On the non-existence of Steiner \((v, k, 2)\) trades with certain volumes

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Abstract

In this note, we prove that there does not exist a Steiner \((v, k, 2)\) trade of volume \(m\), where \(m\) is odd, \(2k + 3 \leq m \leq 3k - 4\), and \(k \geq 7\). This completes the spectrum problem for Steiner \((v, k, 2)\) trades.

1 Introduction

A \((v, k, t)\) trade \(T = \{T_1, T_2\}\) of volume \(m = m(T)\) consists of two disjoint collections \(T_1\) and \(T_2\), each containing \(m\) \(k\)-subsets, called blocks, of some set \(V\), such that each \(t\)-subset of \(V\) is contained in the same number of blocks in \(T_1\) and \(T_2\). The set of elements of \(V\) contained in \(T_1\) is denoted by \(F(T_1)\). Note that there may exist elements of \(V\) which occur in no block of \(T_1\). In this paper since we are not concerned with the value of \(v\) we write \((k, t)\) trade instead of \((v, k, t)\) trade.

Definition 1 A \((k, t)\) trade \(T = \{T_1, T_2\}\) is called Steiner \((k, t)\) trade if any \(t\)-subset of \(F(T_1)\) occurs at most once in \(T_1\).

Definition 2 The spectrum \(S(k, t)\) of Steiner \((k, t)\) trade is

\[ S(k, t) = \{m \mid \text{there exists a Steiner } (k, t) \text{ trade of volume } m\}. \]

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It is well-known that $S(3, 2) = \{0, 4, 6, 7, 8, \ldots\}$ (see [7]), $S(4, 2) = \{0, 6, 8, 9, 10, \ldots\}$ (see [2]) and $S(4, 3) = \{0, 8, 12, 14, 15, 16, \ldots\}$ (see [5] and references therein). In [3], Gray and Ramsay show that $S(5, 2) = \{0, 8, 10, 12, 13, 14, \ldots\}$ and $S(6, 2) = \{0, 10, 12, 14, 15, 16, \ldots\}$. They also prove that:

**Theorem 3** (See [3]) If $0 < m < 2k - 2$ or $m = 2k - 1$, then $m \notin S(k, 2)$. If $m = 0$, or $m \geq 3k - 3$, or $m$ is even and $2k - 2 \leq m \leq 3k - 4$, then $m \in S(k, 2)$.

(2) (See [4]) $2k + 1 \in S(k, 2)$ precisely when $k \in \{3, 4, 7\}$.

So for $k \geq 7$, the inclusion of odd volumes between $2k + 3$ and $3k - 4$ in $S(k, 2)$ has remained undetermined. In this note we prove there does not exist a Steiner $(k, 2)$ trade $T$ with $m = m(T)$ odd and $2k + 3 \leq m \leq 3k - 4$ for $k \geq 7$. This completely settles the spectrum problem for Steiner $(k, 2)$ trades.

## 2 Preliminary Results

First we state some results of [6] and [3].

**Definition 4** For an $s$-subset $S$ and trade $T = \{T_1, T_2\}$, let $r_S(T_1)$ be the number of blocks in $T_1$ which contain $S$. If $S = \{x\}$, we write $r_x$ for $r_{\{x\}}(T_1)$.

**Lemma 5** (See [6]) If $S$ is an $s$-subset, $1 \leq s < t$, and $T$ is a $(k, t)$ trade, then

$$r_S(T) \neq 1, \ m(T) - 1.$$  

**Lemma 6** (See [3]) Suppose $T = \{T_1, T_2\}$ is a Steiner $(k, 2)$ trade with $r_\alpha = 2$ for some $\alpha \in F(T_1)$ and $m(T) < 4k - 10$. If $B_1$ and $B_2$ are the two blocks of $T_1$ containing $\alpha$, then there exist (distinct) elements $x \in B_1$ and $y \in B_2$ such that at least $k - 1$ blocks of $T_1$ (including $B_1$) contain $x$ but not $y$, and at least $k - 1$ blocks of $T_1$ (including $B_2$) contain $y$ but not $x$.

