set $V(D) = \{1, 2, \cdots, n\}$ and arc set $E(D) = \{(n-2, n-1), (n-1, n), (n, n-4), (n-3, n-4), (n-4, n-5), \cdots, (4, 3), (3, 2), (3, n-2), [2, 1], (1, n-3)\}$.

It is obvious that $A \in Q \mathcal{B}_k^0$. Take $X_0 = V(D) \setminus \{2, 4, \cdots, 2(n-k)\}$. Then $|X_0| = k$ and it is not difficult to verify that there is no walk of length $3n - 2k - 3$ from any vertex of $X_0$ to the vertex $n$. By (3.1) we have $F(D, k) = 3n - 2k - 2$. This implies that $3n - 2k - 2 \in E_{nk}^0$.

(2) $2 \leq k \leq \frac{n+2}{2}$ and $A \in Q \mathcal{B}_n^0$. Let $D$ be the associated digraph of $A$ whose shortest odd cycle length is $r$ ($3 \leq r \leq n-1$) and $C_2 = [u, v]$ the $2$-cycle of $D$. Let $X$ be an $k$-vertex subset of $D$ and $y$ any vertex of $D$. In the following we only need to prove that there is a vertex $x \in X$ and a walk of length $3n - 2k - 3$ from $x$ to $y$.

Let $q = \min\{d(u, y), d(v, y)\}$. If $q \leq n-3$, then we can take a vertex $v$ of $C_2$ such that there is a walk of length $n-3$ from $v$ to $y$. Consider that digraph $D^2$. Since $v$ is a loop of $D^2$, there is a vertex $x$ in $X$ such that there exists a walk of length $n-k$ from $x$ to $v$ in $D^2$. Hence there is a walk of length $2(n-k)$ from $x$ to $v$ in $D$. According to above arguments, there is a walk of length $2(n-k) + n-3 = 3n - 2k - 3$ from $x$ to $y$.

If $q = n-2$. Let $d(v, y) = n-2$. We consider two cases.

Case 1: There are not adjacent vertices of $D$ in $X$. Let $x_0$ be the vertex of $X$ which is closest to $v$. Then for each positive integer $p$ with $p \geq d(x_0, v) + n - 2 + \phi(2, r)$, there exists a walk of length $p$ from $x_0$ to $y$.

Subcase 1: $\{u, v\} \cap X \neq \emptyset$. If $u \in X$, then for each positive odd integer $p \geq n-1$, there is a walk of length $p$ from $u$ to $y$. This implies that there is a walk of length $3n - 2k - 3$ from $u$ to $y$. If $v \in X$, noticing that $n-2 + \phi(2, r) \leq n-2 + 2(n-k) = 3n - 2k - 4$, then there is a walk of length $3n - 2k - 3$ from $v$ to $y$.

Subcase 2: $\{u, v\} \cap X = \emptyset$ and there exists $C_r$ such that $V(C_r) \cap X = \emptyset$. Then $r \leq n-k$. Since $d(x_0, v) + n - 2 + \phi(2, r) \leq n-k + n-2 + n-k-1 = 3n - 2k - 3$, there is a walk of length $3n - 2k - 3$ from $x_0$ to $y$.

Subcase 3: $\{u, v\} \cap X = \emptyset$ and there exists $C_r$ such that $V(C_r) \cap X \neq \emptyset$. Let $|V(C_r) \cap X| = m (2 \leq m \leq k)$. Then $d(x_0, v) \leq n-k-(m-1)$. When $m < k$ we have $n-k-(r-m) \geq k-m-1$, namely, $r \leq n-2k+2m+1$.

If $m \leq k-2$, then $d(x_0, v) + n - 2 + \phi(2, r) \leq 3n - 3k + m - 1 \leq 3n - 2k - 3$. If $m = k$, then $d(x_0, v) + n - 2 + \phi(2, r) \leq n-k-(k-1)+n-2+n-2 = 3n-2k-3$. If $m = k-1$, noticing $r \neq n-1$, then $d(x_0, v) + n - 2 + \phi(2, r) \leq n-k-(k-2)+n-2+n-4 < 3n - 2k - 3$. Hence, there is a walk of length $3n - 2k - 3$ from $x_0$ to $y$.

Case 2: There are adjacent vertices of $D$ in $X$. Let $l = \min\{d(j, v) \mid j \in X\}$ and there exist $i \in X$ such that $(i, j) \in E(D)$.

