\textit{n-Extendability of Line Graphs, Power Graphs, and Total Graphs}

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Abstract

A graph \( G \) that has a perfect matching is \( n \)-extendable if every matching of size \( n \) lies in a perfect matching of \( G \). We show that when the connectivity of a line graph, power graph, or total graph is sufficiently large then it is \( n \)-extendable. Specifically: if \( G \) has even size and is \((2n + 1)\)-edge-connected or \((n + 2)\)-connected, then its line graph is \( n \)-extendable; if \( G \) has even order and is \((n + 1)\)-connected, then \( G^2 \) is \( n \)-extendable; if \( G \) has even order and is connected, then \( G^{2n+1} \) is \( n \)-extendable; if the total graph \( T(G) \) has even order and is \((2n + 1)\)-connected, then \( T(G) \) is \( n \)-extendable.

1 Introduction and terminology

All graphs considered in this paper are finite, undirected, connected and simple.
The vertex set and edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$ respectively. The cardinalities of $V(G)$ and $E(G)$ are called respectively the order and size of $G$. The line graph $L(G)$ of a graph $G$ is the graph whose vertex set is $E(G)$ and in which two vertices are joined if and only if they are adjacent edges in $G$. The iterated line graph $L^m(G)$ is defined recursively by $L^1(G) = L(G)$ and $L^m(G) = L(L^{m-1}(G))$ for $m > 1$. A power graph $G^k$ (the $k$th power of a graph $G$) is the graph whose vertices are those of $G$ and in which two distinct vertices are joined whenever the distance between them in $G$ is at most $k$. The vertices and edges of a graph are called elements. Two elements of a graph are neighbours if they are either incident or adjacent. The total graph $T(G)$ has vertex set $V(G) \cup E(G)$ and two vertices of $T(G)$ are adjacent whenever they are neighbours in $G$. The iterated total graph $T^m(G)$ is defined recursively by $T^1(G) = T(G)$ and $T^m(G) = T(T^{m-1}(G))$ for $m > 1$. The subdivision graph $S(G)$ of a graph $G$ is the graph obtained by replacing all edges of $G$ with paths of length two. The inserted vertices are called the subdivision vertices of $S(G)$. We use $P_{n+1}$ to denote a path of length $n$. The number of components of $G$ of odd order is denoted by $o(G)$. A matching of $G$ is a set edges no two of which are adjacent. The matching is perfect if it contains all the vertices of $G$. For the terminology and notation not defined in this paper, the reader is referred to [3].

We will need the following well known condition for the existence of a perfect matching.

**Tutte's Theorem** ([10]) A graph $G$ has a perfect matching if and only if for every subset $S$ of vertices, $|S| \leq o(G - S)$.

Let $n$ and $2m$ be positive integers with $n \leq m - 1$ and let $G$ be a graph with $2m$ vertices having a perfect matching (of size $m$). The graph $G$ is said to be $n$-extendable if every matching of size $n$ in $G$ lies in a perfect matching.

The $n$-extendability of symmetric graphs was studied in [1], [7], and [8]. In this paper we investigate the $n$-extendability of some locally dense graphs, namely, line graphs, power graphs and total graphs. The following lemma is useful.

**Lemma 1** ([4]) (1) If a line graph is connected and has even order, then it has a perfect matching. (2) If $G$ is a connected graph of even order, then $G^2$ has a perfect matching. (3) If a total graph is connected and has even order, then it has a perfect matching.

We show that when the connectivity of line graphs, power graphs and total graphs is sufficiently large, then they are $n$-extendable.

## 2 Line graphs

In this section, a necessary and sufficient condition for a line graph to be $n$-extendable is given. The next two lemmas follow immediately from the definition of a line graph.

**Lemma 2** If $D \subseteq E(G)$ then $L(G - D) = L(G) - D$. 

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Lemma 3 If $D \subseteq E(G)$ then the number of non-trivial components of $G - D$ equals the number of components of $L(G) - D$.

Theorem 4 Let $G$ be a graph of even size. Then $L(G)$ is $n$-extendable if and only if, for any collection $Q_1, Q_2, \ldots, Q_n$ of edge-disjoint $P_3$’s in $G$, $G - E(Q_1) - E(Q_2) - \cdots - E(Q_n)$ does not have a component of odd size.