**Lemma 7** (See [3]) Suppose $T = \{T_1, T_2\}$ is a Steiner $(k, 2)$ trade, $k > 3$, and there exist distinct elements $x, y \in F(T_1)$ such that $r_x + r_y \geq m(T)$. Then $r_x = r_y = m(T)/2$.

We also make use of the following lemma in the next sections.

**Lemma 8** Let $x, y, z$ and $k$ be integers with $k \geq 3$ and $\phi(x, y, z) = xz + yz - xy$. If

(1) $x + y + z = k - 1$; and

(2) $0 \leq x \leq y \leq z \leq k - 2$,

then $\phi(x, y, z) \geq k - 2$ if $k \neq 4, 7$ and $\phi(x, y, z) \geq k - 3$ if $k = 4$ or 7. Furthermore, these minimum values are obtained only at

(i) $(x, y, z) = (0, 1, k - 2)$ for $k = 3$ or $k \geq 8$;
(ii) \((x, y, z) \in \{(0, 1, 3), (1, 1, 2)\}\) for \(k = 5\);

(iii) \((x, y, z) \in \{(0, 1, 4), (1, 2, 2)\}\) for \(k = 6\);

(iv) \((x, y, z) = (1, 1, 1)\) for \(k = 4\); and

(v) \((x, y, z) = (2, 2, 2)\) for \(k = 7\);

**Proof**  If \(x = 0\) then \(\phi(0, y, z) = yz\), \(0 \leq y \leq z\) and \(y + z = k - 1\). So \(\phi(0, y, z) \geq k - 2\). Moreover, \(\phi(0, 1, k - 2) = k - 2\). Now let \(x \geq 1\). Then (2) becomes \(1 \leq x \leq y \leq z \leq k - 2\) and so \(k \geq 4\). From (1) we have \(z = k - 1 - x - y\) so

\[
\psi(x, y) = \phi(x, y, k - 1 - x - y) = (x + y)(k - 1 - (x + y)) - xy
\]

and \(x + 2y \leq k - 1\). If we assume \(x, y\) are real numbers and \(0 \leq \lambda \leq y - x\) then

\[
\psi(x + \lambda, y - \lambda) = (x + y)(k - 1 - (x + y)) - (x + \lambda)(y - \lambda)
\]

\[
= (x + y)(k - 1 - (x + y)) - xy + \lambda(x - y) + \lambda^2
\]

\[
= (x + y)(k - 1 - (x + y)) - xy + \lambda(x - y + \lambda)
\]

\[
\leq \psi(x, y).
\]

So the minimum of \(\psi(x, y)\) occurs at \(x = y\). Letting \(y = x\) in \(\psi(x, y)\) we find \(\psi(x, x) = -5x^2 + 2(k - 1)x\) and by (1) and (2) we have \(1 \leq x \leq (k - 1)/3\). So

\[
\psi(x, x) \geq \min(\psi(1, 1), \psi((k - 1)/3, (k - 1)/3)).
\]

Now if \(x = 1\) then

\[
\phi(1, 1, k - 3) = 2k - 7 \geq k - 2 \text{ for } k \geq 5
\]

and if \(x = (k - 1)/3\) then

\[
\phi((k - 1)/3, (k - 1)/3, (k - 1)/3) = (x - 1)^2/9 \geq k - 2 \text{ for } k \geq 9.
\]

The case \(k \in \{4, 5, 6, 7, 8\}\) is left for the reader. \(\square\)

### 3  Steiner \((k, 2)\) trades with \(k \geq 8\)

In this section we prove that for \(k \geq 8\) there does not exist a Steiner \((k, 2)\) trade \(T\) with \(m(T)\) odd and \(2k + 3 \leq m(T) \leq 3k - 4\). We begin with the following crucial lemma.

**Lemma 9**  Suppose \(T = \{T_1, T_2\}\) be a Steiner \((k, 2)\) trade with \(k \geq 8\). If there exists an \(\alpha \in F(T_1)\) with \(r_\alpha = 3\) then \(m(T) \geq 3k - 3\).