Subcase 1: $l \leq 2(n-k) - 1$. Since $v \in V(C_2)$, there is a vertex $x$ in $X$ such that there exists a walk of length $2(n-k) - 1$ from $x$ to $v$. Therefore there is a walk of length $3n - 2k - 3$ from $x$ to $y$.

Subcase 2: $l = 2(n-k)$ and $r \leq 2(n-k) - 1$. Then $v \in X$ and there is a walk of length $p$ from $v$ to $y$ for each positive integer $p$ with $p \geq n-2 + \phi(2, r)$. Therefore there is a walk of length $3n - 2k - 3$ from $v$ to $y$.

Subcase 3: $l = 2(n-k)$ and $r \geq 2(n-k) + 1$. Then $v \in X$, $u \not\in X$, $k = \frac{n+2}{2}$, $r = n-1$ and $3n - 2k - 3 = 2n - 5$. It is obvious that at least one of $u$ and $v$ is on
$C_{n-1}$ for each odd cycle $C_{n-1}$. If there exists a vertex $x$ in $X$ such that $d(x, y)$ is even and $2 \leq d(x, y) \leq n - 4$, since $x$ is in $V(C_{n-1})$, then there is a walk of length $p$ from $x$ to $y$ for each positive odd integer $p \geq d(x, y) + n - 1$. This implies that there is a walk of length $2n - 5$ from $x$ to $y$. Otherwise, it is obvious that $y \in X$ and $X = V(D)\{u, i | d(i, y) \leq n - 4\}$. We consider two cases.

(a) If there exists $C_{n-1}$, such that $y \in V(C_{n-1}).$ Noticing that $y \in X$, there is a walk of length $p$ from $y$ to $y$ for each positive odd integer $p \geq n - 1$. Therefore there is a walk of length $2n - 5$ from $y$ to $y$.

(b) If $y \notin V(C_{n-1})$ for each odd cycle $C_{n-1}$. Since $D$ is a strongly connected digraph, there exists $C_m$ ($4 \leq m \leq n$), such that $y \in V(C_m)$. If $m = n$, letting $x$ be vertex such that $d(x, y) = n - 5$, then $x \in X$ and there is a walk of length $2n - 5$ from $x$ to $y$. If $m = n - 2$, letting $x$ be vertex such that $d(x, y) = n - 3$, then $x \in X$ and there is a walk of length $2n - 5$ from $x$ to $y$. If $m \leq n - 4$, then there is a walk of length $p$ from $y$ to $y$ for each positive odd integer $p \geq m + n - 1$. Therefore there is a walk of length $2n - 5$ from $y$ to $y$.

This completes the proof of the theorem.

Theorem 3.6 Let $n, k$ be positive integers with $2 \leq k \leq n - 2$. Then

$$\{4, 5, \ldots, 2n - k - 2\} \subseteq E_{nk}^0.$$  \hspace{1cm} (3.20)

Proof. Suppose $4 \leq m \leq n$. Let $D_3(m), D_4(m)$ be the digraphs of order $n$ with vertex sets $V(D_3(m)) = V(D_4(m)) = \{1, 2, \ldots, n\}$ and arc sets $E(D_3(m)) = \{[1, 2], [1, 3], [2, 3], [3, 4], [4, 5], \ldots, (m - 1, m), \ldots, (m, m + 1), (m, m + 2), \ldots, (m, n), (m + 1, 1), (m + 2, 1), \ldots, (n, 1)\}$, $E(D_4(m)) = \{[1, 2], [1, 3], [2, 3], [3, 4], [4, 5], \ldots, (m - 1, m), \ldots, (m, m + 1), (m, m + 2), \ldots, (m, n), (m + 1, 1), (m + 1, 3), \ldots, (n, 3)\}$.

It is obvious that the adjacency matrices of $D_3(m)$ and $D_4(m)$ belong to $QB_n^0$.

(1) Firstly, we will prove that if $4 \leq m \leq k + 2$ then

$$F(D_3(m), k) = m.$$ \hspace{1cm} (3.21)

Take $X_0 = \{3, 4, 5, \ldots, k + 2\}$. Then $|X_0| = k$ and it is not difficult to verify that there is no walk of length $m - 1$ from any vertex of $X_0$ to the vertex $n$. So we have $F(D_3(m), k) \geq m$.

On the other hand, let $X$ be any $k$–vertex subset of $D_3(m)$. If $\{1, 2, 3\} \cap X \neq \emptyset$, then $\exp_{D_3(m)}(X) \leq m$. If $\{1, 2, 3\} \cap X = \emptyset$, then $\{m + 1, m + 2, \ldots, n\} \cap X \neq \emptyset$ and $\exp_{D_3(m)}(X) \leq m$.

