The intersection problem for small $G$-designs

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Abstract

A $G$-design of order $n$ is a pair $(P, B)$ where $P$ is the vertex set of the complete graph $K_n$ and $B$ is an edge-disjoint decomposition of $K_n$ into isomorphic copies of the simple graph $G$. Following design terminology, we call these copies "blocks". Given a particular graph $G$, the intersection problem asks for which $k$ is it possible to find two $G$-designs $(P, B_1)$ and $(P, B_2)$ of order $n$, with $|B_1 \cap B_2| = k$, that is, with precisely $k$ common blocks. Here we complete the solution of this intersection problem for several $G$-designs where $G$ is "small", so that now it is solved for all connected graphs $G$ with at most four vertices or at most four edges.

1 Introduction and preliminaries

Let $G$ be a simple graph which is some subgraph of $K_n$, the complete undirected graph on $n$ vertices. A $G$-design of order $n$ is a pair $(V, B)$ where $V$ is the vertex set of $K_n$ and $B$ is an edge-disjoint decomposition of $K_n$ into copies of the simple graph $G$. Following design terminology, we refer to these copies of $G$ as blocks. Thus, for example, a Steiner triple system is a $K_3$-design and a balanced incomplete block design with block size four and index $\lambda = 1$ is a $K_4$-design. The number of blocks, $|B|$, is $\binom{n}{2}/|E(G)|$ where $E(G)$ is the edge-set of $G$; this number clearly must be an integer.

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The intersection problem for $G$-designs asks for what values of $k$ is it possible to find two $G$-designs $(V, B_1)$ and $(V, B_2)$, of the same order $|V|$ and based on the same set $V$, with $|B_1 \cap B_2| = k$; that is, having precisely $k$ common blocks. This problem was first considered for Steiner triple systems or $K_3$-designs (see [8]), and since then the intersection problem has been considered for many different types of combinatorial structures; see [3] for a recent survey.

A $(p, q)$ graph is one with $p$ vertices and $q$ edges. We list below all non-trivial connected simple $(p, q)$ graphs with $\min(p, q) \leq 4$.

$q = 1$  $\longrightarrow K_2$

$q = 2$  $\longrightarrow P_3$

$q = 3$  $\longrightarrow K_3$, $\longrightarrow S_3$, $\longrightarrow P_4$

$q = 4$  $\longrightarrow C_4$, $\longrightarrow S_4$, $\longrightarrow D$, $\longrightarrow Y$, $\longrightarrow P_5$

$q = 5$  $\longrightarrow K_4 - e$

$q = 6$  $\longrightarrow K_4$

Clearly a $K_2$-design is unique; each block is an edge! And so for this design we cannot find two distinct designs, let alone a pair of designs intersecting in a specified number of blocks! So we leave this trivial case.

As mentioned above, the intersection problem for $K_3$-designs was dealt with in [8]. The intersection problem for $C_4$-designs appears in [4], for $(K_4 - e)$-designs in [5] and for $K_4$-designs (with a few exceptions) in [6].

The remaining cases, namely the graphs $P_3$, $P_4$, $P_5$, $S_3$, $S_4$, $D$ and $Y$, we deal with below. We use the notation of [2] for names of these graphs, and the following diagram shows how we label the blocks.

\[ D \]
\[ a \]
\[ b \]
\[ c \]
\[ d \]

$(a, b, c) \rightarrow d$ or $(b, a, c) \rightarrow d$

\[ Y \]
\[ a \]
\[ b \]
\[ c \]
\[ e \]
\[ d \]

$(a, b, c; d, e)$ or $(a, b, c; e, d)$

\[ P_n \]
\[ a_1 \]
\[ a_2 \]
\[ \ldots \]
\[ a_n \]

$(a_1, a_2, \ldots, a_n)$ or $(a_n, a_{n-1}, \ldots, a_1)$

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In what follows we let $IG(n)$ denote the set of integers $k$ for which there exist two $G$-designs $(P, B_1)$ and $(P, B_2)$ with $|P| = n$ and $|B_1 \cap B_2| = k$. Also if $G$ is a graph with $q$ edges, let

$$JG(n) = \begin{cases} \{0, 1, 2, \ldots, \frac{1}{q} \binom{n}{2} - 2, \frac{1}{q} \binom{n}{2}\} & \text{if } q \mid \binom{n}{2}; \\ \emptyset & \text{otherwise}. \end{cases}$$

In other words, $JG(n)$ denotes the intersection numbers one expects to achieve with a $G$-design of order $n$.

We also modify this notation slightly and let $IG(H)$ and $JG(H)$ denote respectively the achievable and expected intersection numbers for a $G$-decomposition of the graph $H$.

We also need the following definition. If $S$ is a set of positive integers and $h$ is some positive integer, then $h \ast S$ denotes the set of all integers which can be obtained by adding any $h$ elements of $S$ together (repetitions of elements of $S$ allowed). For example, $2 \ast \{0, 1, 3\} = \{0, 1, 2, 3, 4, 6\}$.

Subsequently we shall need to decompose certain bipartite and tripartite graphs into edge-disjoint copies of the graphs $G$. Consider the following example.

**Example 1.1 Decompositions of $K_{4,4}$ into copies of $P_5$.**

Let $K_{4,4}$ have vertex set $\{1_1, 2_1, 3_1, 4_1\} \cup \{1_2, 2_2, 3_2, 4_2\}$, and let $P = \{A, B, C, D\}$ where

- $A = (1_2, 1_1, 2_2, 2_1, 3_2)$,
- $B = (1_2, 4_1, 4_2, 3_1, 3_2)$,
- $C = (1_1, 3_2, 4_1, 2_2, 3_1)$,
- $D = (1_1, 4_2, 2_1, 1_2, 3_1)$.

These cover the 16 edges of $K_{4,4}$, and so form a $P_5$-decomposition of $K_{4,4}$.

Now $C$ and $D$ cover the same edges as

- $C' = (1_2, 2_1, 4_2, 1_1, 3_2)$, $D' = (1_2, 3_1, 2_2, 4_1, 3_2)$,

while $B$, $C$ and $D$ together cover the same edges as

- $\hat{B} = (2_2, 4_1, 3_2, 1_1, 4_2)$, $\hat{C} = (1_2, 2_1, 4_2, 3_1, 3_2)$, $\hat{D} = (2_2, 3_1, 1_2, 4_1, 4_2)$.

Moreover, the permutation $(1\ 2)$ applied to the subscripts of blocks $A$, $B$, $C$ and $D$ yields a different $P_5$-decomposition of $K_{4,4}$ having no blocks in common with $P$; call these blocks $\overline{P}$.

Thus we see that $|P \cap \overline{P}| = 0$, $|P \cap \{A, \hat{B}, \hat{C}, \hat{D}\}| = 1$, $|P \cap \{A, B, C', D'\}| = 2$, $|P \cap \overline{P}| = 4$. (Clearly it is not possible to have two decompositions which have all but one block in common.) We record these intersection numbers for $P_5$-decompositions of $K_{4,4}$ as

$$IP_5(K_{4,4}) = \{0, 1, 2, 4\}.$$
More generally, if $K$ is a collection of graphs, then a $K$-decomposition of the graph $H$, $(V, B)$, is an edge-disjoint decomposition of $H$ with vertex set $V$ into a set of subgraphs $B$, with each subgraph isomorphic to some graph in $K$. If $K = \{G\}$, then we call this a $G$-decomposition of $H$, and if also $H = K_n$, then it is a $G$-design of order $n$.

The following lemma will be most useful in the rest of this paper.

**Lemma 1.1** Let $G$ be a graph with $q$ edges and suppose $(V, B)$ is a $\{K_m, H\}$-decomposition of $K_n$, with $\alpha > 0$ blocks isomorphic to $K_m$. If $IG(m) = JG(m)$ and $IG(H) \supseteq \{0, r\}$ with $|E(H)| = qr$ and $q(r + 1) \leq \alpha \left(\frac{m}{2}\right)$, then $IG(n) = JG(n)$.

**Proof.** First a $G$-design of order $n$ can be constructed by replacing each of the blocks $B \in B$ that is isomorphic to $K_m$ by a $G$-design of order $m$, and replacing each of the blocks $B \in B$ that is isomorphic to $H$ by a $G$-decomposition of $H$.

