On Finite Graphs that are Self-complementary and Vertex-transitive*

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Abstract

Self-complementary graphs have been extensively investigated and effectively used in the study of Ramsey numbers. It is well-known and easily proved that there exist regular self-complementary graphs of order $n$ if and only if $n \equiv 1 \pmod{4}$. So it is natural to ask whether a similar result for self-complementary vertex-transitive graphs exists. More precisely, for which positive integers $n$ do there exist self-complementary vertex-transitive graphs of order $n$? Recently, several long-standing open problems concerning this question have been solved, but the general question is still open. In this article we give a brief survey in this area and introduce several new problems.

1 Introduction

The complement $ar{\Gamma}$ of a graph $\Gamma$ is the graph with vertex set $V \bar{\Gamma}$ such that two vertices $x, y$ are adjacent if and only if $x, y$ are not adjacent in $\Gamma$. A graph is said to be self-complementary if it is isomorphic to its complement. The study of self-complementary graphs has a rich history (see for instance [10, 11, 14, 22, 26, 28, 31]). Various enumeration and constructive results for such graphs have been obtained (see for example [11, 13, 28]), and various graph theoretic properties of such graphs have been investigated (see for example [8, 27, 33]). In a remarkable paper [21], L. Lovász proved that the Shannon capacity of a vertex-transitive self-complementary graph of order $n$ is equal to $\sqrt{n}$. Also this class of graphs has been used to study Ramsey numbers (see for instance [10, 12, 16, 29]).

A graph $\Gamma$ is said to be vertex-transitive if $\text{Aut} \Gamma$ is transitive on $V \Gamma$. For a finite group $G$, let $G^\# = G \setminus \{1\}$, the set of all non-identity elements of $G$. For a subset $S \subseteq G^\#$, the Cayley digraph $\text{Cay}(G, S)$ is the digraph $\Gamma$ with vertex set $G$ such that

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$(x, y)$ is an edge of $\Gamma$ if and only if $yx^{-1} \in S$. If $S = S^{-1} := \{ s^{-1} \mid s \in S \}$, then Cay$(G, S)$ may be viewed as an undirected graph, and if $G$ is cyclic then Cay$(G, S)$ is a circulant. It is easily checked that the group $G$ acting by right multiplication (that is, $g : x \rightarrow xg$) is a subgroup of Aut$\Gamma$ and acts regularly on $V\Gamma = G$. In particular, Cayley graphs are vertex-transitive. However, the converse is not necessarily true (see, for example, [23]). The objective of this article is the study of graphs that are both self-complementary and vertex-transitive.

One of the principal motivations for studying self-complementary vertex-transitive graphs is to investigate Ramsey numbers. In the study of diagonal Ramsey numbers, one wishes to construct a 'big' graph $\Gamma$ such that neither $\Gamma$ nor $\overline{\Gamma}$ contains $K_m$ for a given $m$ so that $\Gamma$ and $\overline{\Gamma}$ should not be very different. Moreover, as $K_m \not\subset \Gamma$ is a global property of $\Gamma$, the edges of $\Gamma$ should be distributed 'homogeneously'. In the extreme case, $\Gamma$ is self-complementary and vertex-transitive. This is one of the reasons why more constructions of self-complementary vertex-transitive graphs is desirable.

Recently, several long-standing open questions regarding self-complementary vertex-transitive graphs have been solved, and some new open problems have naturally arisen. In this paper, we shall give a brief survey on this topic. For convenience, in the ensuing sections we shall call a graph $\Gamma$ an SCVT-graph if $\Gamma$ is self-complementary and vertex-transitive.

2 Constructing SCVT-graphs

It follows from the definition that if

$$\Gamma = \text{Cay}(G, S), \text{ and } S^\sigma = G^\# \setminus S \text{ for some } \sigma \in \text{Aut}(G),$$

then $\Gamma$ is self-complementary. We shall see that this property can produce many SCVT-graphs. In fact, all known constructions of SCVT-graphs are based on this property. For brevity, we shall say a group $G$ and $\sigma \in \text{Aut}(G)$ with this property is an R-group and an R-automorphism, respectively (R stands for Reflexible, refer to [14, p.627]). We shall also call such a graph Cay$(G, S)$ an SC-CI-graph of $G$ (where CI stands for Cayley Isomorphism, which comes from the isomorphism problem for Cayley graphs, refer to [2]).

