Covering a Bipartite Graph with Cycles Passing through Given Edges

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Abstract

We propose a conjecture: for each integer $k \geq 2$, there exists $N(k)$ such that if $G = (V_1, V_2; E)$ is a bipartite graph with $|V_1| = |V_2| = n \geq N(k)$ and $d(x) + d(y) \geq n + k$ for each pair of non-adjacent vertices $x$ and $y$ of $G$ with $x \in V_1$ and $y \in V_2$, then for any $k$ independent edges $e_1, \ldots, e_k$ of $G$, there exist $k$ vertex-disjoint cycles $C_1, \ldots, C_k$ in $G$ such that $e_i \in E(C_i)$ for all $i \in \{1, \ldots, k\}$ and $V(C_1 \cup \cdots \cup C_k) = V(G)$. If this conjecture is true, the condition on the degrees of $G$ is sharp. We prove this conjecture for the case $k = 2$ in the paper.

1 Introduction

Let $k$ be a positive integer and let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n \geq 2$. It is well known [1, 3] that if $d(x) + d(y) \geq n + 1 + k$ for each pair of non-adjacent vertices $x$ and $y$ of $G$ with $x \in V_1$ and $y \in V_2$, then for any forest $F$ with at most $k$ edges and consisting of vertex-disjoint paths of $G$, $G$ has a hamiltonian cycle passing through all the edges of $F$. We propose the following conjecture.

Conjecture A For each integer $k \geq 2$, there exists $N(k)$ such that if $G = (V_1, V_2; E)$ is a bipartite graph with $|V_1| = |V_2| = n \geq N(k)$ and $d(x) + d(y) \geq n + k$ for each pair of non-adjacent vertices $x$ and $y$ of $G$ with $x \in V_1$ and $y \in V_2$, then for any $k$ independent edges $e_1, \ldots, e_k$ of $G$, there exist $k$ vertex-disjoint cycles $C_1, \ldots, C_k$ in $G$ such that $e_i \in E(C_i)$ for all $i \in \{1, \ldots, k\}$ and $V(C_1 \cup \cdots \cup C_k) = V(G)$.

If this conjecture is true, the condition on the degrees of $G$ is sharp. To see this, let $G = (X, Y; E)$ be a bipartite graph obtained from the complete bipartite graph $K_{n-1,n}$ by adding a new vertex $x_0$ to $K_{n-1,n}$ such that $N_G(x_0) = \{x_1, x_2, \ldots, x_k\}$ where $x_1, x_2, \ldots, x_k$ are $k$ vertices of $K_{n-1,n}$ whose degrees in $K_{n-1,n}$ are $n-1$. Then for each pair of non-adjacent vertices $x$ and $y$ of $G$ with $x \in X$ and $y \in Y$, we have $x_0 \in \{x, y\}$ and $d(x) + d(y) = n + k - 1$. Let $e_1, \ldots, e_k$ be $k$ independent edges in $G$ such that $e_i$ is incident with $x_i$ for all $i \in \{1, \ldots, k\}$ and $e_1 = x_0x_1$. Clearly,
every cycle passing through \( e_1 \) must contain at least three vertices in \( \{x_0, x_1, \ldots, x_k\} \). Therefore \( G \) does not possess \( k \) vertex-disjoint cycles satisfying the requirement.

In this paper, we prove the conjecture for the case \( k = 2 \). To state the result, let \( F \) be a graph obtained from \( K_{4,4} \) by removing three independent edges from \( K_{4,4} \). We prove the following:

**Theorem B** Let \( G = (V_1, V_2; E) \) be a bipartite graph with \( |V_1| = |V_2| = n \geq 4 \). Suppose \( d(x) + d(y) \geq n + 2 \) for each pair of non-adjacent vertices \( x \) and \( y \) of \( G \) with \( x \in V_1 \) and \( y \in V_2 \). Then for any two independent edges \( e_0 \) and \( e_1 \) of \( G \), \( G \) has two vertex-disjoint cycles \( C_0 \) and \( C_1 \) such that \( e_i \in E(C_i) \) for each \( i \in \{0, 1\} \) and \( V(C_0 \cup C_1) = V(G) \), unless \( G \) is isomorphic to \( F \).