Secondly, if $q \left(\frac{m}{2}\right)$, then for any positive integer $x$,

$$x \ast JG(m) = \left\{0, 1, 2, \ldots, \frac{x}{q} \left(\frac{m}{2}\right) - 2, \frac{x}{q} \left(\frac{m}{2}\right)\right\},$$

and for all $x \geq r + 1$,

$$\{0, 1, 2, \ldots, x - 2, x\} \cup \{0, r\} = \{0, 1, 2, \ldots, x + r - 2, x + r\}.$$ 

Thus if $B$ contains $\alpha$ blocks isomorphic to $K_m$ and $\beta$ blocks isomorphic to $H$, then

$$IG(v) \supseteq \alpha \ast JG(m) + \beta \ast \{0, r\} = \{0, 1, 2, \ldots, z - 2, z\}$$

where $z = \alpha \left(\frac{m}{2}\right) + \beta r$. But $B$ is a decomposition of $K_n$ so we also have $\alpha \left(\frac{m}{2}\right) + \beta qr = \binom{n}{2}$. Thus $z = \frac{1}{q} \binom{n}{2}$, as required. Hence $IG(n) = JG(n)$. \hfill \square

In what follows, the graph $H$ in Lemma 1.1 will usually be a complete bipartite or tripartite graph.

## 2 Paths on 3, 4 and 5 vertices

### 2.1 The path $P_3$

Note that a $P_3$-design of order $n$ contains $n(n - 1)/4$ blocks and so we must have $n \equiv 0$ or 1 (mod 4).

**Example 2.1** $IP_3(K_{2,2}) = \{0, 2\}$.

Take designs $(P, B_i)$, $i = 1, 2$, where the vertex set of $K_{2,2}$ is $P = \{a, b\} \cup \{c, d\}$, and $B_1 = \{(a, c, b), (a, d, b)\}$, $B_2 = \{c, a, d\}, (c, b, d)$). Since $|B_1 \cap B_2| = 0$ we have $IP_3(K_{2,2}) = \{0, 2\}$. \hfill \square
EXAMPLE 2.2 $IP_3(4) = \{0, 1, 3\}$.

We use designs $(P, B_i)$, $i = 1, 2, 3$, where $P = \{a, b, c, d\}$ and

\[
B_1 = \{(a, b, c), (a, c, d), (a, d, b)\}, \\
B_2 = \{(a, b, c), (d, a, c), (b, d, c)\}, \\
B_3 = \{(a, b, d), (a, d, c), (a, c, b)\}.
\]

Here $|B_1 \cap B_2| = 1$, $|B_1 \cap B_3| = 0$ and of course $|B_1 \cap B_1| = 3$. The result follows. □

EXAMPLE 2.3 $IP_3(K_{1, 2n}) = \{0, 1, 2, \ldots, n - 2, n\}$.

The verification of this is immediate. □

Now let $n = 4m$, and take the vertex set of $K_n$ to be $\{(i, j) \mid 1 \leq i \leq 2m, \ j = 1, 2\}$. Take $K_4$ blocks $\{(2i - 1, j), (2i, j) \mid j = 1, 2\}$, for $1 \leq i \leq m$, and $K_{2, 2}$ blocks $\{(a, 1), (a, 2)\} \cup \{(b, 1), (b, 2)\}$ where $1 \leq a < b \leq 2m$ and $\{a, b\} \neq \{2i - 1, 2i\}$ for $1 \leq i \leq m$. The result is a $\{K_4, K_{2, 2}\}$-decomposition of $K_{4m}$ and consequently by Lemma 1.1 we have $IP_3(4m) = JP_3(4m)$.

Now let $n = 4m + 1$, and let the vertex set of $K_n$ be $\{1, 2, \ldots, 4m, \infty\}$. We may use $P_3$-designs of order $4m$ on $\{1, 2, \ldots, 4m\}$ and use Example 2.3 to find $P_3$-decompositions of $K_{1, 4m}$ on $\{\infty\} \cup \{1, 2, \ldots, 4m\}$. Thus

\[
IP_3(4m + 1) \supseteq IP_3(4m) + IP_3(K_{1, 4m}) = \{0, 1, 2, \ldots, m(4m + 1) - 2, m(4m + 1)\} = JP_3(4m + 1).
\]

We have now proved

**Theorem 2.1** The intersection numbers for $P_3$-designs are given by $IP_3(n) = JP_3(n) = \{0, 1, \ldots, b - 2, b\}$ where $b = n(n - 1)/4$, the total number of blocks in a $P_3$-design of order $n$. □

### 2.2 The path $P_4$

A $P_4$-design of order $n$ contains $n(n - 1)/6$ blocks so that $n \equiv 0$ or $1 \pmod{3}$, $n \geq 4$. So let $n = 3m$ or $3m + 1$. First we give some necessary examples.

**Example 2.4** $IP_4(4) = \{0, 2\}$.

Let $V = \{1, 2, 3, 4\}$, $B_1 = \{(1, 2, 3, 4), (2, 4, 1, 3)\}$, $B_2 = \{(1, 4, 3, 2), (3, 1, 2, 4)\}$. Then $(V, B_1)$, $(V, B_2)$ are both $P_4$-designs, and $|B_1 \cap B_2| = 0$; the result follows. □

**Example 2.5** $IP_4(K_{3, 3}) \supseteq \{0, 3\}$.

Let $K_{3, 3}$ have vertex set $V = \{1, 2, 3\} \cup \{4, 5, 6\}$. Two disjoint decompositions are $B_1 = \{(1, 4, 2, 5), (2, 6, 3, 4), (3, 5, 1, 6)\}$, $B_2 = \{(2, 5, 3, 6), (3, 4, 1, 5), (1, 6, 2, 4)\}$. The result follows. □
Example 2.6 $IP_4(6) = \{0, 1, 2, 3, 5\}$.

Let $K_6$ have vertex set $V = \{0, 1, 2, 3, 4, 5\}$, and let $A = \{(0, 1, 2, 3), (3, 0, 5, 2), (0, 4, 3, 1)\}$, $B = \{(0, 2, 4, 5), (3, 5, 1, 4)\}$, and $C = \{(0, 4, 3, 1), (3, 5, 1, 4)\}$. Then $(V, A \cup B)$ is one $P_4$-design of order 6. Note that the blocks $A$ trade with $A' = \{(1, 0, 5, 2), (4, 0, 3, 2), (4, 3, 1, 2)\}$, and the blocks $C$ trade with $C' = \{(0, 4, 1, 3), (1, 5, 3, 4)\}$. Let $X = A \cup B$, and let $\alpha$ denote the permutation $(14)(35)$ and $\beta$ the permutation $(15)(34)$. The following table lists intersection numbers.

<table>
<thead>
<tr>
<th>blocks</th>
<th>intersection size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X, \ X\alpha$</td>
<td>0</td>
</tr>
<tr>
<td>$X, \ X\beta$</td>
<td>1</td>
</tr>
<tr>
<td>$X, {A', B}$</td>
<td>2</td>
</tr>
<tr>
<td>$X, ((X \setminus C) \cup C')$</td>
<td>3</td>
</tr>
<tr>
<td>$X, \ X$</td>
<td>5</td>
</tr>
</tbody>
</table>

Example 2.7 $IP_4(7) = \{0, 1, 2, 3, 4, 5, 7\}$.

Let $K_7$ have vertex set $V = \{0, 1, 2, 3, 4, 5, 6\}$. Let $A = \{(0, 1, 3, 6), (1, 2, 4, 0)\}$, $B = \{(2, 3, 5, 1), (3, 4, 6, 2), (4, 5, 0, 3)\}$ and $C = \{(5, 6, 1, 4), (6, 0, 2, 5)\}$. Then $(V, X)$, where $X = (A \cup B \cup C)$, is a $P_4$-design of order 7. Moreover, $A$, $B$ and $C$ trade with $A' = \{(2, 1, 3, 6), (1, 0, 4, 2)\}$, $B' = \{(1, 5, 0, 3), (6, 4, 5, 3), (6, 2, 3, 4)\}$ and $C' = \{(4, 1, 6, 0), (0, 2, 5, 6)\}$ respectively. Let $\alpha$ denote the permutation $(06)(13)$. The following table lists intersection numbers.

