ON MAXIMAL SETS OF ONE-FACTORS

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ABSTRACT: A set $S$ of edge-disjoint one-factors in a graph $G$ is said to be maximal if there is no one-factor of $G$ which is edge-disjoint from $S$, and if the union of $S$ is not all of $G$. Maximal sets of one-factors of $K_{2n}$ have been investigated and until very recently only results for particular cases have been obtained. In this paper we present a new technique for solving the problem.

1. INTRODUCTION

We consider graphs which are undirected, finite, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [1]. Thus $G$ is a graph with vertex set $V(G)$, edge set $E(G)$, $\nu(G)$ vertices and $\varepsilon(G)$ edges. $K_n$ denotes the complete graph on $n$ vertices and $K_{n,m}$ denotes the complete bipartite graph with bipartitioning sets of size $n$ and $m$.

A 1-factor of a graph $G$ is a 1-regular spanning subgraph. A 1-factorization of $G$ is a set of (pairwise) edge-disjoint one factors which between them contain each edge of $G$. It is very well known (see [3]) that $K_{2n}$ and $K_{n,n}$ have 1-factorizations for all $n$.

A set $F$ of edge-disjoint 1-factors in a graph $G$ is said to be maximal if there is no 1-factor which is edge-disjoint from $F$ and if $F$

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is not all of $G$. Thus if we write $\overline{F}$ for the complement in $G$ of the union of members of $F$, then $F$ is maximal if and only if $\overline{F}$ is a non-empty graph with no 1-factor. We call $\overline{F}$ the leave of $F$. Observe that if $G$ is regular, then $\overline{F}$ is regular. If $\overline{F}$ is $d$-regular, then $F$ is called a maximal set of deficiency $d$ or simply a $d$-set. The existence of $d$-sets in $K_{2n}$ for $n > 2$ was shown by Cousins and Wallis [4].

Caccetta and Wallis [2] established that $3$-sets exist in $K_{2n}$ for every $2n \geq 16$. This was accomplished by first establishing properties which reduced the problem to one of finding $3$-sets in $K_{2n}$ for $16 \leq 2n \leq 28$, and then exhibiting the required $3$-sets. In this paper we generalize these methods. In particular, we prove that if $K_{2n}$ has a $d$-set, then $K_{4n-2t}$ has a $d$-set for each $0 \leq t \leq n - \frac{1}{2}(d + 1)$. We apply this result to show that $5$-sets exist in $K_{2n}$ for every $2n \geq 22$.

Recently, Rees and Wallis [6] solved the problem of determining the spectrum of maximal sets of 1-factors in $K_{2n}$. Our approach is, however, quite different and has the potential to yield a simpler and more intuitive proof. Our main result is of interest in its own right.

2. PRELIMINARIES

In this section we discuss three results which we make use of in the proof of our main theorem. A matching $M$ in a graph $G$ is a subset of $E(G)$ in which no two edges have a common vertex. We begin by stating a lemma proved in Rees and Wallis [6].

**Lemma 2.1.** Let $K_{m,n}$ be the complete bipartite graph with bipartition $(X, Y)$, where $|X| = m$, $|Y| = n$ and $m \leq n$. Let $Y_1, Y_2, \ldots, Y_n$ be any collection of $m$-subsets of $Y$ such that each vertex $y \in Y$ is contained in exactly $m$ of the $Y_j$'s. Then there is an edge-decomposition of $K_{m,n}$ into matchings $M_1, M_2, \ldots, M_n$ where for each $j = 1, 2, \ldots, n$ $M_j$ is a matching with $m$ edges from $X$ to $Y_j$. \hfill \Box

The edge-chromatic number $\chi'(G)$ of a graph $G$ is the minimum
number of colours needed to colour the edges of $G$. Our next lemma is a special case of a theorem of Folkman and Fulkerson [5]. The proof we give was given to us in a personal communication by Rees.

Lemma 2.2. If $G$ is a graph with $c.k$ edges and $c \geq \chi'(G)$, then the edge set of $G$ admits a decomposition into $c$ matchings, each with $k$ edges.

Proof: Let $\mathcal{C}$ be the set of all proper $c$-colourings of $G$. Note that $\mathcal{C} \neq \emptyset$ since $c \geq \chi'(G)$. For $K \in \mathcal{C}$, define

$$n(K) = \sum_{i=1}^{c} |e_i - k|,$$

where $e_i$ is the number of edges in the $i^{th}$ matching (i.e. $i^{th}$ colour class) of $K$, $i = 1, 2, \ldots, c$.

Let

$$n_0 = \min\{n(K) : K \in \mathcal{C}\},$$

and let $K_0$ be a colouring for which $n(K_0) = n_0$. We will prove that $n_0 = 0$, i.e. $K_0$ is a decomposition of $G$ into $c$ matchings, each with $k$ edges. Suppose that this is not the case and $n(K_0) > 0$. Then there is a matching $M_1$ for which $e_1 = |M_1|$ is not $k$. Now since $\chi(G) = ck$, there must be matchings $M_1$ and $M_2$ say, with $e_1 = |M_1| < k$ and $e_2 = |M_2| > k$.