In [32], all reflexible abelian groups are characterized, and non-abelian groups of order prime-cube are investigated. It was proved in [3] that for an R-automorphism $\sigma$, $(\sigma^2)$ has $2m$ orbits $\Delta_1, \Delta_2, \ldots, \Delta_{2m}$ on $G^\#$ for some positive integer $m$ such that $\Delta_i^\sigma = \Delta_{i+1}$ for all odd $i$. Using these simple properties, a method of constructing SCVT-graphs can be described as follows (see [3]).

Construction 2.1 Let $\mathcal{C}(G, \sigma)$ be the class of the graphs which are constructed using the following steps:

(i) find an R-group $G$ and an R-automorphism $\sigma$ of $G$;

(ii) find $(\sigma^2)$-orbits $\Delta_1, \Delta_2, \ldots, \Delta_{2m}$ on $G^\#$ such that $\Delta_i^\sigma = \Delta_{i+1}$ for all odd $i$;
(iii) set $S = S(\varepsilon_1, \varepsilon_3, \ldots, \varepsilon_{2m-1}) = \bigcup_{\text{odd } i} \Delta_{i}^{\varepsilon_i}$ where $\varepsilon_i = 0$ or $1$, and set $\Gamma = \text{Cay}(G, S)$.

Since $(\Delta_{i}^{\varepsilon_i})^\sigma = \Delta_{i}^{\varepsilon_i - \varepsilon_i}$, we have $S^\sigma = \bigcup_{\text{odd } i} \Delta_{i}^{\varepsilon_i - \varepsilon_i} = G^\# \setminus S$ so that $\Gamma$ is self-complementary. Together with the next simple proposition, Construction 2.1 can produce various SCVT-graphs. An automorphism $\sigma$ of a group $G$ will be said to be fixed-point free if $\sigma$ fixes no element of $G^\#$.

**Proposition 2.2** ([3, Proposition 2.3]) Suppose that $G$ is a finite group of odd order.

1. Let $\sigma \in \text{Aut}(G)$ be such that $\sigma^2$ is fixed-point free and each $g \in G$ is conjugate under $\langle \sigma^2 \rangle$ to $g^{-1}$. Then $G$ is an $R$-group and $\sigma$ is an $R$-automorphism of $G$.

2. Let $G = G_1 \times G_2 \times \ldots \times G_s$ be such that each $G_i$ is an $R$-group and $\sigma_i$ is an $R$-automorphism of $G_i$ of order 4. Then $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_s) \in \text{Aut}(G)$ is an $R$-automorphism of $G$.

The next are some examples of constructing SCVT-graphs.

**Example 2.3** (Sachs [31]) Let $p$ be a prime such that 4 divides $p-1$, and let $G \cong \mathbb{Z}_{p^d}$ where $d \geq 1$. Then $\text{Aut}(G) \cong \mathbb{Z}_{p^{d-1}(p-1)}$, and by Proposition 2.2 (1), each $\sigma \in \text{Aut}(G)$ of order divisible by 4 is an $R$-automorphism of $G$. Therefore, by Construction 2.1 and Proposition 2.2, if $n = p_1^{d_1} p_2^{d_2} \ldots p_s^{d_s}$ such that 4 divides $p_i - 1$ for each $i$ then there exist circulant SC-CI-graphs of order $n$.

**Example 2.4** (see Zelinka (1979), Suprunenko (1985) or Rao (1985)) Let $p$ be a prime, and let $G \cong \mathbb{Z}_p^d$ such that 4 divides $p^d - 1$. Then $\text{GL}(1, p^d) < \text{Aut}(G)$, and by Proposition 2.2 (1), each $\sigma \in \text{GL}(1, p^d)$ of order divisible by 4 is an $R$-automorphism of $G$. Therefore, by Construction 2.1 and Proposition 2.2, if $n = p_1^{d_1} p_2^{d_2} \ldots p_s^{d_s}$ such that 4 divides $p_i^{d_i} - 1$ for all $i$ then there exist SC-CI-graphs of order $n$.