<table>
<thead>
<tr>
<th>blocks</th>
<th>intersection size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X, A' \cup B' \cup C'$</td>
<td>0</td>
</tr>
<tr>
<td>$X, \ X\alpha$</td>
<td>1</td>
</tr>
<tr>
<td>$X, A \cup B' \cup C'$</td>
<td>2</td>
</tr>
<tr>
<td>$X, A' \cup B \cup C'$</td>
<td>3</td>
</tr>
<tr>
<td>$X, A \cup B' \cup C$</td>
<td>4</td>
</tr>
<tr>
<td>$X, A' \cup B \cup C$</td>
<td>5</td>
</tr>
<tr>
<td>$X, \ X$</td>
<td>7</td>
</tr>
</tbody>
</table>

Example 2.8 $IP_4(9) = \{0, 1, 2, \ldots, 9, 10, 12\}$.

Take a $P_4$-design of order 6, on $\{0, 1, 2, 3, 4, 5\}$, and adjoin elements $H, J$ and $K$, and also the blocks

$$X = \{(0, H, 1, J), (2, H, 3, J), (4, H, 5, J), (1, K, 0, J), (3, K, 2, J), (5, K, J, H), (H, K, 4, J)\}.$$

Now using $IP_4(6)$ we have $\{7, 8, 9, 10, 12\} \subseteq IP_4(9)$. Also applying the permutation $(HJ)$ to the set $X$ changes all the blocks in $X$, so again using $IP_4(6)$ we have $\{0, 1, 2, 3, 5\} \subseteq IP_4(9)$.  

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Thus it remains to show that 4 and 6 are in $IP_4(9)$. To do this, let $D$ denote the
design with blocks $X \cup A \cup B$ where $A$ and $B$ are as in Example 2.6 above. Then
$|D \cap D\gamma| = 4$ where $\gamma$ is the permutation $(0,3)(1,2)$. Finally, let
\[ T = \{(0, 2, 4, 5), (3, 0, 5, 2), (0, 4, 3, 1), (3, 5, 1, 4)\} \]
which has trade
\[ T' = \{(3, 1, 4, 5), (0, 3, 5, 1), (3, 4, 2, 0), (4, 0, 5, 2)\}. \]
Then $|D\gamma \cap ((D \setminus T) \cup T')| = 6$, which completes the intersection numbers for designs of order 9.

Now let $n = 3m + 1$ and let the vertex set of $K_n$ be $V = \{(i, j) \mid 1 \leq i \leq m, j = 1, 2, 3\} \cup \{\infty\}$. There is a $\{K_7, K_4, K_{3,3}\}$-decomposition of $K_n$ with: one $K_7$
block $\{\infty\} \cup \{(i, j) \mid i = 1, 2; j = 1, 2, 3\}$; $K_4$ blocks $\{\infty\} \cup \{(i, j) \mid j = 1, 2, 3\}$,
for $3 \leq i \leq m$; $K_{3,3}$ blocks $\{(i, j) \mid j = 1, 2, 3\} \cup \{(i', j) \mid j = 1, 2, 3\}$, for all
$1 \leq i < i' \leq m$, excluding $\{i, i'\} = \{1, 2\}$. Then using Examples 2.7, 2.4, 2.5 and
a slight generalization of Lemma 1.1, it follows that $IP_4(3m + 1) = JP_4(3m + 1) =
\{0, 1, 2, \ldots, t - 2, t\}$ where $t = m(3m + 1)/2$, the total number of blocks in a $P_4$-design of order $3m + 1$.

Next let $n = 3m$. The cases $m$ even and $m$ odd are treated separately. When $m$
is even let $n = 6M$ and let the vertex set of $K_n$ be $\{(i, j) \mid 1 \leq i \leq 2M; j = 1, 2, 3\}$. There is a $\{K_8, K_{3,3}\}$-decomposition of $K_n$ with $K_8$
blocks $\{(2i - 1, j), (2i, j) \mid j = 1, 2, 3\}$ for $1 \leq i \leq M$ and $K_{3,3}$ blocks $\{(i_1, j) \mid j = 1, 2, 3\} \cup \{(i_2, j) \mid j = 1, 2, 3\}$ for
all $1 \leq i_1 < i_2 \leq 2M$ excluding $\{i_1, i_2\} = \{2i - 1, 2i\}, 1 \leq i \leq M$.

The result $IP_4(6M) = JP_4(6M)$ then follows from Examples 2.6, 2.5 and Lemma 1.1.

When $m$ is odd let $n = 6M + 3$, and let the vertex set of $K_n$ be $\{(i, j) \mid 1 \leq i \leq 2M + 1, j = 1, 2, 3\}$. There is a $\{K_9, K_6, K_{3,3}\}$-decomposition of $K_n$ with: one $K_9$
block $\{(i, j) \mid i, j = 1, 2, 3\}$; $K_6$ blocks $\{(2i, j), (2i + 1, j) \mid j = 1, 2, 3\}$ for $i = 2, \ldots, M$;
$K_{3,3}$ blocks $\{(a, j) \mid j = 1, 2, 3\} \cup \{(b, j) \mid j = 1, 2, 3\}$ for all pairs $(a, b)$ with $a \neq b$
and with $a$ and $b$ not both in $\{1, 2, 3\}$ or in $\{2i, 2i + 1\}, 2 \leq i \leq M$.

Then from Examples 2.8, 2.6, 2.5 and Lemma 1.1, we have $IP_4(6M + 3) =
JP_4(6M + 3)$.

We have now proved

**Theorem 2.2** The intersection numbers for $P_4$-designs are given by $IP_4(n) =
\{0, 1, \ldots, b - 2, b\}$ where $b = n(n - 1)/6$. \square

### 2.3 The path $P_5$

The graph $P_5$ has 4 edges, and so a suitable decomposition of $K_n$ will contain
$n(n - 1)/8$ blocks; consequently we must have $n \equiv 0$ or 1 (mod 8). The only ingredients needed are decompositions of $K_{4,4}$, $K_8$ and $K_9$, and of course their intersection numbers too.
Now let the vertex set of $K_n$ be $V = \{(i, j) \mid 1 \leq i \leq 2m, 1 \leq j \leq 4\}$ or $V \cup \{\infty\}$, according as $n = 8m$ or $8m + 1$.

In the former case there is a $\{K_8, K_{4,4}\}$-decomposition of $K_n$ with $K_8$ blocks $\{(2i - 1, j), (2i, j) \mid 1 \leq j \leq 4\}$ for $1 \leq i \leq m$, and $K_{4,4}$ blocks $\{(a, j) \mid 1 \leq j \leq 4\} \cup \{(b, j) \mid 1 \leq j \leq 4\}$ for all $1 \leq a < b \leq 2m$ and $\{a, b\} \neq \{2i - 1, 2i\}$ for $1 \leq i \leq m$. In the latter case there is a $\{K_9, K_{4,4}\}$-decomposition of $K_n$; the $K_9$ blocks have $\{\infty\}$ adjoined to each of the $K_8$ blocks above, otherwise blocks are the same as when $n = 8m$.

In Example 1.1 we showed that $IP_5(K_{4,4}) = \{0, 1, 2, 4\}$. We also need the following two examples.

**Example 2.9** $IP_5(8) = \{0, 1, 2, 3, 4, 5, 7\}$.

On the vertex set $\mathbb{Z}_7 \cup \{\infty\}$, developing the base block $\beta = (\infty, 0, 1, 3, 6)$ modulo 7 generates a $P_5$-decomposition of $K_7$. For each $i \in \mathbb{Z}_7$ the blocks $A_i = \{\beta + i, \beta + i + 1\}$ trade with $A_i' = \{(6, 3, 1, 0, 4) + i, (0, \infty, 1, 2, 4) + i\}$, and $B = \{\beta + 4, \beta + 5, \beta + 6\}$ trades with $B' = \{(3, 0, 5, \infty, 4), (\infty, 6, 5, 2, 0), (0, 6, 1, 4, 5)\}$. We observe that $A_0$, $A_2$ and $A_4$ are mutually disjoint and that $B$ is disjoint from $A_0$ and $A_2$. Consequently $IP_5(8) = \{0, 1, 2, 3, 4, 5, 7\}$. $\square$

**Example 2.10** $IP_5(9) = \{0, 1, 2, 3, 4, 5, 6, 7, 9\}$.

On the vertex set $\mathbb{Z}_9$, a $P_5$-design is generated by developing the base block $\beta = (0, 1, 3, 7, 4)$ (modulo 9). For each $i \in \mathbb{Z}_9$ the blocks $A_i = \{\beta + i, \beta + i + 2\}$ trade with $A_i' = \{(0, 5, 3, 7, 4) + i, (2, 3, 1, 0, 6) + i\}$ and the blocks $B_i = \{\beta + i, \beta + i + 1, \beta + i + 2\}$ trade with $B_i' = \{(0, 1, 2, 3, 5) + i, (1, 3, 7, 4, 2) + i, (6, 0, 5, 8, 4) + i\}$.

