A Structural Method for Hamiltonian Graphs

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Abstract

In this paper, we shall introduce a special structure for graphs and show that a graph $G$ is hamiltonian if and only if $G$ has such a special structure. Using this result, we can prove a new weakened version of Fan's condition for hamiltonian graphs, which generalizes a recent result of Bedrossian, Chen and Schelp (1993).

1 Preliminaries and Main Results

We consider only finite undirected graphs without loops or multiple edges. The set of vertices of a graph $G$ is denoted by $V(G)$ or just by $V$; the set of edges by $E(G)$ or just by $E$. We use $|G|$ as a symbol for the cardinality of $V(G)$. If $H$ and $S$ are subsets of $V(G)$ or subgraphs of $G$, we denote by $N_H(S)$ the set of vertices in $H$ which are adjacent to some vertex in $S$, and set $d_H(S) = |N_H(S)|$. If $S = \{u\}$ and $H = G$, then let $N_G(u) = N(u)$ and set $d_G(u) = d(u)$. For $D \subseteq V(G)$, $G[D]$ denotes the subgraph of $G$ induced by $D$. For basic graph-theoretic terminology, we refer the reader to [3].

Definition 1. Let $H$ be a subgraph of $G$ and $x, y \in V(G) \setminus V(H)$. $\{x, y\}$ is called a pair of useful vertices of $H$ if $G[H \cup \{x, y\}]$ contains a hamiltonian path connecting $x$ and $y$.

Definition 2. A graph $G$ is call $L$-decomposable if $G$ can be separated into $k + 1$ pairwise disjoint subgraphs $G_0, G_1, \ldots, G_k$ such that the following four conditions are satisfied:

1) $G_0$ is complete.

2) For any $1 \leq i \leq k$, there exists a subset $S_i \subseteq N_{G_0}(G_i)$ with at least two vertices which contains a vertex $z$ such that for every $y \neq z \in S_i$, $\{x, y\}$ is a pair of useful vertices of $G_i$.

3) For any three distinct $S_i, S_j, S_l$, we have $S_i \cap S_j \cap S_l = \emptyset$.

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4) For any positive integer \( r \leq k \), \( |\bigcup_{1 \leq j \leq r} S_j| = r \) if and only if \( |V(G_0)| = k = r \).

If \( G \) is L-decomposable, then we say the partition \( G_0, G_1, \ldots, G_k \) which satisfies the four conditions above a \textit{L-decomposition} of \( G \). In Section 2, we shall prove the following structural theorem.

**Theorem 1.** A graph \( G \) is hamiltonian if and only if \( G \) has a L-decomposition.

Theorem 1 has some applications. We shall give some examples here. In order to do this, we need some additional terminology and notations.

In Figure 1, we define four kinds of graphs, C-graph, F-graph, B-graph and N-graph.

![C-graph, F-graph, B-graph, N-graph](image)

Figure 1.

Let \( S, T \) be two induced subgraphs of \( G \) with \( \max\{|S|, |T|\} < |G| \). A graph \( G \) of order \( n \) is said to satisfy property \( ST(n) \) if for any pair of vertices \( x \) and \( y \) at distance two in \( S \) or \( T \), \( \max\{d(x), d(y)\} \geq n/2 \). If \( G \) contains no \( S \) as an induced subgraph, we call \( G \) \( S \)-free. If \( G \) contains neither \( S \) nor \( T \) as an induced subgraph, we call \( G \) \( ST \)-free.

The closure of a graph \( G \) denoted by \( \overline{G} \), is the graph obtained from \( G \) by recursively joining pairs of nonadjacent vertices whose degree sum is at least \( |V(G)| \) until no such pair remains. Let \( V_0 = \{x : d(x) \geq n/2, x \in V(G)\} \).

The following result is due to Bondy and Chvátal.

**Theorem 2[2].** A graph \( G \) is hamiltonian if and only if \( \overline{G} \) is hamiltonian.

Now, using Theorems 1 and 2, we can easily prove the following two theorem known before.