The graphs given in Examples 2.3 and 2.4 are all Cayley graphs on abelian groups. The first family of self-complementary Cayley graphs on non-nilpotent groups was constructed in [3] as follows.

**Proposition 2.5** Let $p$ be an odd prime and let $q$ be a primitive prime of $p^4 - 1$. Let $G = \mathbb{Z}_p^4 \rtimes \mathbb{Z}_q \leq \text{AGL}(1, p^4)$. Then there exist Cayley graphs on $G$ which are self-complementary.

## 3 Self-complementary circulants

By Example 2.3, there exist self-complementary circulants of order $n$ for all $n$ each of whose prime divisors is congruent to 1 modulo 4. H. Sachs (1962) posed a conjecture that only for such $n$ do there exist self-complementary circulants of order $n$. The following theorem is a solution of this conjecture.
Theorem 3.1 (Fronček, Rosa and Širáň (1996)) For a positive integer $n$, there exist self-complementary circulants of order $n$ if and only if all prime divisors of $n$ are congruent to 1 modulo 4.

The proof of the theorem in [14] is by counting the so-called closed walks, which is graph theoretic. In [4], a group theoretic proof is given, which is based on the well-known result of Burnside and Schur that the cyclic groups of composite order are Burnside groups (see [34, Theorem 25.3]).

Therefore, for all possible orders $n$, Construction 2.1 can produce some self-complementary circulants of order $n$. Further, by Muzychuk (1995), Construction 2.1 can produce all self-complementary circulants of square-free orders. The special cases of orders $p$ and $pq$ ($p, q$ are distinct primes) are well studied, and enumeration results for self-complementary circulant digraphs of such orders are obtained using Polya’s theorem in [6, 7, 9], also refer to B. Alspach’s review for [7]. Here we only mention a simple consequence of Construction 2.1, which is given in [3].

Corollary 3.2 Let $p$ be a prime such that 4 divides $p - 1$, and write $p - 1 = 2^t t$ such that $t$ is odd. Then there are

$$\sum_{2^k \leq t} 2^{(p-1)/2^k}$$

SCVT-graphs of order $p$.

For self-complementary circulants of prime-power order, a characterization result is obtained in [3], which is based on Construction 2.1 and the wreath product of graphs. The wreath product $\Gamma_1 \wr \Gamma_2$ of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is the graph with vertex set $V_1 \times V_2$ such that $\{(a_1, a_2), (b_1, b_2)\}$ is an edge if and only if either $\{a_1, b_1\} \in E_1$ or $a_1 = b_1$ and $\{a_2, b_2\} \in E_2$.

Theorem 3.3 ([3, Theorem 1.5]) Let $\Gamma$ be a self-complementary circulant of order $p^n$ for some prime $p$ and some $n \geq 2$. Then one of the following holds:

(i) $\Gamma \in C(Z_{p^n}, \sigma)$ for some $\sigma \in \text{Aut}(Z_{p^n})$ whose order divides $p - 1$;

(ii) $\Gamma = \Gamma_0 \wr \Gamma_1$, where $\Gamma_0$ and $\Gamma_1$ are self-complementary circulants of order $p^{n_0}$ and $p^{n_1}$, respectively, such that $n_0 + n_1 = n$.

4 The existence problem

It is well-known and easily proved that there exists a self-complementary graph of order $n$ if and only if $n$ is congruent to 0 or 1 modulo 4, and that there exists a regular self-complementary graph of order $n$ if and only if $n$ is congruent to 1 modulo 4.

Since vertex-transitive graphs are regular, the following natural question was proposed by Zelinka (1979).
Question 4.1 Is it true that for every \( n \equiv 1 \pmod{4} \) there exists an SCVT-graph of order \( n \)?