Moreover, the blocks $C = \{\beta + 5, \beta + 7, \beta + 8\}$ trade with the blocks $C' = \{(7, 8, 3, 6, 5), (3, 0, 8, 6, 2), (8, 1, 5, 2, 0)\}$. The following table lists the disjoint trades which may be used in order to achieve the required intersection values.

<table>
<thead>
<tr>
<th>trades</th>
<th>intersection achieved</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_0, B_3, B_6$</td>
<td>0</td>
</tr>
<tr>
<td>$A_0, A_1, A_4, A_5$</td>
<td>1</td>
</tr>
<tr>
<td>$C, A_0, A_1$</td>
<td>2</td>
</tr>
<tr>
<td>$B_0, B_3$</td>
<td>3</td>
</tr>
<tr>
<td>$A_0, B_6$</td>
<td>4</td>
</tr>
<tr>
<td>$A_0, A_1$</td>
<td>5</td>
</tr>
<tr>
<td>$B_0$</td>
<td>6</td>
</tr>
<tr>
<td>$A_0$</td>
<td>7</td>
</tr>
<tr>
<td>nothing</td>
<td>9</td>
</tr>
</tbody>
</table>

Now applying Lemma 1.1 yields the following result for $P_5$-designs.

**Theorem 2.3** The intersection numbers for $P_5$-designs are given by $IP_5(n) = \{0, 1, \ldots, b - 2, b\}$ where $b = n(n - 1)/8$. $\square$
3 Stars with 3 and 4 edges

3.1 $S_3$-designs

The number of blocks in an $S_3$-design of order $n$ is $n(n - 1)/6$, and so $n \equiv 0$ or 1 (mod 3), and $n \geq 6$. ($S_3$ involves four vertices, and it is easy to see that $K_4$ has no $S_3$-decomposition.)

We start with the following example.

**Example 3.1** $IS_3(K_{3,3}) = \{0, 3\}$.

Let $K_{3,3}$ have vertex set $\{1, 2, 3\} \cup \{4, 5, 6\}$. The following two $S_3$-decompositions are disjoint.

\[
D_1 = \{(1 : 4, 5, 6), (2 : 4, 5, 6), (3 : 4, 5, 6)\}, \\
D_2 = \{(4 : 1, 2, 3), (5 : 1, 2, 3), (6 : 1, 2, 3)\}.
\]

Moreover, it is straightforward to see that $1 \notin IS_3(K_{3,3})$. \hfill \square

One slight difficulty in this case (and, indeed, for $S_m$-designs in general) is that the expected full set of intersection numbers for a design of order 6 (or $2m$ in general) cannot be achieved. In the case of $S_3$-designs, each block involves 4 vertices, and it is impossible to find a trade consisting of two blocks when the design is of order 6. The smallest trade involves seven vertices, such as $\{(x : a, b, c), (x : d, e, f)\}$ trading with $\{(x : a, b, d), (x : c, e, f)\}$. We do however achieve the other expected intersection numbers, as the following example shows.

**Example 3.2** $IS_3(6) = \{0, 1, 2, 5\}$.

Let $V = \{0, 1, 2, 3, 4, 5\}$ and take

\[
B = \{(0 : 5, 1, 2), (1 : 5, 2, 3), (2 : 5, 3, 4), (3 : 5, 4, 0), (4 : 5, 0, 1)\}.
\]

Let $\alpha = (012)$, $\beta = (345)$ and $\gamma = (01)$ be permutations on $V$. The result then follows from the table below.

<table>
<thead>
<tr>
<th>blocks</th>
<th>intersection</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B \cap B\alpha$</td>
<td>0</td>
</tr>
<tr>
<td>$B \cap B\beta$</td>
<td>1</td>
</tr>
<tr>
<td>$B \cap B\gamma$</td>
<td>2</td>
</tr>
<tr>
<td>$B \cap B$</td>
<td>5</td>
</tr>
</tbody>
</table>

\hfill \square

Three more necessary examples follow.

**Example 3.3** $IS_3(7) = \{0, 1, 2, 3, 4, 5, 7\}$.
Take the vertex set \(\{0, 1, 2, 3, 4, 5, 6\}\), and blocks \(B \cup \{(6 : 0, 1, 2), (6 : 3, 4, 5)\} = B \cup Y\) where \(B\) is as in Example 3.2. The permutations \(\alpha, \beta\) and \(\gamma\) of Example 3.2 fix \(Y\). Hence \(\{2, 3, 4, 7\} \subseteq IS_3(7)\). Moreover, \(Y\) trades with \(Y' = \{(6 : 0, 1, 3), (6 : 2, 4, 5)\}\), and so \(0 \in IS_3(7)\). Also \(|(B \cup Y) \cap (B\beta \cup Y')| = 1\) and \(|(B \cup Y) \cap (B \cup Y')| = 5\), so the result follows. □

**Example 3.4** \(IS_3(9) = \{0, 1, \ldots, 10, 12\}\).

Let the vertex set be \(Z_9\), and take blocks \(B\) as follows.

<table>
<thead>
<tr>
<th>block</th>
<th>in subset(s)</th>
<th>block</th>
<th>in subset(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0 : 1, 3, 6))</td>
<td>(X)</td>
<td>((6 : 1, 2, 7))</td>
<td>(Y, T)</td>
</tr>
<tr>
<td>((1 : 2, 4, 7))</td>
<td>(Y)</td>
<td>((7 : 2, 0, 8))</td>
<td>(T)</td>
</tr>
<tr>
<td>((2 : 0, 5, 8))</td>
<td></td>
<td>((8 : 0, 1, 6))</td>
<td>(X, T)</td>
</tr>
<tr>
<td>((3 : 1, 4, 6))</td>
<td>(X)</td>
<td>((3 : 2, 7, 8))</td>
<td></td>
</tr>
<tr>
<td>((4 : 2, 5, 7))</td>
<td>(Y)</td>
<td>((4 : 0, 8, 6))</td>
<td>(X)</td>
</tr>
<tr>
<td>((5 : 0, 3, 8))</td>
<td>(Z)</td>
<td>((5 : 1, 6, 7))</td>
<td>(Y, Z)</td>
</tr>
</tbody>
</table>

The set \(X\) trades with \(X' = \{(1 : 0, 3, 8), (0 : 3, 4, 8), (6 : 0, 3, 8), (4 : 3, 6, 8)\}\); the set \(Y\) trades with \(Y' = \{(2 : 1, 4, 6), (1 : 4, 5, 6), (7 : 1, 4, 6), (5 : 4, 6, 7)\}\); the set \(Z\) trades with \(Z' = \{(5 : 0, 3, 7), (5 : 8, 1, 6)\}\); and the set \(T\) trades with \(T' = \{(6 : 1, 2, 8), (7 : 0, 2, 6), (8 : 0, 1, 7)\}\). Also \(Z\) and \(T\) are disjoint. The intersection values now follow from the table below, where numbers in parentheses are permutations on \(Z_9\).