**Proof**  Let \(B_1, B_2\) and \(B_3\) be the three blocks in \(T_1\) which contain the element \(\alpha\) and let \(C_1, C_2\) and \(C_3\) be the three blocks in \(T_2\) which contain the element \(\alpha\) (see Table 1). Note that \(B_1 \cup B_2 \cup B_3 = C_1 \cup C_2 \cup C_3\).
Define $X_{ij} = (C_i \cap B_j) \setminus \{\alpha\}$ and $x_{ij} = |X_{ij}|$ for $1 \leq i, j \leq 3$. Then it follows that $\sum_{i=1}^{3} x_{ij} = k - 1$ and $\sum_{j=1}^{3} x_{ij} = k - 1$. Moreover, since $T_1$ and $T_2$ are distinct we have $x_{ij} \leq k - 2$. We also define

$$P_i = \{\{\beta, \gamma\} \mid \beta \in X_{ir}, \gamma \in X_{is}, \text{ and } 1 \leq r < s \leq 3\}$$

for $1 \leq i \leq 3$. Note that each $P_i$ is the edge set of the complete tripartite graph, $G_i$ say, with parts $X_{ij}$, $1 \leq j \leq 3$. Now let $A \in T_1 \setminus \{B_1, B_2, B_3\}$ and $P = \{\{x, y\} \mid x, y \in A\}$. Since each element of $P_i$ occurs exactly in one block of $T_1$ we have $|P \cap P_i| = 0, 1, \text{ or } 3$. Moreover, if $|P \cap P_i| = 3$ then these three pairs form a triangle. Therefore there must be at least

$$x_{i_1}x_{i_2} + x_{i_1}x_{i_3} + x_{i_2}x_{i_3} - 2(\text{maximum number of triangles in } G_i)$$

blocks in $T_1$ to cover the pairs in $P_i$. Assuming $x_{i_1} \leq x_{i_2} \leq x_{i_3}$, the maximum number of triangles in $G_i$ is $x_{i_1}x_{i_2}$. So there must be at least

$$x_{i_1}x_{i_2} + x_{i_1}x_{i_3} + x_{i_2}x_{i_3} - 2x_{i_1}x_{i_2} = x_{i_1}x_{i_3} + x_{i_2}x_{i_3} - x_{i_1}x_{i_2}$$

blocks in $T_1$ to cover the elements of $P_i$. On the other hand, no element of $P_i$ and $P_j$ can occur in the same block of $T_1$ for $i \neq j$. Now applying Lemma 8 we see

$$m(T) \geq 3 + (k - 2) + (k - 2) + (k - 2) = 3k - 3.$$ 

This completes the proof.

**Lemma 10** Suppose $T = \{T_1, T_2\}$ is a Steiner $(k, 2)$ trade, $k \geq 8$ and $m(T) \leq 3k - 4$. Then each block of $T_1$ contains an element which occurs in exactly two blocks.

**Proof** First note that if $r_\alpha = 3$ for some $\alpha \in F(T_1)$ then by Lemma 9 $m(T) \geq 3k - 3$ which is a contradiction. Now consider the block $\{a_1, a_2, a_3, \ldots, a_k\} \in T_1$. If $r_{a_i} > 3$ for all $1 \leq i \leq k$ then since each element $a_1, a_2, a_3, \ldots, a_k$ occurs at least four times in the blocks of $T_1$ and no pair of these elements can occur in more than one block of $T_1$ it follows that $m(T) \geq 3k + 1$. This is also a contradiction. So each block of $T_1$ contains an element which occurs exactly in two blocks.

**Theorem 11** Let $m$ be odd, $k \geq 8$ and $2k + 3 \leq m \leq 3k - 4$. Then there does not exist a Steiner $(k, 2)$ trade of volume $m$. 