We discuss only finite simple graphs and use standard terminology and notation from [2] except as indicated. Let \( G \) be a graph. For a vertex \( u \in V(G) \) and a subgraph \( H \) of \( G \), \( N(u, H) \) is the set of neighbors of \( u \) contained in \( H \), i.e., \( N(u, H) = N_G(u) \cap V(H) \). We let \( d(u, H) = |N(u, H)| \). Thus \( d(u, G) \) is the degree of \( u \) in \( G \). For a subset \( U \) of \( V(G) \), \( G[U] \) denotes the subgraph of \( G \) induced by \( U \). Let \( e \) be an edge of \( G \). An \( e \)-subgraph of \( G \) is a subgraph \( H \) of \( G \) such that \( e \in E(H) \). If \( P \) is an \( e \)-path, we define \( \sigma(e, P) = \min(|E(P')|, |E(P'')|) \) where \( P' \) and \( P'' \) are two components of \( P - e \). If \( \sigma(e, P) = 0 \), we say \( e \) is an endedge of \( P \). We use \( l(C) \) and \( l(P) \) to denote the length of a cycle \( C \) and the length of a path \( P \), respectively. For a path \( P \) of an odd length, say \( P = x_1x_2 \ldots x_{2q} \), we define \( E_0(P) = \{x_1x_2, x_{2q-1}x_{2q}\} \cup \{x_ix_{i+1} | i = 2, 4, \ldots, 2q-2\} \) and \( E_1(P) = \{x_jx_{j+1} | j = 3, 5, \ldots, 2q-3\} \), and moreover, let \( r(e, P) = 0 \) if \( e \in E_0(P) \) and \( r(e, P) = 1 \) if \( e \in E_1(P) \).

## 2 Lemmas

The following lemmas are Ore-type lemmas in bipartite graphs. The proofs of them can be found in or easily deduced from [1, 3, 4]. Let \( G = (V_1, V_2; E) \) be a given bipartite graph in the following.

**Lemma 2.1** Let \( e \) be an edge and \( P = x_1x_2 \ldots x_{2q} \) an \( e \)-path in \( G \). Let \( y \in V(G) - V(P) \) such that \( \{x_{2q}, y\} \not\subseteq V_i \) for every \( i \in \{1, 2\} \). If \( d(x_{2q}, P) + d(y, P) \geq q + 1 + r(e, P) \), then \( G \) has an \( e \)-path \( P' \) such that \( V(P') = V(P) \cup \{y\} \). Moreover, if \( e \neq x_1x_2 \), then \( P' \) is a path from \( y \) to \( x_1 \).

**Proof.** Clearly, the lemma holds if \( yx_{2q} \in E \). So we may assume \( yx_{2q} \not\in E \). As \( d(y, P) > 0 \), it is also easy to see that if \( e = x_1x_2 \) and \( x_1x_{2q} \not\in E \), then the lemma holds. Hence we may assume that if \( e = x_1x_2 \), then \( x_1x_{2q} \not\in E \). Let \( I = \{x_{i+1}|x_ix_{2q} \in E\} \). Then \( |N(y, P) \cap I| = |N(y, P)| + |I| - |N(y, P) \cup I| \geq q + 1 + r(e, P) - q = 1 + r(e, P) \). If \( r(e, P) = 0 \) then there exists \( x_{i+1} \in N(y, P) \cap I \). Clearly, \( x_ix_{i+1} \not\in e \). On the other hand if \( r(e, P) = 1 \) then there exist \( i \) and \( j \) with \( i \neq j \) such that \( \{x_{i+1}, x_{j+1}\} \subseteq N(y, P) \cap I \). We may assume w.l.o.g. that \( x_ix_{i+1} \not\in e \). In either case, \( P' = yx_{i+1}x_{i+2} \ldots x_{2q}x_ix_{i-1} \ldots x_1 \) is the desired path.