<table>
<thead>
<tr>
<th>blocks</th>
<th>intersection</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B \cap B(6 7 8))</td>
<td>0</td>
</tr>
<tr>
<td>(B \cap B(4 7 5 8))</td>
<td>1</td>
</tr>
<tr>
<td>(B \cap B(4 5)(7 8))</td>
<td>2</td>
</tr>
<tr>
<td>(B \cap B(7 8))</td>
<td>3</td>
</tr>
<tr>
<td>(B \cap ((B \setminus (X \cup Y)) \cup X' \cup Y'))</td>
<td>4</td>
</tr>
<tr>
<td>(B \cap B(4 5))</td>
<td>5</td>
</tr>
<tr>
<td>(B \cap ((B \setminus (X \cup Z)) \cup X' \cup Z'))</td>
<td>6</td>
</tr>
<tr>
<td>(B \cap ((B \setminus (X \cup T)) \cup X' \cup T'))</td>
<td>7</td>
</tr>
<tr>
<td>(B \cap ((B \setminus X) \cup X'))</td>
<td>8</td>
</tr>
<tr>
<td>(B \cap ((B \setminus T) \cup T'))</td>
<td>9</td>
</tr>
<tr>
<td>(B \cap ((B \setminus Z) \cup Z'))</td>
<td>10</td>
</tr>
<tr>
<td>(B \cap B)</td>
<td>12</td>
</tr>
</tbody>
</table>

□

**Example 3.5** \(IS_3(10) = \{0, 1, \ldots, 13, 15\}\).

Take \(Z_{10}\) and blocks \(B\) of Example 3.4 above, together with \(P = \{(9 : 0, 1, 2), (9 : 3, 4, 5), (9 : 6, 7, 8)\}\). The blocks in \(P\) are fixed by the above permutations (except for \((4 7 5 8)\)) and by the trades on \(B\), so \(\{3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 15\} \subseteq IS_3(10)\). Also \(P\) trades with \(P' = \{(9 : 0, 1, 3), (9 : 2, 4, 6), (9 : 5, 7, 8)\}\), and so in particular \(\{0, 1, 5\} \subseteq IS_3(10)\) also. Finally we see that \(2 \in IS_3(10)\), using \(1 \in IS_3(9)\) and the
trade \( \{9 : 0, 1, 2\}, \{9 : 3, 4, 5\} \) with \( \{9 : 0, 1, 3\}, \{9 : 2, 4, 5\} \). This completes the example. \(\square\)

In the general situation we deal with four cases: \( n = 6m \), \( n = 6m + 1 \), \( n = 6m + 3 \) and \( n = 6m + 4 \). In each case the vertex set is \( V = \{(i, j) \mid 1 \leq i \leq 2m, j = 1, 2, 3\} \), or \( V \cup \{\infty\} \), or \( V' = V \cup \{(2m + 1, j) \mid j = 1, 2, 3\} \) or \( V' \cup \{\infty\} \) (respectively).

First, when \( n = 6m + 1 \), there is a \( K_7, K_{3,3} \)-decomposition of \( K_n \) with \( K_7 \) blocks \( \{\infty\} \cup \{(2i - 1, j), (2i, j) \mid j = 1, 2, 3\} \) for \( 1 \leq i \leq m \) and \( K_{3,3} \) blocks \( \{(a, j) \mid j = 1, 2, 3\} \cup \{(b, j) \mid j = 1, 2, 3\} \), for all \( 1 \leq a < b \leq 2m \), excluding \( \{a, b\} = \{2i - 1, 2i\}, 1 \leq i \leq m \).

From Lemma 1.1 it follows that \( IS_3(6m + 1) = JS_3(6m + 1) \).

Secondly, when \( n = 6m + 4 \), we use a \( K_{10}, K_7, K_{3,3} \)-decomposition of \( K_n \), with one \( K_{10} \) block and \( m - 1 \) \( K_7 \) blocks. Once again Lemma 1.1 then yields \( IS_3(6m + 4) = JS_3(6m + 4) \).

Thirdly, when \( n = 6m \), in order to achieve the intersection number \( b - 2 \), with all but two blocks in common, since \( 5 - 2 = 3 \not\in IS_3(6) \), we use a \( K_9, K_6, K_{3,3} \)-decomposition of \( K_n \) with two \( K_9 \) blocks and \( m - 3 \) \( K_6 \) blocks. This assumes that \( m \geq 3 \), so \( n \geq 18 \); the case of order 12, therefore, must be considered separately.

Then, for \( m \geq 3 \), as before we obtain \( IS_3(6m) = JS_3(6m) \).

Fourthly, when \( n = 6m + 3 \), we use a \( K_9, K_6, K_{3,3} \)-decomposition of \( K_n \) with one \( K_9 \) block and \( m - 1 \) \( K_6 \) blocks, and obtain \( IS_3(6m + 3) = JS_3(6m + 3) \).

It now remains to consider the case of order 12.

**Example 3.6** \( IS_3(12) = \{0, 1, \ldots, 20, 22\} \).

First, all intersection numbers except 20 (that is, \( b - 2 \)) can be achieved with the following construction using two designs of order 6 and four lots of decompositions of \( K_{3,3} \). Let \( A, B, C \) and \( D \) each stand for a set of three vertices. Then on sets \( \{A, B\} \) and \( \{C, D\} \), place \( S_3 \)-designs of order 6, and on the sets \( \{A\} \cup \{C\}, \{A\} \cup \{D\}, \{B\} \cup \{C\} \), and \( \{B\} \cup \{D\} \), place \( S_3 \)-decompositions of \( K_{3,3} \). The result is an \( S_3 \)-design of order 12, and we see that

\[
IS_3(12) \supseteq 2 \cdot IS_3(6) + 4 \cdot IS_3(K_{3,3})
\]

which includes all required intersection numbers except 20.

Secondly, in order to obtain this intersection number, note that in the above construction, one of the four decompositions of \( K_{3,3} \) is on the sets \( \{A\} \cup \{C\} \) while another is on the sets \( \{A\} \cup \{D\} \); so there will be two blocks of the form \((x : u, v, w)\) and \((x : r, s, t)\). These may be traded with \((x : u, v, t)\) and \((x : r, s, w)\); so we have \( 20 \in IS_3(12) \) as required. \(\square\)

The results in this subsection have shown

**Theorem 3.1** \( IS_3(n) = \{0, 1, \ldots, b - 2, b\} \) where \( n \equiv 0 \) or 1 \( (\mod 3) \), \( n \geq 6 \) and \( b = \frac{n(n - 1)}{6} \), except that \( 3 \not\in IS_3(6) \). \(\square\)
3.2 $S_4$-designs

Since the number of blocks in an $S_4$-design of order $n$ is $n(n-1)/8$, we must have $n \equiv 0 \text{ or } 1 \pmod{8}$. First note that once we have intersection numbers $IS_4(8m)$, we can easily obtain $IS_4(8m+1)$. For in order to construct an $S_4$-design of order $8m+1$ from one of order $8m$ we may simply adjoin one new vertex, say $x$, and $2m$ new blocks of the form \{$(x, a, b, c, d) \mid a, b, c, d \in V$\} where $V$ is the vertex set of the design of order $8m$. Moreover, by judicious interchange of the $2m$ elements, we see that we may construct two $S_4$-designs of order $8m+1$ so that

$$IS_4(8m+1) \supset IS_4(8m) + \{0, 1, 2, \ldots, 2m - 2, 2m\}.$$

Now consider the following examples.

**Example 3.7** $IS_4(K_{4,4}) \supset \{0, 4\}$.

Imitate the construction in Example 3.1 above, but taking four vertices rather than three in each partite set.

**Example 3.8** $IS_4(8) = \{0, 1, 2, 3, 4, 7\}$.

With vertex set $\{0, 1, 2, 3, 4, 5, 6, 7\}$, let blocks $B$ be as follows.

$$(0 : 1, 2, 3, 7), (1 : 2, 3, 4, 7), (2 : 3, 4, 5, 7), (3 : 4, 5, 6, 7), (4 : 5, 6, 0, 7), (5 : 6, 0, 1, 7), (6 : 0, 1, 2, 7).$$

The following table shows the intersection values achieved by applying the given permutations to the vertices.

<table>
<thead>
<tr>
<th>permutation</th>
<th>intersection size</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0 1 2 3)</td>
<td>0</td>
</tr>
<tr>
<td>(0 1 2)</td>
<td>1</td>
</tr>
<tr>
<td>(0 1 2)</td>
<td>2</td>
</tr>
<tr>
<td>(0 2)</td>
<td>3</td>
</tr>
<tr>
<td>(0 1)</td>
<td>4</td>
</tr>
<tr>
<td>identity</td>
<td>7</td>
</tr>
</tbody>
</table>

*Example 3.9* $IS_4(9) = \{0, 1, 2, 3, 4, 5, 6, 7, 9\}$.