Proof. Suppose $L(G)$ is $n$-extendable. Any edge disjoint $P_3$’s $Q_1, Q_2, \ldots, Q_n$ of $G$ correspond to $n$ independent edges $e_i = u_iv_i$ of $L(G)$ ($i = 1, 2, \ldots, n$). So $L(G) - \{u_1, v_1, \ldots, u_n, v_n\}$ has a perfect matching and therefore does not have any odd components. But each component of $L(G) - \{u_1, v_1, \ldots, u_n, v_n\}$ is the line graph of some component of $G - E(Q_1) - E(Q_2) - \cdots - E(Q_n)$. Hence no component of $G - E(Q_1) - E(Q_2) - \cdots - E(Q_n)$ has an odd number of edges.

For the converse, let edges $e_i = u_iv_i$ ($i = 1, 2, \ldots, n$) form a matching of $L(G)$. These edges correspond to $n$ edge disjoint $P_3$’s $Q_1, Q_2, \ldots, Q_n$ of $G$. By Lemma 1, the line graph of each component of $G - E(Q_1) - E(Q_2) - \cdots - E(Q_n)$ has a perfect matching. Thus $L(G) - \{u_1, v_1, \ldots, u_n, v_n\}$ has a perfect matching and $L(G)$ is $n$-extendable. □

Corollary 5 If a graph $G$ has even size and is $(2n + 1)$-edge-connected, then $L(G)$ is $n$-extendable.

Proof. Let $Q_1, Q_2, \ldots, Q_n$ be $n$ edge-disjoint $P_3$’s of $G$. Since $G$ is $(2n + 1)$-edge-connected, $G - E(Q_1) - E(Q_2) - \cdots - E(Q_n)$ is connected and therefore has no component with an odd number of edges. The result now follows from Theorem 4. □

The connectivity in Corollary 5 is the least possible. Let $F$ and $H$ be two disjoint graphs both isomorphic to $K_{2n+3}$ if $K_{2n+3}$ has odd size or to $K_{2n+3}$ with one edge deleted if $K_{2n+3}$ has even size. Join $F$ and $H$ by $n$ $P_3$’s such that the middle vertices of the $P_3$’s are $n$ different vertices of $F$ and the end vertices of the $n$ $P_3$’s are $2n$ different vertices of $H$. The resulting graph is $2n$-edge-connected, but deleting the edges of the $n$ $P_3$’s gives a component of odd size. By Theorem 4, its line graph is not $n$-extendable.

We have another version of Corollary 5.

Corollary 6 If $L(G)$ has even order and is $(2n + 1)$-connected, then $L(G)$ is $n$-extendable.

Corollary 7 If a graph $G$ has even size and is $(n + 2)$-connected, then $L(G)$ is $n$-extendable.

Proof. Suppose that $L(G)$ is not $n$-extendable. By Theorem 4 there are $n$ edge disjoint $P_3$’s $Q_1, Q_2, \ldots, Q_n$ of $G$ such that $G' = G - E(Q_1) - E(Q_2) - \cdots - E(Q_n)$ has a component of odd size and is therefore disconnected. Let $w_j$ be the middle vertex of $Q_j$ for $1 \leq j \leq n$. Let $W = \{w_1, \ldots, w_n\}$. Note that the $w_i$’s are not necessarily distinct. Let $v_1, \ldots, v_m$ be the distinct vertices of $W$. Suppose each $v_i$ is repeated $l_i$
times in $W$. $G$ is $(n+2)$-connected, so $G-W$ is connected. Also, since $G'$ has at least two components of odd size, there is a component $C$ of odd size that contains vertices only from $W$. Without loss of generality, let $V(C) = \{v_1, \ldots, v_r\}$. Note that $r \geq 2$ since $C$ has odd size. Assume that $l_i$ is the least of the $l_i$'s for $1 \leq i \leq r$ and that $v_1 = w_1 = \cdots = w_{l_i}$. The end vertices of $Q_1, Q_2, \ldots, Q_{l_i}$ and the vertices $v_2, \ldots, v_m$ form a cut set of order $2l_1 + (r-1) + (m-r) \leq 2l_1 + (1 + l_3 + \cdots + l_r) + (l_{r+1} + \cdots + l_m) \leq 1 + l_1 + l_2 + \cdots + l_m = n + 1$, contradicting the fact that $G$ is $(n+2)$-connected. □

The connectivity in Corollary 7 is also the least possible. Let $F$ be $K_n$ where $n = 4i + 2$ for some $i$. Let $H$ be $K_{2n}$ with one edge deleted. Both $F$ and $H$ have an odd number of edges. Join $F$ to $H$ with $n P_3$'s such that the middle vertices of the $n P_3$'s are the $n$ different vertices of $F$ and the end vertices of the $n P_3$’s are the $2n$ different vertices of $H$. The resulting graph is $(n+1)$-connected but deleting the edges of the $n P_3$’s gives a component of odd size. By Theorem 4, its line graph is not $n$-extendable.