**Lemma 2.2** Let \( e \) be an edge and \( P = x_1x_2 \ldots x_{2q} \) an \( e \)-path with \( q \geq 2 \) in \( G \). If \( d(x_1, P) + d(x_{2q}, P) \geq q + 1 + r(e, P) \), then \( G \) has an \( e \)-cycle \( C \) with \( V(C) = V(P) \).
**Proof.** Clearly, the lemma holds if \(x_1x_{2q} \in E\). So we may assume \(x_1x_{2q} \notin E\). As in the proof of Lemma 2.1, the condition implies that there exist \(x_i\) and \(x_j\) for some \(\{i, j\} \subseteq \{1, 3, \ldots, 2q - 1\}\) such that \(\{x_1x_{i+1}, x_2x_i, x_{i+1}x_{j+1}, x_2x_j\} \subseteq E\) with \(i \neq j\) if \(r(e, P) = 1\). As \(x_1x_{2q} \notin E\), we see that \(e \notin \{x_i x_{i+1}, x_j x_{j+1}\}\) if \(r(e, P) = 0\). We may assume w.l.o.g. that \(e \neq x_i x_{i+1}\) if \(i \neq j\). Then \(C' = x_1x_2 \ldots x_ix_{2q}x_{2q-1} \ldots x_{i+1}x_1\) is the desired cycle. \(\square\)

**Lemma 2.3** Let \(e\) be an edge and \(C\) an e-cycle in \(G\). Let \(y \in V(G) - V(C)\). If \(d(y, C) \geq 2\), then \(G[V(C) \cup \{y\}]\) contains an e-cycle \(C'\) such that \(l(C') < l(C)\), unless \(d(y, C) = 2\), then \(N(y, C) = \{x', x''\}\) and \(C\) has a subpath \(x'zx''\) with \(z\) not incident with \(e\).

**Proof.** Say \(C = x_1x_2 \ldots x_{2q}x_1\) with \(e = x_1x_{2q}\). Let \(\{x_i, x_j\} \subseteq N(y, C)\) such that \(1 \leq i < j \leq 2q\) and \(xy \notin E\) for all \(x \in V(C) - \{x_i, x_{i+1}, \ldots, x_j\}\). Clearly, \(C' = x_1 \ldots x_iyx_j \ldots x_{2q}x_1\) is an e-cycle. If \(l(C') \neq l(C)\), then \(j = i + 2\). This proves the lemma. \(\square\)

**Lemma 2.4** Let \(e\) be an edge, \(C\) an e-cycle and \(P\) a path with two endvertices \(u \in V_1\) and \(v \in V_2\) in \(G\) such that \(V(C) \cap V(P) = \emptyset\). Let \(l(C) = 2q\). If \(d(u, C) + d(v, C) \geq q + 1\), then \(G\) has an e-cycle \(C'\) with \(V(C') = V(C \cup P)\).

**Proof.** Let \(C = x_1x_2 \ldots x_{2q}x_1\) with \(e = x_1x_{2q}\) and \(x_1 \in V_1\). The condition implies that \(\{x_{i+1}, x_{i+1}u\} \subseteq E\) for some \(i \in \{1, 3, \ldots, 2q - 1\}\). Then \(x_1x_{2q}x_{2q-1} \ldots x_{i+1}uPvx_i\) \(x_{i-1} \ldots x_1\) is the desired cycle. \(\square\)

### 3 Proof of the Theorem

Let \(G = (V_1, V_2; E)\) be a bipartite graph with \(|V_1| = |V_2| = n \geq 4\) such that \(d(x) + d(y) \geq n + 2\) for each pair of non-adjacent vertices \(x\) and \(y\) of \(G\) with \(x \in V_1\) and \(y \in V_2\). Suppose that there exist two independent edges \(e_0\) and \(e_1\) of \(G\) such that \(G\) does not have two vertex-disjoint cycles \(C_0\) and \(C_1\) with \(e_i \in E(C_i)\) for each \(i \in \{0, 1\}\) and \(V(C_0 \cup C_1) = V(G)\). Then we shall prove that \(G\) is isomorphic to \(F\).

Say \(e_1 = uv\). Clearly, \(d(x, G - u - v) + d(y, G - u - v) \geq n + 2 - 2 = (n - 1) + 1\) for each pair of non-adjacent vertices \(x\) and \(y\) of \(G - u - v\). Thus by Lemma 2.2, \(G - u - v\) is hamiltonian. Hence \(G - u - v\) has an \(e_0\)-cycle \(C\). Choose an \(e_0\)-cycle \(C\) in \(G - u - v\) such that

\[
l(C)\text{ is minimal.} \tag{1}\]

Subject to (1), we choose \(C\) such that

\[
The length of a longest path of \(G - V(C)\) containing \(e_1\) is maximal. \tag{2}\]