As indicated in the remark preceding Example 3.7,

$$IS_4(9) \supset IS_4(8) + \{0, 2\}$$

$$= \{0, 1, 2, 3, 4, 7\} + \{0, 2\}$$

$$= \{0, 1, 2, 3, 4, 5, 6, 7, 9\}.$$
Example 3.10 $IS_4(16) = \{0, 1, \ldots, 28, 30\}$.

First, all intersection numbers except 28 (that is, $(b - 2)$) can be achieved with the following construction using two designs of order 8 and four lots of decompositions of $K_{4,4}$. Let $A$, $B$, $C$ and $D$ each stand for a set of four vertices. Then on the sets $\{A, B\}$ and $\{C, D\}$, place $S_4$-designs of order 8, and on the sets $\{A\} \cup \{C\}$, $\{A\} \cup \{D\}$, $\{B\} \cup \{C\}$, and $\{B\} \cup \{D\}$, place $S_4$-decompositions of $K_{4,4}$. The result is an $S_4$-design of order 16, and we see that

$$IS_4(16) \supseteq 2 \cdot IS_4(8) + 4 \cdot IS_4(K_{4,4})$$

which includes all required intersection numbers except 28.

Secondly, in order to obtain this intersection number, take another design of order 16 with vertex set $\mathbb{Z}_{15} \cup \{\infty\}$ and 30 blocks as follows:

$$(i : i + 1, i + 2, i + 3, i + 4), \quad (i : i + 5, i + 6, i + 7, \infty), \quad i \in \mathbb{Z}_{15}.$$

The two blocks $(0 : 1, 2, 3, 4), (0 : 5, 6, 7, \infty)$ trade with $(0 : 5, 6, 7, 4), (0 : 1, 2, 3, \infty)$, changing just two blocks, and thus showing that $28 \in IS_4(16)$ as required.

Again, using the remark at the start of this subsection, using the above example it is easy to obtain $IS_4(17) = \{0, 1, \ldots, 32, 34\}$.

Now the general construction for order $8m$ uses a $\{K_{16}, K_8, K_{4,4}\}$-decomposition of $K_{8m}$ with one $K_{16}$ block and $m - 2$ $K_8$ blocks. Explicitly, let the vertex set be $\{(i, j) | 1 \leq i \leq 2m, 1 \leq j \leq 4\}$, and let the $K_{16}$ block be $\{(i, j) | 1 \leq i, j \leq 4\}$, the $K_8$ blocks be $\{(2i - 1, j), (2i, j) | 1 \leq j \leq 4\}$ for $3 \leq i \leq m$, and the $K_{4,4}$ blocks be $\{(a, j) | 1 \leq j \leq 4\} \cup \{(b, j) | 1 \leq j \leq 4\}$ for all $a \neq b$ where $a$ and $b$ are not both first components of elements in the same $K_{16}$ or $K_8$ blocks. Then $IS_4(8m) = JS_4(8m)$.

The only difference for order $8m + 1$ is that, since $IS_4(9)$ includes all intersection numbers expected, including "$b - 2"$, we may merely use a $\{K_9, K_{4,4}\}$-decomposition of $K_{8m+1}$, in order to achieve $IS_4(8m + 1) = JS_4(8m + 1)$.

We have now proved

Theorem 3.2 The intersection numbers for $S_4$-designs are given by $IS_4(n) = \{0, 1, \ldots, b - 2, b\}$ where $n \equiv 0$ or 1 (mod 8), except that 5 $\not\equiv IS_4(8)$.

\[\square\]

4 \hspace{1cm} D, a triangle with pendant edge

Once again, since $D$ has four edges, we find that a $D$-design of order $n$ contains $n(n - 1)/8$ blocks and so $n \equiv 0$ or 1 (mod 8). However, since $D$ contains an odd cycle (a triangle!) there is no $D$-decomposition of any bipartite graph, so in this case we require a $D$-decomposition of a tripartite graph.

Example 4.1 $ID(K_{2,2,2}) \supseteq \{0, 3\}$, and $ID(K_{4,4,4}) \supseteq \{0, 3, 6, 9, 12\}$.
For $K_{2,2,2}$, take the vertex sets $\{1, 1'\} \cup \{2, 2'\} \cup \{3, 3'\}$. Then disjoint $D$-decompositions are given by $\{(1, 3', 1'), (3, 2', 1')\}$, $\{(1, 3', 2'), (3', 2, 1)\}$, $\{(1', 2, 3), (1', 2, 3')\}$. Thus $\{0, 3\} \subseteq ID(K_{2,2,2})$.

Now let the vertex sets for $K_{4,4,4}$ be $\{A, D\} \cup \{B, E\} \cup \{C, F\}$, where each letter here is itself a set of two points. Then we may take four decompositions of $K_{2,2,2}$ on the four sets $A \cup B \cup F$, $A \cup E \cup C$, $D \cup B \cup C$ and $D \cup E \cup F$, yielding 12 blocks for a $D$-decomposition of $K_{4,4,4}$. Then using the intersection values for $ID(K_{2,2,2})$ we obtain $ID(K_{4,4,4}) \supseteq \{0, 3, 6, 9, 12\}$. \qed

For the general construction, we take the vertex set $V = \{(i, j) \mid 1 \leq i \leq 2m, 1 \leq j \leq 4\}$ if $n = 8m$, or $V \cup \{\infty\}$ if $n = 8m + 1$.

Then if $2m \equiv 0$ or $2 \pmod{6}$, $2m \geq 6$, we may use a GDD with group size 2 and block size 3 on $\{1, 2, \ldots, 2m\}$, while if $2m \equiv 4 \pmod{6}$, $2m \geq 10$, we may use a GDD with one group of size 4 and the rest of size 2, and block size 3 on $\{1, 2, \ldots, 2m\}$. These exist; see for instance Lemma 2.1 in [1], or the general result in [7]. Then for each group $\{x_1, \ldots, x_g\}$ of the GDD, place a $D$-design on the set $\{(x_i, j) \mid 1 \leq i \leq g, 1 \leq j \leq 4\}$ or on this set together with $\infty$. Since the group sizes are 2 or 4, this means we require $D$-designs of orders 8, 9, 16 and 17. And for each block $\{a, b, c\}$ of the GDD, place a $D$-decomposition of $K_{4,4,4}$ on $\{(a, j) \mid 1 \leq j \leq 4\} \cup \{(b, j) \mid 1 \leq j \leq 4\} \cup \{(c, j) \mid 1 \leq j \leq 4\}$.

It now remains to deal with orders 8, 9, 16 and 17.

**Example 4.2** $ID(8) = \{0, 1, \ldots, 5, 7\}$.

Take the vertex set $\{\infty\} \cup \mathbb{Z}_7$, and blocks $B = \{(i, 1 + i, 3 + i) - \infty \mid i \in \mathbb{Z}_7\}$. Note the following trades.

$X = \{(1, 2, 4) - \infty, (3, 4, 6) - \infty\}$ trades with $X' = \{(1, 2, 4) - 3, (\infty, 4, 6) - 3\}$,

$Y = \{(2, 3, 5) - \infty, (4, 5, 0) - \infty\}$ trades with $Y' = \{(2, 3, 5) - 4, (\infty, 5, 0) - 4\}$,

$Z = \{(5, 6, 1) - \infty, (0, 1, 3) - \infty\}$ trades with $Z' = \{(5, 6, 1) - 0, (\infty, 1, 3) - 0\}$,

$A = \{(0, 1, 3) - \infty, (2, 3, 5) - \infty, (5, 6, 1) - \infty\}$ trades with $A' = \{(0, 3, 1) - \infty, (2, 5, 3) - \infty, (6, 1, 5) - \infty\}$.

Here $X$, $Y$ and $Z$ are pairwise disjoint, and $A$ is also disjoint from $X$. Thus we achieve the following intersection values, where $\alpha$ below denotes the permutation $(1 \infty)$ applied to $B$.