We turn now to the iterated line graph $L^m(G)$.

**Lemma 8** ([5]) (1) If $G$ is $k$-connected, then $L(G)$ is $k$-connected. (2) If $G$ is $k$-edge-connected, then $L(G)$ is $(2k-2)$-edge-connected.

**Corollary 9** If $G$ is $(n+2)$-connected and $L^m(G)$ has even order, then $L^m(G)$ is $n$-extendable.

Proof. This follows from Corollary 7 and Lemma 8(1). □

If we relax the connectivity of $G$, then $L^m(G)$ is still $n$-extendable for sufficiently large $m$.

**Corollary 10** Let $k, m, n$ be positive integers and $2^m \geq (4n-2)/k$. If $G$ is $(k+2)$-edge-connected and $L^m(G)$ has even order then $L^m(G)$ is $n$-extendable.

Proof. From Lemma 8(2), $L^{m-1}(G)$ is $(2^{m-1}k + 2)$-edge-connected. The result now follows from Corollary 5. □

**Corollary 11** Let $k, m, n$ be positive integers and $2^m \geq (4n-2)/k$. If $G$ is $(k+2)$-connected and $L^m(G)$ has even order then $L^m(G)$ is $n$-extendable.

Proof. This follows from Corollary 10 since $G$ is at least $(k+2)$-edge-connected. □

### 3 Power graphs

In this section, we prove that when the connectivity of a graph $G$ is sufficiently large, $G^2$ is $n$-extendable. We also show that for any connected graph $G$, $G^r$ is $n$-extendable for sufficiently large $r$.

**Lemma 12** Let $G$ be a $k$-connected graph. Then $G^m$ is $km$-connected if $km$ is less than the order of $G$. □
Proof. Suppose \( S \) is a cutset of \( G^m \) and \( S \) contains less than \( km \) vertices. Let \( u \) and \( v \) be vertices separated in \( G^m \) by \( S \). Since \( G \) is \( k \)-connected, there are at least \( k \) internal vertex disjoint paths in \( G \) from \( u \) to \( v \). They must all contain a vertex from \( S \). There are fewer than \( m \) vertices from \( S \) in one of these paths. By choosing a different \( u \) and \( v \) if necessary, we can assume that all internal vertices of this path lie in \( S \). Thus, in \( G^m \), \( u \) and \( v \) are adjacent; a contradiction. \( \square \)

The following result shows that if the connectivity of a graph \( G \) is large, the square of \( G \) is \( n \)-extendable.

**Theorem 13** If \( G \) is \( k \)-connected with even order and \( k > n \), then \( G^r \) is \( n \)-extendable for \( r \geq 2 \).

Proof. Suppose \( G^r \) is not \( n \)-extendable. There are \( n \) independent edges \( e_i = u_iv_i \) (\( i = 1, 2, \ldots, n \)) which do not lie in any perfect matching of \( G^r \). Let \( H = G^r - \{u_1, v_1, \ldots, u_n, v_n\} \). By Lemma 12, \( H \) is connected. By Tutte’s Theorem, there is a cutset \( S \) of \( H \) such that \( o(H - S) > |S| \). By parity, \( o(H - S) = |S| + 2m \) for some positive integer \( m \). Let \( S' = S \cup \{u_1, v_1, \ldots, u_n, v_n\} \). Then \( |S'| = |S| + 2n \) and \( o(G^r - S') = o(H - S) = |S| + 2m \).