**Theorem 3[4].** Let \( G \) be a 2-connected graph of order \( n \). If each pair of vertices \( x \) and \( y \) at distance 2 satisfies \( \max\{d(x), d(y)\} \geq n/2 \), then \( G \) is hamiltonian.

**Theorem 4[1].** Let \( G \) be a 2-connected graph of order \( n \). If \( G \) satisfies property \( CF(n) \), then \( G \) is hamiltonian.

To prove Theorems 3 and 4, we assume, by contradiction, that \( G \) is a counterexample with as many as possible edges. By Theorem 2, \( G[V_0] \) is a complete subgraph of \( G \). Let \( G_0 \) be a induced complete subgraph of \( G \) with as many as possible vertices and \( V_0 \subseteq V(G_0) \). Let \( G_1, G_2, \ldots, G_k \) be the components of \( G \setminus G_0 \). We can easily
verify that $G_0, G_1, \ldots, G_k$ is a L-decomposition of $G$ under conditions of Theorem 3 or Theorem 4, which leads to a contradiction by Theorem 1.

In section 3, we shall prove the following more general theorem by using Theorems 1 and 2.

**Theorem 5.** Let $G$ be a 2-connected graph of order $n$. If $G$ satisfies property $CB(n)$, then $G$ is hamiltonian.

## 2 The Proof of Theorem 1

If $G$ is a hamiltonian graph, let $C = c_1 c_2 \cdots c_n c_1$ be a hamiltonian cycle of $G$. Set $G_0 = G[\{c_1, c_2\}]$ and $G_1 = G[\{c_3, \ldots, c_n\}]$. Then $G_0, G_1$ satisfy the four conditions of Definition 2. Thus $G$ has a L-decomposition.

Conversely, let $G_0, G_1, \ldots, G_k$ be a L-decomposition of $G$. By Definition 2, $G_0$ is a complete subgraph with $|G_0| \geq 2$ and for any $1 \leq i \leq k$, there exists some $S_i \subseteq N_{G_0}(G_i)$ which satisfies the conditions 2)–4) of Definition 2. By condition 2), $S_i$ contains a vertex $x_i$ such that for any $y \in S_i \setminus \{x_i\}$, $\{x_i, y\}$ is a pair of useful vertices of $G_i$ for all $1 \leq i \leq k$. Using the following Claim we will give a structural proof of the sufficiency.

**Claim.** $G_0$ contains either a cycle $C = u_i \cdots u_{i_k} u_i$ with $|V(C)| = |G_0|$ (when $|G_0| = 2$, $C$ is just an edge) such that

$$\{u_{ij}, u_{ij+1}\} = \{x_{ij}, y_{ij}\}, j = 1, \ldots, k, \ j \mod k \quad (*)$$

or $q$ pairwise disjoint paths $P_i = u_i u_{i_2} \cdots u_{i_{r_i+1}}$, $i = 1, 2, \ldots, q$

$$\{u_{ij}, u_{ij+1}\} = \{x_{ij}, y_{ij}\}, j = 1, 2, \ldots, r_i \quad (**)$$

and

$$u_i, \ldots, u_{i_{r_i+1}} \notin \bigcup_j \bigcup_{i \neq i_j} S_j \quad (***)$$

where $y_{ij} \in S_i \setminus \{x_{ij}\}$.

In fact, let $P = u_i \cdots u_{i_{r_i+1}}$ be a longest path satisfying the equation (**). Then $u_{i_2} \cdots u_{i_r} \notin \bigcup_j \bigcup_{i \neq i_j} S_j$ by condition 3). If $u_i, u_{i_{r_i+1}} \notin \bigcup_j \bigcup_{i \neq i_j} S_j$, then $P$ is desired. Otherwise, there exists a subset $S_{r_i+1}$ such that $\{u_i, u_{i_{r_i+1}}\} \cap S_{r_i+1} \neq \emptyset$. By the maximality of $P$ and $|S_{r_i+1}| \geq 2$, we have that $S_{r_i+1} = \{u_i, u_{i_{r_i+1}}\}$. Since $|\bigcup_{1 \leq j \leq r_i} S_j| \geq r$ for any $r \leq k$, we need only to consider the following two cases.