Let \(P\) be a longest \(e_1\)-path in \(H\). Subject to (1) and (2), we further choose \(C\) and \(P\) such that

\[
s(e_1, P)\text{ is minimal.} \tag{3}\]

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Note that $C$ does not have a chord by (1). Let $C = x_1x_2 \ldots x_{2s}x_1$ with $x_1 \in V_1$ and $e_0 = x_1x_2$, and $H = G - V(C)$. By our assumption on $G$, $H$ does not have a hamiltonian cycle passing through $e_1$. Let $P = y_1y_2 \ldots y_m$. W.l.o.g., say $y_1 \in V_1$. We claim

**Claim 1.** $V(P) = V(H)$, i.e., $m = 2n - 2s$.

Suppose $m < 2n - 2s$. We distinguish two cases: $m$ is even or $m$ is odd.

**Case a:** $m$ is even, say $m = 2t$.

Choose a vertex $y_0$ from $H - V(P)$ such that $y_0 \in V_1$. By Lemma 2.1 and (2), $d(y_0, P) + d(y_{2t}, P) \leq t + r(e_1, P)$. Then we have $d(y_0, H) + d(y_{2t}, H) \leq \frac{1}{2}|V(H)| + r(e_1, P)$. It follows that $d(y_0, C) + d(y_{2t}, C) \leq s + 2 - r(e_1, P)$. Suppose first that $d(y_0, C) + d(y_{2t}, C) \geq s + 2$. Then we have $d(y_0, C) \geq 2$. By Lemma 2.3 and (1), we must have $d(y_0, C) = 2$, and consequently, $d(y_{2t}, C) = s$. Furthermore, $N(y_0, C) = \{x_i, x_{i+2}\}$ for some $i \in \{2, 4, \ldots, 2s - 2\}$. Then $C' = C - x_{i+1} + y_0x_i + y_0x_{i+2}$ is an $e_0$-cycle with $l(C') = l(C)$ and $P' = P + y_{2t}x_{i+1}$ is an $e_1$-path with $l(P') = l(P)$ + 1, contradicting (2). Therefore, we must have $r(e_1, P) = 1$ and $d(y_0, C) + d(y_{2t}, C) = s + 1$. It follows that $t \geq 3$ and $d(y_0, P) + d(y_{2t}, P) = t + 1$. In particular, $d(y_0, P) > 0$. If $G$ has an $e_1$-cycle $C'$ with $V(C') = V(C)$, then $C' + y_0$ has an $e_1$-path $P'$ with $V(P') = V(P) \cup \{y_0\}$, contradicting (2). Therefore by Lemma 2.2, we have $d(y_0, P) + d(y_{2t}, P) \leq t + 1$. It follows that $d(y_1, C) + d(y_{2t}, C) \geq n + 2 - t - 1 \geq s + 2$. By Lemma 2.3 and (1), $d(y_1, C) \leq 2$ and $d(y_{2t}, C) \leq 2$. We conclude that $d(y_1, C) = d(y_{2t}, C) = s = 2$. W.l.o.g., say $|V(P_1)| \leq |V(P_2)|$ where $P_1$ and $P_2$ are two components of $P - e_1$. Then $C'' = C - x_3 + y_1$ is an $e_0$-cycle with $l(C'') = l(C)$ and $P'' = P - y_1 + y_{2t}x_3$ is an $e_1$-path with $l(P'') = l(P)$ and $\sigma(e_1, P'') = \sigma(e_1, P) - 1$, contradicting (3).

**Case b:** $m$ is odd, say $m = 2t + 1$.

We have $y_{2t+1} \in V_1$. Then either $e_1 = y_{2t-1}y_{2t}$ or $e_1 = y_{2t+1}y_{2t}$ for some $i \in \{1, 2, \ldots, t\}$. W.l.o.g., say the former holds. Then $r(e_1, P - y_1) = 0$ and $\sigma(e_1, P - y_1) > 0$ if $e_1$ is on $P - y_1$. Choose $y_0$ from $H - V(P)$ such that $y_0 \in V_2$. By Lemma 2.1 and (2), if $d(y_0, P - y_1) + d(x_{2t+1}, P - y_1) \geq t + 1$, then $G$ has a path $P'$ from $y_0$ to $y_2$ such that $V(P') = V(P)$ \cup \{y_0\}$, and moreover, $P'$ is an $e_1$-path when $e_1$ is on $P - y_1$. Thus $P' + y_2y_1$ is an $e_1$-path, contradicting (2). Hence $d(y_0, P) + d(y_{2t+1}, P) = d(y_0, P - y_1) + d(y_{2t+1}, P - y_1) \leq t$. It follows that $d(y_0, C) + d(y_{2t+1}, C) \geq n + 2 - t - d(y_0, H - V(P)) \geq s + 3$. Thus $d(y_0, C) \geq 3$. By Lemma 2.3, this is in contradiction with (1). So the claim is true.