As \( G \) is \( k \)-connected, each component of \( G^r - S' \) is adjacent in \( G \) to at least \( k \) vertices of \( S' \). Suppose no two odd components of \( G^r - S' \) in \( G \) have a common neighbour in \( S' \). Then there are at least \((|S| + 2m)k \) vertices in \( S' \). But \( S' \) has only \(|S| + 2n < (|S| + 2m)k \) vertices. So at least two odd components \( C_1 \) and \( C_2 \) have in \( G \) a common neighbour \( v \) in \( S' \). Then there is vertex \( u \) in \( C_1 \) and a vertex \( w \) in \( C_2 \) such that \( u \) and \( w \) are both adjacent to \( v \). In \( G^r \), \( u \) and \( w \) are adjacent. So \( u \) and \( w \) are in the same component of \( G^r - S' \), contradicting the fact that \( C_1 \) and \( C_2 \) are different components of \( G^r - S' \). \( \square \)

The connectivity bound is sharp. Let \( F = K_{n+1} \) if \( n \) is even or \( K_{n+2} \) if \( n \) is odd. Let \( H \) be isomorphic to \( F \). Let \( e_i = u_iv_i \) (\( i = 1, 2, \ldots, n \)) be \( n \) independent edges which are vertex disjoint from \( F \) and \( H \). Join each \( u_i \) to every vertex of \( F \) and join each \( v_i \) to every vertex of \( H \). The resulting graph \( G \) is \( n \)-connected. But \( G^2 - \{u_1, v_1, \ldots, u_n, v_n\} \) has an odd component and therefore no perfect matching. Thus \( G^2 \) is not \( n \)-extendable.

If we relax the connectivity of \( G \), then its power graph \( G^r \) is still \( n \)-extendable for sufficiently large \( r \).

**Theorem 14** If \( G \) is \( k \)-connected with even order and \( 1 \leq k \leq n \), then \( G^r \) is \( n \)-extendable for \( r \geq 2(n - k) + 3 \).

Proof. Proceed as in the first paragraph of the proof for Theorem 13. Let \( C_1, C_2, \ldots, C_t \) be the components of \( G^r - S' \). Let \( N_i \) be the set of vertices of \( S' \) that are adjacent in \( G \) to vertices of \( C_i \). Since \( G \) is \( k \)-connected, each \( N_i \) contains at least \( k \) vertices. Also, the \( N_i \) are pairwise disjoint otherwise one of the components \( C_i \) contains a vertex \( u \) that is distance two from a vertex \( v \) in some other component \( C_j \) but then \( u \) and \( v \) would be in the same component of \( G^r \). Since \( G \) is connected, there is a path \( P \) in \( G \) from a vertex \( w_i \) in \( N_i \) to a vertex \( w_j \) in \( N_j \) (\( j \neq i \)). By
assuming $P$ has the minimum length among all such paths, $P$ is contained in $S'$ and the internal vertices of $P$ have no vertex in $N_I$ for $1 \leq l \leq t$. Since $|S'| = |S| + 2n$ and $t \geq |S| + 2m$, the order of $P$ is at most $|S| + 2n - k(|S| + 2m) + 2 \leq |S| + 2n - k(|S| + 2) + 2 = 2(n - k) - |S|(k - 1) + 2 \leq 2(n - k) + 2$. There is a vertex $z_i$ in $C_i$ and a vertex $z_j$ in $C_j$ adjacent to $w_i$ and $w_j$ respectively. Then $z_iPz_j$ is a path of length at most $2(n - k) + 3$. So $z_i$ and $z_j$ are adjacent in $G^r$, contradicting the fact that $C_i$ and $C_j$ are different components of $G^r - S'$. □

The bound on $r$ in Theorem 14 is the least possible. Let $G = u_0u_1\ldots u_{2n}u_{2n+1}$ be a path. Let $e_i = u_{2i-1}u_{2i}$ ($i = 1, 2, \ldots n$). Since $G^{2n} - \{u_1, u_2, \ldots u_{2n}\}$ has an odd component ($u_0$ or $u_{2n+1}$) it does not have a perfect matching. We can replace $u_0$ or $u_{2n+1}$ by odd components, and the resulting graph will still be a counterexample.

4 Total graphs

In this section we show that when the connectivity of a total graph $T(G)$ is sufficiently large, then $T(G)$ is $n$-extendable. We quote three useful lemmas.

Lemma 15 ([2]) For any graph $G$, $T(G) = (S(G))^2$.

Lemma 16 Let $G$ be a connected graph and let $w$ be a vertex in a cutset $R$ of $T(G)$.

1. If $w$ is a subdivision vertex of $S(G)$, then $w$ is adjacent to at most two components of $T(G) - R$. (2) If $R$ contains no subdivision vertices of $S(G)$, then $w$ is adjacent to exactly one component of $T(G) - R$.

Proof. This follows immediately from Lemma 15. □

Theorem 17 If $T(G)$ is $(2n + 1)$-connected and has even order, then $T(G)$ is $n$-extendable.