**Case 1.** $|\bigcup_{1 \leq j \leq r_i+1} S_j| = r_i + 1$.

Then $|V(G_0)| = k = r_i + 1$ by condition 4). Thus $C = u_i \cdots u_{i_{r_i+1}} u_i$ is a cycle of $G_0$ with $|V(C)| = |G_0|$ satisfying $(*)$.

**Case 2.** $|\bigcup_{1 \leq j \leq r_i+1} S_j| > r_i + 1$.

By condition 3), there is a $l \in \{1, \ldots, r_i\}$ such that $|S_{l_i}| \geq 3$. We assume without loss of generality that $\{x_{i_l}, y_{i_l}, z_{i_l}\} \subseteq S_{l_i}$ satisfying $x_{i_l} = u_{i_l}$, $y_{i_l} = u_{i_l+1}$ and $z_{i_l} \notin$
\( V(P) \). Then we can construct a new path \( P' = z_{i_1}u_{i_1}u_{i_1-1}\cdots u_{i_1}u_{i_r+1}u_r\cdots u_{i_r+1} \) which is longer than \( P \) and satisfies \((**)\) when the subscripts are rewritten. This contradiction completes the proof of the Claim.

Now, from the Claim above, if \( G_0 \) contains a cycle \( C \) with \( V(C) = |G_0| \) satisfying \((*)\), then it is easy to check that \( G \) is hamiltonian. Otherwise, by the Claim above, \( G_0 \) contains \( q \) pairwise disjoint paths \( P_i = u_{i_1}u_{i_2}\cdots u_{i_{r+1}}, i = 1, 2, \ldots, q \) which satisfy both \((**)\) and \((***)\), and we have \( \sum_{i=1}^{q} r_i = k \). Since \( G_0 \) is a complete subgraph of \( G \), we can easily check that \( G \) has a hamiltonian cycle.

Therefore, Theorem 1 is true. \( \diamond \)

Theorem 1 has the following consequence.

**Corollary 1.** Let \( G_0 \) be a complete subgraph of \( G \) with \( |G_0| \geq 2 \). If \( G_0 \) contains a pair of useful vertices of each component of \( G \setminus G_0 \) and \( G[N(G_0)] \) is C-free, then \( G \) is hamiltonian.

**Proof.** Let \( G_1, \ldots, G_k \) be all the components of \( G \setminus G_0 \) and set \( G^* = G[N(G_0)] \). By Theorem 1, it is sufficient to show that \( G_0, G_1, \ldots, G_k \) is a L-decomposition of \( G \).

By the hypothesis, we can choose \( S_i \subseteq V_{G_0}(G_i) \) such that \( S_i \) satisfies 2) of Definition 2 and \( |S_i| \) is as large as possible. Since \( G^* \) is C-free, 3) of Definition 2 is satisfied. Thus we only need to show that 4) of Definition 2 is also satisfied.

In fact, let \( r \leq k \) be any positive integer. Since \( G^* \) is C-free, we have \( |V(G_0)| \geq k \geq r \). If \( |V(G_0)| = k = r \), then \( |\bigcup_{1 \leq i \leq r} S_i| = r \). Conversely, if \( |\bigcup_{1 \leq j \leq r} S_{i_j}| = r \), then \( |S_{i_j}| = 2 \) \( (j = 1, 2, \ldots, r) \) and each vertex \( x \in \bigcup_{1 \leq j \leq r} S_{i_j} \) is a common vertex of some two pairs of useful vertices. Let \( x \in S_{i_1} \cap S_{i_2} \) and \( y \in N_{G_{i_1}}(x), z \in N_{G_{i_2}}(x) \). When \( |G_0| > r \), then there exists some \( w \in V(G_0) \setminus (\bigcup_{1 \leq j \leq r} S_{i_j}) \). Since \( G^* \) is C-free, we have \( wy \in E \) or \( wz \in E \). Therefore, either \( S_{i_1} \cup \{w\} \) or \( S_{i_2} \cup \{w\} \) still satisfies 2) of Definition 2, which contrary to the choice of \( S_i \), or \( S_i \). Thus \( |V(G_0)| = k = r \). This completes the proof of Corollary 1. \( \diamond \)

### 3 The Proof of Theorem 5

In order to prove Theorem 5, we need the following theorem.