<table>
<thead>
<tr>
<th>trades</th>
<th>blocks changed</th>
<th>intersection achieved</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B\alpha$</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>$X$, $Y$, $Z$</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>$X$, $A$</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>$X$, $Y$</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>$A$</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$X$</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>nothing</td>
<td>0</td>
<td>7</td>
</tr>
</tbody>
</table>

\[ \square \]
EXAMPLE 4.3 $ID(9) = \{0, 1, \ldots, 7, 9\}$.
With vertex set $\mathbb{Z}_9$, let $D = \{(i, i+3, i+6) | i \in \mathbb{Z}_9\}$. The following trades are disjoint:

$X = \{(1, 2, 5) - 7, (4, 5, 8) - 1\}$ trades with $X' = \{(8, 4, 5) - 7, (2, 5, 1) - 8\}$,
$Y = \{(2, 3, 6) - 8, (5, 6, 0) - 2\}$ trades with $Y' = \{(3, 6, 2) - 0, (0, 5, 6) - 8\}$,
$Z = \{(0, 1, 4) - 6, (3, 4, 7) - 0, (6, 7, 1) - 3\}$ trades with
$Z' = \{(3, 1, 7) - 6, (0, 7, 4) - 3, (6, 4, 1) - 0\}$.

Now denote permutations by $\alpha = (01), \beta = (125), \gamma = (1234)$, and let $T = \{(7, 8, 2) - 4, (8, 0, 3) - 5\}$. The following table then completes this example.

<table>
<thead>
<tr>
<th>blocks</th>
<th>intersection size</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D \cap D\gamma$</td>
<td>0</td>
</tr>
<tr>
<td>$D \cap D\beta$</td>
<td>1</td>
</tr>
<tr>
<td>$D \cap {X' \cup Y' \cup Z' \cup T}$</td>
<td>2</td>
</tr>
<tr>
<td>$D \cap D\alpha$</td>
<td>3</td>
</tr>
<tr>
<td>$D \cap {X \cup Y' \cup Z' \cup T}$</td>
<td>4</td>
</tr>
<tr>
<td>$D \cap {X' \cup Y' \cup Z \cup T}$</td>
<td>5</td>
</tr>
<tr>
<td>$D \cap {X \cup Y \cup Z' \cup T}$</td>
<td>6</td>
</tr>
<tr>
<td>$D \cap {X' \cup Y \cup Z \cup T}$</td>
<td>7</td>
</tr>
<tr>
<td>$D \cap D$</td>
<td>9</td>
</tr>
</tbody>
</table>

EXAMPLE 4.4 $ID(16) = \{0, 1, \ldots, 28, 30\}$.
With vertex set $\mathbb{Z}_{15} \cup \{\infty\}$, a design is given by

$\{(i, 1+i, 6+i) - (8+i), (i, 3+i, 7+i) - \infty\}$ where $i \in \mathbb{Z}_{15}$.

Now blocks $A_i$ trade with $A'_i$ for $0 \leq i \leq 6$ where

$A_i = \{(i, 3+i, 7+i) - \infty, (7+i, 10+i, 14+i) - \infty\}$ and
$A'_i = \{(i, 3+i, 7+i) - (10+i), (7+i, \infty+i, 14+i) - (10+i)\}$.

Disjoint from these trades are the following five trades, $B_i$ with $B'_i$, for $0 \leq i \leq 4$, where

$B_i = \{(i, 1+i, 6+i) - (8+i), (5+i, 6+i, 11+i) - (13+i), (10+i, 11+i, 1+i) - (3+i)\}$
and

$B'_i = \{(i, 6+i, 1+i) - (3+i), (5+i, 11+i, 6+i) - (8+i), (10+i, 1+i, 11+i) - (13+i)\}$
(addition in $\mathbb{Z}_{15}$). Thus we have trades on $2a + 3b$ blocks, where $0 \leq a \leq 7$ and $0 \leq b \leq 5$. This means that we may trade $2a + 3b = c$ blocks for $2 \leq c \leq 29$. Thus $\{1, 2, \ldots, 28\} \subseteq ID(16)$. And trivially $30 \in ID(16)$. Finally, to show $0 \in ID(16)$, let

$X = \{(6, 9, 13) - \infty, (13, 1, 5) - \infty, (14, 2, 6) - \infty\}$

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which trades with

\[ X' = \{(14, 2, 6) - 9\lambda, (1, 15, 13) - 9, (13, 6, \infty) - 15\}. \]

Thus trading \( \{B_i\}_{i=0}^{4} \cup \{A_i\}_{i=0}^{5} \cup \{X\} \) will change all the blocks, so \( 0 \in ID(16) \). This concludes the example. \( \square \)

**Example 4.5** \( ID(17) = \{0, 1, \ldots, 32, 34\} \).

Let the vertex set be \( Z_{17} \). Then a design is given by

\[ D = \{(i, i + 3, i + 8) - (i + 12), (i, i + 1, i + 7) - (i + 9) \mid i \in Z_{17}\}. \]

Let permutations on \( Z_{17} \) be given by

\[ \alpha_0 = (0 1 2 3 4 5 6 7 8), \quad \alpha_1 = (0 1 2 3 4 5 6 7 8 9 10), \]
\[ \alpha_2 = (0 1)(2 3 4 5 6), \quad \alpha_3 = (0 1)(2 3 4 5), \quad \alpha_4 = (0 1 2 3 4). \]

Then \( |D \cap D\alpha_i| = i, 0 \leq i \leq 4 \), so \( \{0, 1, 2, 3, 4\} \subseteq ID(17) \). For the remaining intersection values we consider trades as follows.

The set \( A_i = \{(1, 4, 9) - 13, (13, 16, 4) - 8\} + i \pmod{17} \) trades with \( A_i' = \{(16, 4, 13) - 9, (9, 1, 4) - 8\} + i \pmod{17}, 0 \leq i \leq 4 \). Disjoint from this are the blocks

\[ B_i = \{(1, 2, 8) - 10, (9, 10, 16) - 1\} + i \pmod{17} \]

trading with

\[ B_i' = \{(8, 2, 1) - 16, (9, 16, 10) - 8\} + i \pmod{17}, \]

\( 0 \leq i \leq 7 \). Also let

\[ C_i = \{(0, 3, 8) - 12, (12, 15, 3) - 7, (11, 12, 1) - 3\} + i, \]

which trades with

\[ C_i' = \{(0, 8, 3) - 7, (12, 15, 3) - 1, (1, 11, 12) - 8\} + i, \]

for \( 0 \leq i \leq 4 \).

Note that \( C_0 \) is disjoint from \( A_i, i = 0, 1, 2, 3 \),
\( C_1 \) is disjoint from \( A_i, i = 1, 2, 3, 4 \),
\( C_2 \) is disjoint from \( A_i, i = 0, 2, 3, 4 \),
\( C_3 \) is disjoint from \( A_i, i = 0, 1, 3, 4 \),
\( C_4 \) is disjoint from \( A_i, i = 0, 1, 2, 4 \).

Thus we may obtain trades of sizes 2, 3, \ldots, 28, 29, yielding \( \{5, 6, \ldots, 31, 32\} \subseteq ID(17) \). Finally, \( 34 \in ID(17) \) trivially. This completes the example. \( \square \)

Now combining the results of this section we have

**Theorem 4.1** The intersection numbers for \( D \)-designs are given by \( ID(n) = \{0, 1, \ldots, b - 2, b\} \) where \( b = n(n - 1)/8 \). \( \square \)

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5 The graph $Y$

A $Y$-design of order $n$ contains $n(n - 1)/8$ blocks, and so $n \equiv 0$ or $1 \pmod{8}$. The only ingredients we need are $Y$-designs of orders $8$ and $9$, a $Y$-decomposition of $K_{4,4}$, and their intersection numbers. (In fact, it suffices to use $IY(K_{4,4}) \supseteq \{0, 4\}$.)