Proof. Suppose $T(G)$ is not $n$-extendable. There are $n$ independent edges $e_i = u_iv_i$ ($i = 1, 2, \ldots n$) which do not lie in a perfect matching of $T(G)$. Let $T' = T(G) - \{u_1, v_1, \ldots, u_n, v_n\}$. By Tutte's Theorem, there is a subset $S'$ of vertices of $T'$ such that $o(T' - S') > |S'|$. By parity, $o(T' - S') = |S'| + 2m$ for some positive integer $m$. Let $S = S' \cup \{u_1, v_1, \ldots, u_n, v_n\}$. Then $o(T(G) - S) = o(T' - S') = |S'| + 2m = |S'| - 2n + 2m$. Let $C_1, C_2, \ldots$ denote the odd components of $T(G) - S$.

We now reduce $S$ while keeping the relation $o(T(G) - S) = |S'| - 2n + 2m$ ($m \geq 1$). Let $w$ be a vertex in $S$ and replace $S$ with $S'' = S\setminus\{w\}$.

If $w$ is not adjacent to any odd component, then $o(T(G) - S'') = o(T(G) - S) + 1 = |S''| - 2n + 2(m + 1)$.

Suppose every vertex of $S$ is adjacent to an odd component. If $w$ is a subdivision vertex of $S(G)$, then, by Lemma 16, $w$ is adjacent to at most two odd components. If $w$ is adjacent to two odd components $C_i$ and $C_j$, then the subgraph of $T(G) - S''$ induced by $C_i \cup \{w\} \cup C_j$ is an odd component and $o(T(G) - S'') = |S''| - 2n + 2m$. If $w$ is adjacent to only one odd component $C_i$, then again $o(T(G) - S'') = |S''| - 2n + 2m$.
If $S$ does not contain any subdivision vertex of $S(G)$, then, by Lemma 16, $w$ is adjacent to exactly one odd component and again $o(T(G) - S'') = |S''| - 2n + 2m$.

Repeat the process above until $|S'| = 2n$. Then $o(T(G) - S) = |S'| - 2n + 2m = 2m \geq 2$. Thus $S$ is a cutset of $T(G)$ of order $2n$, a contradiction. □

If we relax the connectivity of $G$ then its iterated total graph $T^r(G)$ is still $n$-extendable for sufficiently large $r$.

**Lemma 18** ([6, 9]) If $G$ is $k$-connected, then $T(G)$ is $2k$-connected.

**Corollary 19** Let $G$ be $k$-connected and $2^r > 2n/k$. The iterated total graph $T^r(G)$ is $n$-extendable if it has even order.

Proof. This follows immediately from Lemma 18 and Theorem 17. □

Note that if $G$ is $k$-connected, then $T(G)$ may be exactly $2k$-connected. Let $w$ be a vertex of degree $k$. Then $w$ has $2k$ neighbours in $T(G)$ which form a cutset. On the other hand the connectivity of $T(G)$ may be considerably higher than $2k$. For example, let $G$ be the graph formed by identifying a vertex from $K_{4p}$ with a vertex of $K_{4p+1}$. Then $G$ is 1-connected but $T(G)$ has even order and is $(8p - 2)$-connected. Thus Theorem 17 is more powerful than Corollary 19.

The connectivity in Theorem 17 and inequality in Theorem 18 are sharp. Let $G$ be a $k$-connected $k$-regular graph. Suppose $2^r k = 2n$. Since $T^1(G)$ is $2^1 k$-regular, $T^i(G)$ is exactly $2^r k$-connected by Lemma 18. By Lemma 15 $T^r(G) = (S(T^{r-1}(G)))^2$. Let $w$ be a vertex in $T^{r-1}(G)$, let $w_i (i = 1, 2, \ldots, 2^{r-1} k)$ be the vertices of $T^{r-1}(G)$ adjacent to $w$ and let $u_i$ be the subdivision vertex on $ww_i$ in $S(T^{r-1}(G)) (i = 1, 2, \ldots, 2^{r-1} k)$. Then the $u_i w_i$ are $2^{r-1} k = n$ independent edges of $T^r(G)$. But $T^r(G) - \{u_i, w_i | i = 1, 2, \ldots, 2^{r-1} k\}$ does not have a perfect matching as $w$ is an isolated vertex. So $T^r(G)$ is not $n$-extendable.

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**References**


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