**Theorem 6**[5]. If \( G \) is 3-connected and CN-free, then for any distinct vertices \( x, y \) of \( G \), there exists a hamiltonian path connecting \( x \) and \( y \).

Now, set \( V_0 = \{x \in V(G) : d(x) \geq n/2\} \). By Theorem 2, we may assume that \( G[V_0] \) is a complete subgraph of \( G \) if \( V_0 \neq \emptyset \). Let \( G_0 \) be a complete subgraph of \( G \) such that \( V_0 \subseteq V(G_0) \) and \( |V(G_0)| \) is as large as possible. Let \( G_1, \ldots, G_k \) be all the components of \( G \setminus G_0 \). Then by the property \( CB(n) \), \( G[N(G_0)] \) is C-free and \( G_0 \) is CB-free for any \( 1 \leq s \leq k \). By Corollary 1, we need only to show that \( G_0 \) contains a pair of useful vertices of \( G_s \) for \( 1 \leq s \leq k \).

Assume that there is a component \( G_s \) of \( G \setminus G_0 \) such that \( G_0 \) does not contain
any pair of useful vertices of $G$. Let $S$ be a minimal cut vertex set of $G_*$ and $v \in S$. Then by the assumption and Theorem 6, $|S| \leq 2$. Since $G_*$ is C-free, $G_* \setminus S$ has only two components $H_1, H_2$. Let $H = G[V(H_1) \cup V(H_2) \cup \{v\}]$ and $S_{-i} = \{u \in V(H_1) : d_H(u,v) = i\}$ and $S_i = \{u \in V(H_2) : d_H(u,v) = i\}$ for $i \geq 0$. Denote $m := \max\{i : S_i \neq \emptyset\}$ and $n := \max\{i : S_{-i} \neq \emptyset\}$. Clearly, we have $V(G_*) = S \cup \bigcup_{i=0}^{m} S_i$, and $G[S \cup S_j]$ is complete if and only if $|i-j| = 1$ since $G_*$ is CB-free.

If $|S| = 1$, then there exist some $x \in S_m$ and $y \in S_{-n}$ such that neither $x$ nor $y$ is a cut vertex of $G_*$ and $N_{G_0}(x) \neq \emptyset$ and $N_{G_0}(y) \neq \emptyset$. Since $G_*$ is 2-connected. Because of the structure of $G_*$, there exists a path $P$ connecting $x$ and $y$ in $G_*$ with $V(P) = V(G_*)$. Thus by the assumption, $N_{G_0}(x) = N_{G_0}(y)$ and $|N_{G_0}(x)| = 1$, which contrary to the fact that $G[N(G_0)]$ is C-free.

If $|S| = 2$, let $v' \in S$ and $v' \neq v$. Since $G_*$ is 2-connected$,$ $N(v') \cap S_i \neq \emptyset$ for some $1 \leq i \leq m$ and $N(v') \cap S_{-j} \neq \emptyset$ for some $1 \leq j \leq n$. Let $i_0 = \max\{i : N(v') \cap S_i \neq \emptyset\}$ and $j_0 = \max\{j : N(v') \cap S_{-j} \neq \emptyset\}$. By the hypothesis of Theorem 5, we may assume that there exists some $t$ with $0 \leq t \leq m$ such that $N_{G_0}(S_t) \neq \emptyset$.

Since $G_*$ is 2-connected, we have

(a) $|S_i| \geq 2$ for any $m - 1 \geq i \geq i_0$ and $|S_{-j}| \geq 2$ for any $n - 1 \geq j \geq j_0$.