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Proof Let $T = \{T_1, T_2\}$ be a Steiner $(k, 2)$ trade of volume $m$. By Lemma 9, $r_\alpha \neq 3$ for all $\alpha \in F(T_1)$. By Lemma 10, there exists an element $\alpha \in F(T_1)$ with $r_\alpha = 2$. Now since for $k > 7$ we have $3k - 4 < 4k - 10$, by Lemma 6 there exist distinct elements $x$ and $y$ in $F(T_1)$ such that at least $k - 1$ blocks of $T_1$ contain $x$ but not $y$, and at least $k - 1$ blocks of $T_1$ contain $y$ but not $x$, and a block from each of these collections contains $\alpha$.

(i) If there is a block $B$ in $T_1$ with $x, y \in B$, then we need at least $(k - 1) + (k - 1) + (k - 2) + 1 = 3k - 3$ blocks in $T_1$, since by Lemma 5 $r_\beta \geq 2$ for all $\beta \in F(T_1)$, which is a contradiction. So there is no block in $T_1$ containing both $x$ and $y$.

(ii) If each block of $T_1$ contains either $x$ or $y$ then $r_x + r_y \geq m(T)$. Now by Lemma 7 $r_x = r_y = m(T)/2$ which is impossible since $m$ is odd. So there is a block in $T_1$ which contains neither $x$ nor $y$.

(iii) Let $B \in T_1$ and $x, y \notin B$. By Lemma 10 there is an element $\gamma \in B$ with $r_\gamma = 2$. Now since $m(T) < 4k - 10$ by Lemma 6 there exist distinct elements $z$ and $w$ in $F(T_1)$ such that at least $k - 1$ blocks of $T_1$ contain $z$ but not $w$, and at least $k - 1$ blocks of $T_1$ contain $w$ but not $z$, and a block from each of these collections contains $\gamma$. If $\{x, y\} \cap \{z, w\} = \emptyset$ then

$$m(T) \geq (k - 1) + (k - 1) + (k - 3) + (k - 3) > 3k - 4$$

for $k \geq 8$.

So without loss of generality we can assume $\{x, y\} \cap \{z, w\} = \{x\}$, say $x = w$. By (i) the pair $\{x, z\}$ cannot appear in any block of $T_1$. If the pair $\{y, z\}$ does not appear in any block of $T_1$ then

$$m(T) \geq (k - 1) + (k - 1) + (k - 1) = 3k - 3 > 3k - 4.$$

So let $\{y, z\}$ appears in a block of $T_1$. This forces that $m(T) = 3k - 4$ (so $k$ is odd), $r_x = r_y = r_z = k - 1$ and for any block $A \in T_1$ we have $A \cap \{x, y, z\} \neq \emptyset$. Now for any element $\delta \in F(T_1) \setminus \{x, y, z\}$ we must have $r_\delta = 2$. Therefore

$$k(3k - 4) = (k - 1) + (k - 1) + (k - 1) + 2(|F(T_1)| - 3).$$

This is also impossible since left hand side is odd and right hand side is even. This completes the proof. \qed

4 Non-existence of a Steiner $(7, 2)$ trade of volume $17$

In this section we prove that there does not exist a Steiner $(7, 2)$ trade of volume 17. So by $[3]$ and $[4]$ $S(7, 2) = \{0, 12, 14, 15, 18, 19, 20, \ldots\}$.

Lemma 12 Let $T = \{T_1, T_2\}$ be a Steiner $(7, 2)$ trade with $m(T) = 17$. If there exists an $\alpha \in F(T_1)$ with $r_\alpha = 3$ then $r_x \geq 3$ for all $x \in F(T_1)$.

Proof Let $B_1$, $B_2$ and $B_3$ be the three blocks in $T_1$ which contain the element $\alpha$ and let $C_1$, $C_2$ and $C_3$ be the three blocks in $T_2$ which contain the element $\alpha$. 