Let $t = n - s$. Then $m = 2t$ by Claim 1. We divide our proof into the following two cases: $r(e_1, P) = 0$ or $r(e_1, P) = 1$.

**Case 1:** $r(e_1, P) = 0$.

By Lemma 2.2, we have $d(y_1, P) + d(y_{2t}, P) \leq t$. Hence

$$d(y_1, C) + d(y_{2t}, C) \geq s + 2. \quad (4)$$

If $e_1 \neq y_1y_2$ and $e_1 \neq y_{2t-1}y_{2t}$, then by Lemma 2.3 and (1), $d(y_1, C) \leq 2$ and $d(y_{2t}, C) \leq 2$, and consequently, we obtain $d(y_1, C) = d(y_{2t}, C) = s = 2$ by (4).

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Then we may assume w.l.o.g. that $|V(P_1)| \leq |V(P_2)|$ where $P_1$ and $P_2$ are two components of $P - e_1$. Replacing $C$ and $P$ by $C - x_3 + y_1$ and $P - y_1 + y_2 x_3$, we obtain a contradiction with (3). Hence either $e_1 = y_1 y_2$ or $e_1 = y_{2t-1} y_{2t}$. W.l.o.g., say $e_1 = y_{2t-1} y_{2t}$.

If $t = 1$, then $s \geq 3$ as $n \geq 4$. Clearly, for any two vertices $x \in V(C) \cap V_1$ and $y \in V(C) \cap V_2$ with $xy \notin E$, we have $n + 2 \leq d(x) + d(y) \leq 6$, and consequently, this implies that $s = 3$ and $\{xy, yx_1\} \leq E$. Thus $G$ is isomorphic to $F$. Hence we may assume that $t \geq 2$.

We claim that $s = 2$. If this is not true, i.e., $s \geq 3$, then $d(y_1, C) = 2$ and $d(y_{2t}, C) = s$ by (1), (4) and Lemma 2.3. Moreover, $N(y_i, C) = \{x_i, x_{i+2}\}$ for some $i \in \{2, 4, \ldots, 2s - 2\}$. Then $C' = C - x_{i+1} + y_1 x_i + y_1 x_{i+2}$ is an e_0-cycle with $l(C') = l(C)$ and $P' = y_{2t} y_3 \ldots y_2 x_{i+1}$ is an e_1-path with $r(e_1, P') = 0$. Thus $y_2 x_{i+1} \notin E$. By Lemma 2.3 and (1), $d(y_2, C') \leq 2$ and $d(x_{i+1}, C') \leq 2$. It follows that $d(y_2, P') + d(x_{i+1}, P') \geq t + 1$. By Lemma 2.2, $G[V(P')]$ has an e_1-cycle containing all the vertices of $P'$, a contradiction. This shows $s = 2$.

By (4), we have $d(y_1, C) = 2$ and $d(y_{2t}, C) = 2$. Clearly, the theorem holds if $x_3 y_2 \notin E$. Hence we may assume $x_3 y_2 \notin E$. If $x_1 y_2 \notin E$, then we obtain $d(y_2, P') + d(x_3, P') \geq t + 1$ with $P' = y_{2t} y_3 \ldots y_2 x_3$ and $r(e_1, P') = 0$, and by Lemma 2.2, a contradiction follows. Hence we have $x_1 y_2 \notin E$.