Hence (3.21) holds.

(2) Secondly, we will prove that if $k \leq n - 3$ and $k + 3 \leq m \leq n$ then

$$F(D_3(m), k) = 2m - k - 2.$$ \hspace{1cm} (3.22)

Take $X_0 = \{4, 5, \ldots, k + 3\}$. Then $|X_0| = k$ and it is not difficult to verify that there is no walk of length $2m - k - 3$ from any vertex of $X_0$ to the vertex $n$. So we have $F(D_3(m), k) \geq 2m - k - 2$.
On the other hand, let $X$ be any $k$-vertex subset of $D_3(m)$. If $\{1, 2, 3\} \cap X \neq \emptyset$, then $\exp_{D_3(m)}(X) \leq m$. If $\{1, 2, 3\} \cap X = \emptyset$, then $\exp_{D_3(m)}(X) \leq m + 1 - k - 3 + m = 2m - k - 2$.

So (3.22) holds.

(3) Thirdly, we will prove that if $k \leq n - 3$ and $k + 3 \leq m \leq n$ then

$$F(D_4(m), k) = 2m - k - 3.$$  \hspace{1cm} (3.23)

Take $X_0 = \{4, 5, \ldots, k + 3\}$. Then $|X_0| = k$ and it is not difficult to verify that there is no walk of length $2m - k - 4$ from any vertex of $X_0$ to the vertex $n$. So we have $F(D_4(m), k) \geq 2m - k - 3$.

On the other hand, let $X$ be any $k$-vertex subset of $D_4(m)$. If $\{1, 2, 3\} \cap X \neq \emptyset$, then $\exp_{D_4(m)}(X) \leq m$. If $\{1, 2, 3\} \cap X = \emptyset$, then $\exp_{D_4(m)}(X) \leq m + 1 - k - 3 + m - 1 = 2m - k - 3$.

So (3.23) holds.

The theorem now follows from (3.21), (3.22) and (3.23).}

\textbf{Theorem 3.7} If $k = 2$, then $\{2, 3\} \subseteq E^0_{nk}$. If $3 \leq k \leq n - 2$, then $\{1, 2, 3\} \subseteq E^0_{nk}$.

\textbf{Proof.} (1) Suppose $2 \leq k \leq n - 2$. Let $D(A)$ be the digraph of order $n$ with vertex set $V(D(A)) = \{1, 2, \ldots, n\}$ and arc set $E(D(A)) = \{[1, 2], [2, 3], [2, 4], \ldots, [2, n], (3, 1), (4, 1), \ldots, (n, 1)\}$.

It is obvious that $A \in QB^0_n$ and $F(D(A), k) = 2$. So $2 \in E^0_{nk}$.

(2) Suppose $2 \leq k \leq n - 2$. Let $D(A)$ be the digraph of order $n$ with vertex set $V(D(A)) = \{1, 2, \ldots, n\}$ and arc set $E(D(A)) = \{[1, 2], (2, 3), (2, 4), \ldots, (2, n), [3, 1], [4, 1], \ldots, [n, 1]\}$.

It is obvious that $A \in QB^0_n$ and $F(D(A), k) = 3$. So $3 \in E^0_{nk}$.

(3) Suppose $3 \leq k \leq n - 2$. Let $D(A)$ be the digraph of order $n$ with vertex set $V(D(A)) = \{1, 2, \ldots, n\}$ and arc set $E(D(A)) = \{(i, j) \mid i, j = 1, 2, \ldots, n$ and $i \neq j\} \setminus \{(2, 1)\}$.

It is obvious that $A \in QB^0_n$ and $F(D(A), k) = 1$. So $1 \in E^0_{nk}$.

This completes the proof of the theorem.

\textbf{Theorem 3.8} Let $n, k$ be positive integers with $2 \leq k \leq n - 2$.

(1) If $n$ is even and $2 \leq k \leq \frac{n+2}{2}$, then

$$E^0_{nk} = \{1, 2, \ldots, 3n - 2k - 3\} \setminus S.$$  \hspace{1cm} (3.24)

(2) If $n$ is even and $\frac{n+4}{2} \leq k \leq n - 2$ or $n$ is odd, then

$$E^0_{nk} = \{1, 2, \ldots, 3n - 2k - 2\} \setminus S.$$  \hspace{1cm} (3.25)

where $S = \{1\}$ when $k = 2$, otherwise $S = \emptyset$. \hspace{1cm} \blacksquare
References


(Received 3/6/98)