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Note that \( B_1 \cup B_2 \cup B_3 = C_1 \cup C_2 \cup C_3 \). Let \( X_{ij}, x_{ij} \) and \( P_i \) for \( 1 \leq i, j \leq 3 \) be defined as in Lemma 9. So we have \( \sum_{i=1}^{3} x_{ij} = 6 \) and \( \sum_{j=1}^{3} x_{ij} = 6 \). Applying Lemma 8 and the fact \( m(T) = 17 \) forces \( x_{ij} = 2 \) for \( 1 \leq i, j \leq 3 \). This implies that the blocks in \( T_1 \) have one of the two structures as shown in Table 2. Note that for both structures \( C_1 = \{\alpha, 1, 2, 7, 8, 13, 14\}, C_2 = \{\alpha, 3, 4, 9, 10, 15, 16\}, \) and \( C_3 = \{\alpha, 5, 6, 11, 12, 17, 18\}. Moreover \( r_x \geq 3 \) for \( x \in \{1, 2, 3, \ldots, 18\} \).

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Table 2

**Case 1** Let the blocks in \( T_1 \) have Structure 1.

(i) If a block contains an element which occurs exactly in three blocks it cannot contain an element which occurs in more than five blocks.

(ii) There are at least five elements in \( B_1 \) which occur in exactly three blocks.

**Case 2** Let the blocks in \( T_1 \) have Structure 2.

(i) If a block contains an element which occurs exactly in three blocks it cannot contain an element which occurs in more than four blocks.

(ii) There are at least five elements in \( B_1 \) which occur in exactly three blocks.

Now let \( \alpha, \beta, \gamma \in F(T_1) \) with \( r_\alpha = r_\beta = r_\gamma = 3 \). Then we need at least six blocks in \( T_1 \) for these three elements. Now let \( r_\delta = 2 \) for some \( \delta \in F(T_1) \). Since \( m(T) = 17 < 4.7 - 10 = 18 \) by Lemma 6 there exist (distinct) elements \( x \) and \( y \) in \( F(T_1) \) such that at least 6 blocks of \( T_1 \) contain \( x \) but not \( y \), and at least 6 blocks of \( T_1 \) contain \( y \) but
not $x$. So by (i) $m(T) \geq 6 + 6 + 6$ which is a contradiction. Therefore, if $r_\alpha = 3$ for some $\alpha \in F(T_1)$ then $r_x \geq 3$ for all $x \in F(T_1)$. This completes the proof. \[ \square \]

In a similar manner to Theorem 11 we prove the following lemma.

**Lemma 13** Let $T = \{T_1, T_2\}$ be a Steiner $(7, 2)$ trade with $m(T) = 17$. Then $r_x \geq 3$ for all $x \in F(T_1)$.

**Proof** Let $r_\alpha = 2$ for some $\alpha \in F(T)$. Since $m(T) = 17 < 18 = 4.7 - 10$ then by Lemma 6 there exist (distinct) elements $x$ and $y$ in $F(T_1)$ such that at least 6 blocks of $T_1$ contain $x$ but not $y$, and at least 6 blocks of $T_1$ contain $y$ but not $x$.

(i) If there is a block $B$ in $T_1$ with $x, y \in B$ then since by Lemma 5 $r_\alpha \geq 2$ for all $\alpha \in F(T_1)$ we need at least $6 + 6 + 5 + 1 = 18$ blocks in $T_1$ which is impossible. So there is no block in $T_1$ containing both $x$ and $y$.

(ii) If each block of $T_1$ contains either $x$ or $y$ then $r_x + r_y \geq m(T)$. Now by Lemma 7 $r_x = r_y = 17/2$ which is impossible. So there is a block in $T_1$ which contains neither $x$ nor $y$.

(iii) Let $B \in T_1$ and $x, y \notin B$. If each element of $B$ occurs in more than three blocks then $m(T) \geq 3.7 + 1 = 22$ which is impossible. Moreover, by Lemma 12 and the fact that $F(T_1)$ has an element which occurs in exactly two blocks, $B$ contains no element which occurs in exactly three blocks. Therefore there is an element $\beta \in B$ with $r_\beta = 2$. So by Lemma 6 there exist (distinct) elements $z$ and $w$ in $F(T_1)$ such that at least 6 blocks of $T_1$ contain $z$ but not $w$, and at least 6 blocks of $T_1$ contain $w$ but not $z$. If $\{x, y\} \cap \{z, w\} = \emptyset$ then $m(T) \geq 6 + 6 + 4 + 4 = 20$. So without loss of generality we can assume $\{x, y\} \cap \{z, w\} = \{x\}$, say $x = w$. By (i) the pair $\{x, z\}$ cannot appear in any block of $T_1$. If the pair $\{y, z\}$ does not appear in any block of $T_1$ then