Let $a - 1$ be the greatest integer in $\{1, 3, \ldots, 2t - 3\}$ such that $G[\{y_1, y_2, \ldots, y_{2a}\}]$ is isomorphic to $K_{a, a}$, $N(y_i, C) = \{x_i, x_{i+2}\}$ and $N(y_{i+1}, C) = \{x_i\}$ for all $i \in \{1, 3, \ldots, 2a - 1\}$. The above argument shows that $a \geq 1$. We claim $a = t - 1$. On the contrary, assume $a < t - 1$. Let $L = y_{2a+1} y_{2a+2} \ldots y_{2t}$. Clearly, $x_1 y_2 y_3 \ldots y_{2i} y_1 x_2 x_3 x_4 x_1$ is an e_0-cycle in $G$ for all $i \in \{1, 2, \ldots, a\}$. Therefore $x_1 y_2 y_3 \ldots y_{2i} y_1 x_2 x_3 x_4 x_1 \notin E$ for all $i \in \{1, 2, \ldots, a+1\}$. In particular, $G[V(L)]$ does not have a hamiltonian cycle passing through $e_1$. By Lemma 2.2, $d(y_{2a+1}, L) + d(y_{2t}, L) \leq t - a$. As $d(y_{2a+1}) + d(y_{2t}) \geq t + 4$, we see that $N(y_{2a+1}, C) \supseteq \{x_2, x_4\} \cup \{y_2, y_4, \ldots, y_{2a+2}\}$. Clearly, $C'' = x_1 x_2 y_1 \ldots y_{2a+1} x_4 x_1$ is an e_0-cycle in $G$. Let $P'' = y_{2a+2} y_{2a+3} \ldots y_{2t} x_3$. Then $G[V(P'')]$ does not have a hamiltonian cycle passing through $e_1$. In particular, $x_3 y_{2a+2} \notin E$. Since $r(e_1, P'') = 0$, we obtain $d(y_{2a+2}, P'') + d(x_3, P'') \leq t - a$ by Lemma 2.2. As $x_3 y_2 \notin E$ for all $i \in \{1, 2, \ldots, a\}$, we see that $d(y_{2a+2}, P) + d(x_3, P) \leq t + 1$, and consequently, $d(x_3, C) + d(y_{2a+2}, C) \geq 3$. However, it is clear that $d(x_3, C) + d(y_{2a+2}, C) \leq 3$. It follows that $d(y_{2a+2}, P) + d(x_3, P) = t + 1$ and $d(x_3, C) + d(y_{2a+2}, C) = 3$, and consequently, $N(y_{2a+2}) \supseteq \{x_1, y_1, y_3, \ldots, y_{2a+1}\}$. This is a contradiction to the maximality of $a$. This shows that $a = t - 1$. If $t \geq 3$, then $x_1 y_1 y_2 x_1$ and $x_3 y_4 y_3 y_4 \ldots y_{2t} x_3$ are the two desired cycles. Hence $t = 2$. Clearly, we have two desired cycles if $x_2 y_3 \in E$. So $x_2 y_3 \notin E$. As $d(x_2) + d(y_3) \geq 6$, we see that $x_4 y_3 \in E$ and therefore $G$ is isomorphic to $F$.

Case 2: $r(e_1, P) = 1$.

Say $e_1 = y_{2a+1} y_{2a+2}$ for some $2a + 1 \in \{3, 5, \ldots, 2t - 3\}$. Then either $\sigma(e_1, P) = 2a$ or $\sigma(e_1, P) = 2t - 2a - 2$. W.l.o.g., say $\sigma(e_1, P) = 2t - 2a - 2$. Let $C' = y_{2a+1} y_{2a+2} \ldots y_{2t} y_{2a+1}$ and $H' = H - V(C')$. Then $G[V(C \cup H')]$ does not have a hamiltonian cycle passing through $e_0$. It is also easy to see that for every endvertex $u$ of a hamiltonian path of $H'$, $u$ is not adjacent to a vertex of $C' - \{y_{2a+1}, y_{2a+2}\}$ for
otherwise we would have an $e_1$-path $Q$ with $V(P) = V(Q)$ and $\sigma(e_1, Q) < \sigma(e_1, P)$, contradicting (3).