By (a) and the structure of $G_*$, we have

(b) If $|S_m| \geq 2$ then for any two distinct vertices $x$ and $y$ in $S_m$, there exists a path $P$ in $G_*$ connecting $x$ and $y$ with $V(P) = V(G_*)$.

(c) For any $x \in S_{i-1}$ and $y \in S_i$ $(1 \leq i \leq m)$, there exists a path in $G_*$ connecting $x$ and $y$ with $V(P) = V(G_*)$.

Since $|N_{G_0}(G_*)| \geq 2$. By the assumption, (c) and the hypothesis of Theorem 5, we have

(d) $n + m \geq 3$.

Now, we distinguish the following two cases.

Case 1. $0 \leq t < m$, that is there exists some $x \in S_t$ and $y \in V(G_0)$ such that $xy \in E$.

Then by the hypothesis of Theorem 5 and $1 \leq t < m$, there exists a vertex $z \in S_{t-1}$ or $z \in S_{t+1}$ such that $yz \in E$. By the assumption and (c), for any $y' \in V(G_0) \setminus \{y\}$ and $w \in S_{t-1} \cup S_{t+1}$, $y'w \not\in E$. Thus we can find a vertex set $F = \{x, y, z, y', w\}$ such that $G[F]$ is a B-graph and does not satisfy the condition of Theorem 5, a contradiction.

Case 2. For any $0 \leq i \leq m - 1$, $N_{G_0}(S_i) = \emptyset$, that is $t = m$.

Symmetrically, we may assume that for any $0 \leq j \leq n - 1$, $N_{G_0}(S_{-j}) = \emptyset$.

If $N_{G_0}(v') \neq \emptyset$, let $y \in V(G_0)$ such that $v'y \in E$. Then by the hypothesis of Theorem 5, we have $y \in N_{G_0}(S_{i_0})$ or $y \in N_{G_0}(S_{-j_0})$. Thus $i_0 = m$ or $j_0 = n$.  

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Without loss of generality, let $y \in N_{G_0}(S_{i_0})$. When $y \notin N_{G_0}(S_{-j_0})$, then by the hypothesis of Theorem 5, there exists a vertex $y' \in V(G_0) \setminus \{y\}$ such that $y'y' \in E$ or $y' \in N_{G_0}(S_m)$ or $y' \in N_{G_0}(S_{-n})$ whenever $j_0 = n$. By the structure of $G_1$, we can derive that $(y, y')$ is a pair of useful vertices of $G_1$, contrary to the assumption.

When $y \in N_{G_0}(S_{i_0}) \cap N_{G_0}(S_{-j_0})$, that is $i_0 = m$ and $j_0 = n$. Since $G$ is 2-connected, there exists a vertex $y' \in V(G_0)$ such that $y' \in N_{G_0}(S_m) \cup N_{G_0}(S_{-n})$ or $y'y' \in E$. Also by the structure of $G_1$, we can derive that $(y, y')$ is a pair of useful vertices of $G_1$, contrary to the assumption. Hence in rest proof we suppose that $N_{G_0}(y') = \emptyset$.

Since $G$ is 2-connected, there exist $x \neq x' \in S_m \cup S_{-n}$ and $y \neq y' \in V(G_0)$ such that $xy \in E$ and $x'y' \in E$. By the assumption and (b), $(x, x') \not\subseteq S_m$ and $(x, x') \not\subseteq S_{-n}$. Let $x \in S_m$ and $x' \in S_{-n}$. By (d), let $m \geq 2$, then $S_m \in N(y)$ by the hypothesis of Theorem 5.

If $i_0 = m$, then $S_{m-1} \subseteq N(y')$ by the hypothesis of Theorem 5. Thus by the structure of $G_1$, we can derive that $(y, y')$ is a pair of useful vertices of $G_1$, contrary to the assumption. If $i_0 < m$, then $S_{i_0-1} \subseteq N(y')$. Thus we can also get a contradiction as before.

Therefore, Theorem 5 is true.

References


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