$$m(T) \geq 6 + 6 + 6 = 18 > 17.$$ 

So let $\{y, z\}$ appears in a block of $T_1$. Since $m(T) = 17$ we have $r_x = r_y = r_z = 6$ and for any block $A \in T_1$ we have $A \cap \{x, y, z\} \neq \emptyset$. Now for any element $\gamma \in F(T_1) \setminus \{x, y, z\}$ we must have $r_\gamma = 2$. Therefore

$$7 \cdot 17 = 119 = 6 + 6 + 6 + 2(|F(T_1)| - 3).$$

This is also impossible since left hand side is odd and right hand side is even. This completes the proof. \[ \square \]

The proof of the following lemma is left for the reader.

**Lemma 14** Let $T = \{T_1, T_2\}$ be a Steiner $(7, 2)$ trade with $m(T) = 17$.

(i) Any block of $T_1$ contains at most two elements which occur in more than three blocks.

(ii) If a block $B \in T_1$ contains two elements which occur in more than three blocks then $A \cap B \neq \emptyset$ for any block $A \in T_1$.

**Theorem 15** There does not exist a Steiner $(7, 2)$ trade with volume 17.
**Proof** Let $T = \{T_1, T_2\}$ be a Steiner $(7,2)$ trade with $m(T) = 17$. Then by Lemmas 12 and 13 the blocks of $T_1$ have one of the two structures as shown in Table 2.

**Case 1** Let $T_1$ have Structure 1 as shown in Table 2. By Lemma 14 part (i), $|B_{i6} \cap \{1,2,3,\ldots,18\}| = i$, where $0 \leq i \leq 2$. It is straightforward to check that if $i = 0, 1$ then $m(T) > 17$. If $i = 2$ then by Lemma 14 part (ii), $B_{i6} \cap B_j \neq \emptyset$ for $1 \leq j \leq 17$. This is impossible since $B_1 \cap B_2 \cap B_3 = \{a\}$.

**Case 2** Let $T_1$ have Structure 2 as shown in Table 2. By Lemma 14 part (i) $B_4 \cap \{1,2,3,\ldots,18\} = \{1,7\}$. So $B_4 \cap B_j \neq \emptyset$ for $1 \leq j \leq 17$. This is impossible since $B_3 \cap B_4 = \emptyset$.

\[\square\]

## 5 Conclusion

Here we summarize the results on the spectrum of Steiner $(k,2)$ trades.

**Theorem 16** There exists a Steiner $(k,2)$ trade of volume $m$ if and only if

1. $m = 0$;
2. $m \geq 3k - 3$;
3. $m$ is even and $2k - 2 \leq m \leq 3k - 4$; or
4. $m = 2k + 1$ when $k \in \{3,4,7\}$.

The result of this paper contributes to the understanding of Steiner 2-designs for (at least) the following reasons.

1. It lays the foundation for solving the intersection problem for these designs, which heretofore has only been solved for block sizes 3 and 4. (See [1] for a survey on intersection problem.)

2. It aids in the investigation of defining sets (see [8]) for these designs. A defining set must meet every trade; so every bit of information about possible trades gives another small step towards understanding the seemingly unfathomable secrets of defining sets.

3. In practical applications of design theory (e.g. design of experiments), many appropriate designs can be found. As the experiment progresses, an additional constraint on the design might surface. Do we have to scrap it all and start over with a new design, or can we just wiggle the existing design a bit, so as to satisfy the new constraint, and only have to repeat some of the trials? Exactly what is needed here is a small trade. Our results here indicate just what kind of small trades are possible.

**Open problem.** What is the spectrum of Steiner $(k,3)$ trades?
References


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