Let $L = y_1 y_2 \ldots y_{2a}$. We have $d(y_1, C') \leq 1$ and $d(y_{2a}, C') \leq 1$. By Lemma 2.4, we have $d(y_1, C) + d(y_{2a}, C) \leq s$. We claim that $H'$ is hamiltonian. This is obvious if $y_1 y_{2a} \in E$. If $y_1 y_{2a} \notin E$, then $d(y_1, L) + d(y_{2a}, L) \geq t + s + 2 - s - 2 = t$, and therefore by Lemma 2.2, $H'$ is hamiltonian. So the claim is true. Thus $d(y, H') = 0$ for all $y \in V(C') - \{y_{2a+1}, y_{2a+2}\}$. If $d(y_1, L) + d(y_{2t}, L) \geq a + 1$, then there exists $i \in \{1, 3, \ldots, 2a - 1\}$ such that $\{y_1 y_{i+1}, y_i y_{2i}\} \subseteq E$, and consequently, $P' = y_{2a} y_{2a-1} \ldots y_{i+1} y_i y_{2i} y_{2i-1} \ldots y_{2a-2} y_{2a+1}$ is an $e_1$-path with $V(P') = V(P)$ and $0 = \sigma(e_1, P') < \sigma(e_1, P)$, a contradiction. This shows $d(y_1, L) + d(y_{2t}, L) \leq a$. It follows that $d(y_1, P) + d(y_{2t}, P) \leq t + 1$, and consequently, $d(y_1, C) + d(y_{2t}, C) \geq s + 1$. Similarly, we can show that $d(y_{2a}, P) + d(y_{2t-1}, P) \leq t + 1$ and $d(y_{2a}, C) + d(y_{2t-1}, C) \geq s + 1$. In particular, we have obtained $d(y_1, C) > 0$ and $d(y_{2a}, C) > 0$. By Lemma 2.3 and (1), $d(y_{2t-1}, C) + d(y_{2t}, C) \leq 4$. We obtain

\[
2a \geq d(y_1, H') + d(y_{2a}, H') \\
\geq 2(s + t + 2) - [d(y_{2t-1}) + d(y_{2t})] - [d(y_1, C \cup C') + d(y_{2a}, C \cup C')] \\
\geq 2(s + t + 2) - (2(t - a) + 4) - (s + 2) \\
= 2a + s - 2.
\]

It follows that $s = 2$, $d(y_{2t-1}, C) + d(y_{2t}, C) = 4$ and $d(y_1, C) + d(y_{2a}, C) = 2$. Since $d(y_1, C) > 0$ and $d(y_{2a}, C) > 0$, it is clear that if $y_1 x_4 \notin E$ or $y_2 x_1 \notin E$, then $G[V(C \cup L)]$ has a hamiltonian cycle containing $e_0$, a contradiction. If $\{y_1 x_4, y_2 x_1\} \subseteq E$, then $x_1 x_4 y_1 y_2 x_2 x_1$ and $C' - y_{2t-1} y_{2t} + x_3 y_{2t} + x_2 y_{2t-1}$ are the two desired cycles. This proves the theorem.

Remarks. The following example shows $N(3) \geq 7$ if $N(3)$ exists. Let $G$ be a bipartite graph obtained from $K_{6,6}$ with a bipartition $\{\{x_1, \ldots, x_6\}, \{y_1, \ldots, y_6\}\}$ by removing $x_3 y_5, x_3 y_6, x_4 y_3, x_5 y_3$ and $x_4 y_4$ from $K_{6,6}$. Clearly, $d(x) + d(y) \geq 9$ for each pair of non-adjacent vertices $x$ and $y$ of $G$ with $x \in \{x_1, \ldots, x_6\}$ and $y \in \{y_1, \ldots, y_6\}$. But $G$ does not contain three vertex-disjoint cycles passing through $x_1 y_1, x_2 y_2$ and $x_3 y_3$, respectively. Hence $N(3) \geq 7$.

As for general finite simple graphs, we proposed a conjecture in [5] and proved it for the case $k = 2$.

**Conjecture C** [5] *For each integer $k \geq 2$, there exists $N(k)$ such that if $G$ is a graph of order $n \geq N(k)$ and $d(x) + d(y) \geq n + 2k - 2$ for each pair of non-adjacent vertices $x$ and $y$ of $G$, then for any $k$ independent edges $e_1, \ldots, e_k$ of $G$, there exist $k$ vertex-disjoint cycles $C_1, \ldots, C_k$ in $G$ such that $e_i \in E(C_i)$ for all $i \in \{1, \ldots, k\}$ and $V(C_1 \cup \cdots \cup C_k) = V(G)$.*

Moreover, we know that if this conjecture is true, then the condition on the degrees of $G$ is sharp.

Note added in the proof: Conjectures A and C were verified recently for $k = 3$. However, the verification is more tedious than the above proof.

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4 References


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