In [14, Section 4], Fronček, Rosa and Širáň in [14] asked the same question for a product of two distinct primes \( p \) and \( q \), that is, whether there exist SCVT-graphs of order \( pq \) where \( p \) and \( q \) are congruent to 3 (mod 4)? These questions had been open until very recently when a negative answer was given by the author in [19] as follows.

**Theorem 4.2** Let \( n = pq \) where \( p, q \) are distinct primes. Then there exist SCVT-graphs of order \( n \) if and only if \( p, q \equiv 1 \pmod{4} \).

The proof of this theorem in [19] was naturally divided into two steps, and depends on the following lemma.

**Lemma 4.3** Let \( \Gamma \) be an SCVT-graph. Let \( G \) be a subgroup of \( \text{Aut} \Gamma \) which is transitive on \( V \Gamma \). Suppose that \( G \) is of rank \( r \) and has subdegrees \( d_0, d_1, d_2, \ldots, d_{r-2}, d_{r-1} \) which satisfy \( d_i \leq d_{i+1} \) for all \( i \in \{0, 1, \ldots, r - 2\} \). Then \( r \) is odd, \( d_0 = 1 \) and \( d_{i+1} = d_i \) for every odd \( i \in \{1, 2, \ldots, r - 2\} \).

The first step excludes the vertex-primitive case by checking the classification of vertex-primitive graphs of order \( pq \) obtained by Praeger and Xu in [25]. We found that there do not exist vertex-primitive graphs of order \( pq \) which are self-complementary. The second step treats the vertex-imprimitive case by induction on the graph orders. We notice that the classification of vertex-primitive graphs of order \( pq \) of Praeger and Xu is dependent on the classification of finite simple groups, and so is Theorem 4.2.

Therefore, by Theorem 4.2, there do not exist SCVT-graphs of order \( n \) for each \( n \equiv 1 \pmod{4} \). The existence problem for self-complementary vertex-transitive or Cayley graphs naturally arises here (refer to [14, Section 4]).

**Problem 4.4** Determine positive integers \( n \) such that there exist self-complementary vertex-transitive or Cayley graphs of order \( n \).

Let \( SCVT \) and \( SCCay \) denote the set of positive integers \( n \) for which there exist SCVT-graphs and self-complementary Cayley graphs of order \( n \), respectively. Note that Cayley graphs are vertex-transitive. By Example 2.4 and Theorem 4.2, we have

\[
C \subseteq SCCay \subseteq SCVT \subseteq D
\]

where \( C = \{ p_1^{d_1} \cdots p_r^{d_r} | p_i^{d_i} \equiv 1 \pmod{4} \} \) and \( D = \{ n \mid n \equiv 1 \pmod{4} \} \).

For Cayley graphs, Fronček, Rosa and Širáň in [14] proved

**Theorem 4.5** Let \( n = ps \), where \( p \equiv 3 \pmod{4} \) is a prime and \( p > s \). Then no Cayley graphs of order \( n \) are self-complementary.

The general case for Problem 4.4 seems very hard. To approach it, we are particularly interested in the following extreme cases (see [19, Section 3]).
Question 4.6

(1) Does $SCCay = C$, in others words, do there exist self-complementary Cayley graphs of order $n$ with $n \notin C$?

(2) Does $SCVT = SCCay$, in other words, do there exist self-complementary vertex-transitive graphs which are not Cayley graphs?

5 SC-CI-graphs

In Section 2, several families of SCVT-graphs are produced by Construction 2.1, which are all SC-CI-graphs. The question of whether or not every self-complementary Cayley graph $Cay(G, S)$ is an SC-CI-graph of $G$ had been open until very recently when a negative answer was given by R. Jajcay and the author in [17] by producing infinite families of counterexamples. The counterexamples in [17] are constructed using the wreath product of certain self-complementary Cayley graphs.