**Example 5.1** $IY(K_{4,4}) \supseteq \{0, 4\}$.

Let the vertex set be $\{1, 2, 3, 4\} \cup \{5, 6, 7, 8\}$. Then two disjoint decompositions are given by

$$\{(4, 7, 1; 5, 6), (1, 8, 2; 6, 7), (2, 5, 3; 7, 8), (3, 6, 4; 5, 8)\}$$

and

$$\{(8, 3, 5; 1, 2), (5, 4, 6; 2, 3), (6, 1, 7; 3, 4), (7, 2, 8; 1, 4)\}.$$

**Example 5.2** $IY(8) = \{0, 1, 2, 3, 4, 5, 7\}$.

With vertex set $\{\infty\} \cup \mathbb{Z}_7$, take blocks $D = A \cup B \cup C$ where

$$A = \{((0, 1, 3; 6, \infty), (1, 2, 4; 0, \infty)), (2, 3, 5; 1, \infty), (3, 4, 6; 2, \infty)\},$$

$$B = \{((4, 5, 0; 3, \infty), (5, 6, 1; 4, \infty), (6, 0, 2; 5, \infty)\}).$$

Blocks $A$ trade with $A' = \{(6, 3, 1; 0, 2), (3, \infty, 4; 0, 2)\}$,
blocks $B$ trade with $B' = \{(1, 5, 3; 2, 4), (5, \infty, 6; 2, 4)\}$ and
blocks $C$ trade with $C' = \{(4, 5, 0; 3, 2), (4, 1, 6; 5, 0), (5, 2, \infty; 1, 0)\}$.

Now let $\alpha$ denote the permutation $(01)$ and $\beta$ the permutation $(012)$. We obtain the following intersection numbers, which completes the result.

<table>
<thead>
<tr>
<th>blocks</th>
<th>intersection</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D \cap D\beta$</td>
<td>0</td>
</tr>
<tr>
<td>$D \cap D\alpha$</td>
<td>1</td>
</tr>
<tr>
<td>$D \cap {A \cup B' \cup C'}$</td>
<td>2</td>
</tr>
<tr>
<td>$D \cap {A' \cup B' \cup C}$</td>
<td>3</td>
</tr>
<tr>
<td>$D \cap {A \cup B \cup C'}$</td>
<td>4</td>
</tr>
<tr>
<td>$D \cap {A \cup B' \cup C}$</td>
<td>5</td>
</tr>
<tr>
<td>$D \cap D$</td>
<td>7</td>
</tr>
</tbody>
</table>

**Example 5.3** $IY(9) = \{0, 1, \ldots, 7, 9\}$.

Let the vertex set be $\mathbb{Z}_9$, and blocks be $D = \{(0 + i, 1 + i, 3 + i; 6 + i, 7 + i) \mid i \in \mathbb{Z}_9\}$ (addition mod $9$). The blocks

$$A_i = \{(i - 1, i, 2 + i; 5 + i, 6 + i), (i, 1 + i, 3 + i; 6 + i, 7 + i)\}, \quad 1 \leq i \leq 4,$$

then $IY(9)$ is obtained.
trade with
\[ A_i' = \{(5 + i, 2 + i, i; i - 1, i + 1), (2 + i, 6 + i, 3 + i; 1 + i, 7 + i)\}, \ 1 \leq i \leq 4. \]

Also the blocks
\[ B_i = \{(3i - 3, 3i - 2, 3i; 3i + 3, 3i + 4),
\[ (3i - 2, 3i - 1, 3i + 1; 3i + 4, 3i + 5), (3i - 1, 3i, 3i + 2; 3i + 5, 3i + 6)\}, \]
\[ 1 \leq i \leq 3, \ \text{trade with the blocks} \]
\[ B_i' = \{(3i + 3, 3i, 3i - 2; 3i - 3, 3i - 1),
\[ (3i, 3i + 4, 3i + 1; 3i - 1, 3i + 5), (3i + 2, 3i + 5, 3i + 1; 3i + 4, 3i - 1)\}, \]
\[ 1 \leq i \leq 3. \text{ Thus we obtain the required intersection numbers:} \]

<table>
<thead>
<tr>
<th>blocks</th>
<th>intersection</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D \cap {B_1' \cup B_2' \cup B_3'} )</td>
<td>0</td>
</tr>
<tr>
<td>( D \cap {(8, 0, 2, 5, 6) \cup {A_i' \mid 1 \leq i \leq 4}} )</td>
<td>1</td>
</tr>
<tr>
<td>( D \cap {A_1' \cup A_2' \cup A_3' \cup B_3'} )</td>
<td>2</td>
</tr>
<tr>
<td>( D \cap {A_1' \cup A_2' \cup A_3' \cup B_3'} )</td>
<td>3</td>
</tr>
<tr>
<td>( D \cap {A_1' \cup A_2' \cup A_3' \cup B_3'} )</td>
<td>4</td>
</tr>
<tr>
<td>( D \cap {A_1' \cup A_2' \cup A_3' \cup B_3'} )</td>
<td>5</td>
</tr>
<tr>
<td>( D \cap {A_1' \cup A_2' \cup A_3' \cup B_3'} )</td>
<td>6</td>
</tr>
<tr>
<td>( D \cap {A_1' \cup A_2' \cup A_3' \cup B_3'} )</td>
<td>7</td>
</tr>
<tr>
<td>( D \cap D )</td>
<td>9</td>
</tr>
</tbody>
</table>

Thanks to Lemma 1.1 we now have

**Theorem 5.1** The intersection numbers for Y-designs are given by \( IY(n) = \{0, 1, \ldots, b - 2, b\} \) where \( b = n(n - 1)/8. \)

\[ \square \]

6 Summary

The following table summarises the intersection results for G-designs where G is a connected graph on at most four vertices or at most four edges.

In the table, b denotes the number of blocks in a G-design of order n, and the impossible intersection values are \( b - x \) where x is as given. A reference is listed if the result is not in this paper.
<table>
<thead>
<tr>
<th>$G$</th>
<th>$b$</th>
<th>$x$</th>
<th>Comments</th>
<th>Ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_2$</td>
<td>$n(n-1)/2$</td>
<td>all except $b$</td>
<td>unique design!</td>
<td></td>
</tr>
<tr>
<td>$P_3$</td>
<td>$n(n-1)/4$</td>
<td>1</td>
<td>$n \equiv 0, 1 \pmod{4}$</td>
<td></td>
</tr>
<tr>
<td>$P_4$</td>
<td>$n(n-1)/6$</td>
<td>1</td>
<td>$n \equiv 0, 1 \pmod{3}, \quad n \geq 4$</td>
<td></td>
</tr>
<tr>
<td>$P_5$</td>
<td>$n(n-1)/8$</td>
<td>1</td>
<td>$n \equiv 0, 1 \pmod{8}$</td>
<td></td>
</tr>
<tr>
<td>$K_3$</td>
<td>$n(n-1)/6$</td>
<td>1, 2, 3, 5</td>
<td>$n \equiv 1, 3 \pmod{6}, \quad 5, 8 \not\in IK_3(9)$.</td>
<td>[8]</td>
</tr>
<tr>
<td>$D$</td>
<td>$n(n-1)/8$</td>
<td>1</td>
<td>$n \equiv 0, 1 \pmod{8}$</td>
<td></td>
</tr>
<tr>
<td>$Y$</td>
<td>$n(n-1)/8$</td>
<td>1</td>
<td>$n \equiv 0, 1 \pmod{8}$</td>
<td></td>
</tr>
<tr>
<td>$S_3$</td>
<td>$n(n-1)/6$</td>
<td>1</td>
<td>$n \geq 6, n \equiv 0, 1 \pmod{8}, \quad 3 \not\in IS_3(6)$</td>
<td></td>
</tr>
<tr>
<td>$S_4$</td>
<td>$n(n-1)/8$</td>
<td>1</td>
<td>$n \equiv 0, 1 \pmod{8}, \quad 5 \not\in IS_4(8)$</td>
<td></td>
</tr>
<tr>
<td>$C_4$</td>
<td>$n(n-1)/8$</td>
<td>1</td>
<td>$n \equiv 1 \pmod{8}$</td>
<td>[4]</td>
</tr>
<tr>
<td>$K_4 - e$</td>
<td>$n(n-1)/10$</td>
<td>1, 2</td>
<td>$n \equiv 0, 1 \pmod{5}, \quad n \geq 6; \quad 7, 8 \not\in I(11)$</td>
<td>[5]</td>
</tr>
<tr>
<td>$K_4$</td>
<td>$n(n-1)/12$</td>
<td>1, 2, 3, 4, 5, 7</td>
<td>$n \equiv 1, 4 \pmod{12}; \quad 7, 9, 10, 11, 14 \not\in I(16); \quad several\ unknown\ values\ for\ n = 25, 28, 37.$</td>
<td>[6]</td>
</tr>
</tbody>
</table>
References


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