On complementary path decompositions of the complete multigraph

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Abstract: We give a complete solution to the existence problem for complementary $P_3$-decompositions of the complete multigraph, where $P_3$ denotes the path of length 3.

A complementary decomposition $2\lambda K_v \rightarrow (P_3, P_3)$ is an edge decomposition of the complete multigraph $\lambda K_v$ into $P_3$'s with the property that upon taking the complement of each path one obtains a second decomposition of $\lambda K_v$ into $P_3$'s (where the complement of the path $abcd$ is the path $bdac$). The following result was proven by Granville, Moisiasidis and Rees in [1] (and, with a few small exceptions, also follows from the techniques in [3]):

**Theorem 1.** There exists a complementary decomposition $2K_v \rightarrow (P_3, P_3)$ if and only if $v \equiv 1 \pmod{3}$.

In this paper, we give a complete solution to the existence problem for complementary $P_3$-decompositions of the complete multigraph $2\lambda K_v$. Note that if $D$ is such a decomposition then the set $\{(a, b, c, d) : abcd \in D\}$ is an edge decomposition of $2\lambda K_v$ into $K_4$'s, that is, a $(v, 4, 2\lambda)$-BIBD. The following result was proven by Hanani in [2]:

**Lemma 2.** If there exists a complementary decomposition $2\lambda K_v \rightarrow (P_3, P_3)$, then

$$\lambda(v - 1) \equiv 0 \pmod{3}$$

and

$$\lambda v(v - 1) \equiv 0 \pmod{6}.$$

As a consequence of the remarks above and the results in Hanani [2], we need consider only the case $\lambda = 3$.

It is easy to see that the existence of a \((v, 4, \lambda)\)-BIBD implies the existence of a 
\(2\lambda K_v \rightarrow (P_3, P_3)\). From [2] we then have

**Lemma 3.** There exists a complementary decomposition \(6K_v \rightarrow (P_3, P_3)\) if \(v \equiv 0, 1 \pmod{4}\).

For our proof, we also need the following initial block constructions.

**Lemma 4.** There exists a complementary decomposition \(6K_v \rightarrow (P_3, P_3)\) if \(v \equiv 0, 2 \pmod{6}\).

**Proof.** In \(\mathbb{Z}_{v-1} \cup \{\infty\}\), the required initial blocks are

\[
(\infty, 0, 1, 2), \\
(0, 1, \infty, 3), \\
(0, k, 2k, 3k), \quad 2 \leq k \leq \frac{1}{2}(v - 2).
\]

**Lemma 5.** There exists a complementary decomposition \(6K_q \rightarrow (P_3, P_3)\) if \(q \equiv 3 \pmod{4}\) is a prime power.

**Proof.** Let \(d = \frac{1}{2}(q - 1)\), and let \(x\) be a generator of \(GF(q)\); then the required initial blocks are

\[
(x^i, x^{i+1}, x^{i+d}, x^{i+d+1}), \quad 0 \leq i \leq d - 1.
\]

**Lemma 6.** There exists a complementary decomposition \(6K_{15} \rightarrow (P_3, P_3)\).

**Proof.** In \(\mathbb{Z}_{15}\), the required initial blocks are

\[
(0, 1, 5, 10), \\
(0, 5, 10, 3), \\
(0, 12, 14, 8), \\
(0, 2, 3, 11), \text{ two times}, \\
(0, 12, 8, 14), \text{ two times}.
\]

**Theorem 7.** For every integer \(v \geq 4\), there exists a complementary decomposition \(6K_v \rightarrow (P_3, P_3)\).

**Proof.** By Hanani [2], it suffices to show that there exists a complementary decomposition for all \(v \in \{4, 5, \ldots, 12, 14, 15, 18, 19, 23, 27\}\). If \(v \equiv 1 \pmod{3}\), then Theorem 1 gives the result; if \(v \equiv 0, 1 \pmod{4}\), then Lemma 3 gives the result; if \(v \equiv 0, 2 \pmod{6}\), then Lemma 4 gives the result; if \(v \equiv 3 \pmod{4}\) is a prime power, then Lemma 5 gives the result; for the remaining case \(v = 15\), Lemma 6 gives the result. The proof is now complete.

**Theorem 8.** There exists a complementary decomposition \(2\lambda K_v \rightarrow (P_3, P_3)\) if and only if

\[
\lambda(v - 1) \equiv 0 \pmod{3}
\]

and

\[
\lambda v(v - 1) \equiv 0 \pmod{6}.
\]

**Proof.** That these conditions are necessary follows from Lemma 2. We need consider only values of \(\lambda\) which are factors of 3, because if \(\lambda_1 \mid \lambda_2\) then the existence
of a $2\lambda_1 K_v \rightarrow (P_3, P_3)$ implies the existence of a $2\lambda_2 K_v \rightarrow (P_3, P_3)$. Thus we have the following cases:

$$\lambda = 1 \quad v \equiv 1 \pmod{3},$$

$$\lambda = 3 \quad \text{all } v \geq 4.$$

In Theorems 1 and 7 we have established the existence of the required designs.

References


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