Theorem 5.1 Let $p$ be a prime such that $4$ divides $p - 1$, and let $\sigma_1, \sigma_2 \in \text{Aut}(Z_p)$ such that $4$ divides both $o(\sigma_1)$ and $o(\sigma_2)$. Set $\Gamma_i = Cay(Z_p, \sigma_i)$ with $i = 1, 2$, and $\Gamma = \Gamma_1 \wr \Gamma_2$. Then $\Gamma$ is a Cayley graph of $Z_{p^2}$ which is self-complementary, and further $\Gamma$ is an SC-CI-graph of $Z_{p^2}$ if and only if $o(\sigma_1) = o(\sigma_2)$. In particular, if $8$ divides $p - 1$ then there exist self-complementary Cayley graphs $Cay(Z_{p^2}, S)$ which are not SC-CI-graphs of $Z_{p^2}$.

Theorem 5.1 provides self-complementary Cayley graphs $\Gamma$ on $Z_{p^2}$ which are not SC-CI-graphs of $Z_{p^2}$. The following theorem characterizes self-complementary graphs of prime-square order.

Theorem 5.2 ([3]) Let $\Gamma$ be an SCVT-graph of order $p^2$ where $p$ is a prime. Then one of the following holds:

(i) $\Gamma = \Gamma_0 \wr \Gamma_1$ where $\Gamma_i \in C(Z_p, \sigma_i)$ for some $\sigma_i \in \text{Aut}(Z_p)$;

(ii) $\Gamma \in C(Z_{p^2}, \sigma)$ for some $\sigma \in \text{Aut}(Z_{p^2})$ such that $o(\sigma)$ divides $p - 1$;

(iii) $\Gamma \in C(Z_{p^2}, \sigma)$ for some $\sigma \in \text{GL}(2, p)$ such that $o(\sigma)$ divides $p - 1$ or $p + 1$.

By results of [18, 24], the graph $\Gamma = \Gamma_0 \wr \Gamma_1$ in part (i) is also a Cayley graph on $Z_{p^2}$, and therefore, by Godsil [15], $\Gamma$ is an SC-CI-graph of $Z_{p^2}$. Actually, we still do not know whether there exists a self-complementary Cayley graph which cannot be represented as an SC-CI-graph of any group.

By Example 2.4, if $n \in C$ then SCVT-graphs of orders $n$ can be constructed by Construction 2.1. Regarding Problem 4.4, naturally we wish to use Construction 2.1 to construct SCVT-graphs of other orders. Unfortunately, the following result shows that this is not possible, and demonstrates how hard it is to construct SCVT-graphs of order $n$ for $n \notin C$ (if they exist).
Theorem 5.3 ([3, Theorem 1.3]) Let \( \Gamma \) be a self-complementary graph of order \( n \). Then \( \Gamma \) may be represented as an SC-CI-graph of some group if and only if \( n \in C \).

Therefore, using only Construction 2.1 and the wreath product, we cannot construct SCVT-graphs of order \( n \) for \( n \notin C \).

6 The vertex-primitive case

In the final section, we discuss self-complementary vertex-primitive graphs. In Example 2.4, taking \( \sigma \) to be of order \((p^d - 1)/2\), the associated SC-CI-graphs of order \( p^d \) are the Paley graphs. It easily follows that the Paley graphs are arc-transitive. Moreover, H. Zhang (1992) proved that all self-complementary edge-transitive graphs are exactly the Paley graphs. In [37], Zhang also obtained a similar result for self-complementary edge-transitive digraphs.

We notice that Paley graphs are vertex-primitive, and moreover, they are the only known examples of self-complementary vertex-primitive graphs. Let \( \Gamma \) be a vertex-primitive graph of order \( n \) which is self-complementary. Then \( n \equiv 1 \pmod{4} \), in particular, \( n \) is odd. Thus \( \text{Aut } \Gamma \) is a primitive permutation group of odd degree, and such groups are classified by Liebeck and Saxl in [20]. Hence a reasonable approach to Problem 4.4 is to search for a complete solution of the following problem.

Problem 6.1 ([19, Problem 3.3]) Classify self-complementary vertex-primitive graphs.

From the proof of Theorem 4.2 in [19], we know that there do not exist self-complementary vertex-primitive graphs of order a product of two distinct primes. We are inclined to think that the only possible orders of self-complementary vertex-primitive graphs are prime powers.

References


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