Let $H$ be the subgraph of $G$ whose edge set is $M_1 \cup M_2$. Then $H$ is the disjoint union of cycles and paths. Since $e_2 > e_1$, $H$ must contain as a component a path $P$ of odd length which begins and ends with an edge of $M_2$. Now switch the colours in $P$, i.e. those edges of $P$ that were coloured 1 get coloured 2 and vice-versa. Let us call the matchings corresponding to these colour changes $M_1'$ and $M_2'$. This creates a new colouring $K_0'$ of $G$ with corresponding matchings $M_1', M_2', M_3', \ldots, M_c'$. Furthermore,

$$e_1' = |M_1'| = e_1 + 1,$$

and

$$e_2' = |M_2'| = e_2 - 1.$$
Now recalling that $e_1 < k$ and $e_2 > k$, we have

$$|e_1' - k| < |e_1 - k|,$$

and

$$|e_2' - k| < |e_2 - k|.$$  

Hence

$$n(K_0') < n(K_0),$$

and this contradicts the minimality of $n(K_0)$. It thus follows that $n_0 = 0$. This proves the lemma.

We conclude this section by stating a result of Wallis [7].

**Lemma 2.3.** A $d$-regular graph $G$ with no 1-factor and no odd-component satisfies:

$$\nu(G) \geq \begin{cases} 3d + 7, & \text{for odd } d \geq 3 \\ 3d + 4, & \text{for even } d \geq 6 \\ 22, & \text{for } d = 4. \end{cases}$$

No such $G$ exists for $d = 1$ or 2.

**3. MAIN RESULT**

Our main result is essentially a generalization of Theorems 4 and 5 of Caccetta and Wallis [2].

**Theorem 3.1.** Suppose for odd $d$ there exists a $d$-set in $K_{2n}$. Then for each $0 \leq t \leq n - \frac{1}{2}(d + 1)$ there is a $d$-set in $K_{4n-2t}$.

**Proof:** We can write $K_{4n-2t} = K_{2n-2t} \lor K_{2n}$. Let $X$ and $Y$ denote the graphs $K_{2n-2t}$ and $K_{2n}$, respectively. Now $Y$ has a maximal set of $(2n - d - 1)$ 1-factors. Take $2t$ of these 1-factors and let $H$ be the graph formed by the union of these 1-factors.
Applying Lemma 2.2 (with c = 2n and k = t) we decompose the edge-set of H into 2n matchings $M_1, M_2', ..., M_{2n}$, each with t edges. Let $Y_i$ denote the vertices of Y not saturated by the matching $M_i$. Note that since H has regularity 2t, each vertex in Y will be contained in exactly 2n-2t of the $Y_i$'s. Furthermore, each $Y_i$ contains exactly 2n-2t vertices of Y.

Now we apply Lemma 2.1 to the subgraph $K_{2n-2t, 2n}$. This yields 2n disjoint matchings $N_1', N_2', ..., N_{2n}'$, where $N_i'$ joins the vertices of $Y_i$ to the vertices of X. Let

$$L_i = M_i \cup N_i$$

for $i = 1, 2, ..., 2n$.

There remain in Y a set S of $(2n - 1 - d) - 2t$ 1-factors from the original maximal set on Y. Construct $(2n - 1 - d) - 2t$ 1-factors on X (such a set exists since $K_{2p}$ has a 1-factorization) and pair these off with the 1-factors of S to form a set of $(2n - 1 - d - 2t)$ 1-factors $L'_1, L'_2, ..., L'_{2n-1-d-2t}$. Then the set

$$F = \{L_i : i = 1, 2, ..., 2n\} \cup \{L'_j : j = 1, 2, ..., 2n-1-d-2t\}$$

forms a maximal set of 1-factors of deficiency d in $K_{4n-2t}$. Note that the leave $\overline{F}$ of F consists of 2-components one of which is the leave of the maximal set of 1-factors in $K_{2n}$. This completes the proof of the theorem.

As a corollary we have:

**Corollary:** If $K_{2n}$ has a d-set, d odd, then for each even integer $m \geq 2n + d + 1$, $K_m$ has a d-set.

**Proof:** Suppose $K_{2n}$ has a d-set, d odd. Then by Theorem 3.1 there exists a d-set in $K_{2n+d+1}, K_{2n+d+3}, ..., K_{4n}$. Further a d-set in $K_{2n+d+1}$ implies a d-set in $K_{2n+2d+2}, K_{2n+2d+4}, ..., K_{4n+2d+2}$. Now since a d-set in $K_{2n}$ implies (Dirac's Theorem) that $d \leq n$ we have $2n + 2d + 2 \leq 4n + 2$. Hence repeated applications of Theorem 3.1 will in fact cover all even integers $m \geq 2n + d + 1$. This completes the proof of the Corollary.
4. **APPLICATION OF THEOREM 3.1**

We now discuss the application of Theorem 3.1. First we consider the existence of 3-sets in $K_{2n}$. Since, by Lemma 2.3, the smallest 3-regular graph without a 1-factor contains at least 16 vertices, $K_{2n}$ has no 3-set for $2n \leq 14$. A 3-set in $K_{18}$ was shown in [2]. The above result implies that if we can find a 3-set in $K_{18}$, then we have a 3-set in $K_{2n}$ for every $2n \geq 16$. This is the case as shown in [2]. We remark that the proof that $K_{2n}$ has a 3-set for every $2n \geq 16$ in [2] involved the construction of 3-sets in $K_{2n}$ for $16 \leq 2n \leq 28$. Application of Theorem 3.1 eliminates the need to look at the cases $20 \leq 2n \leq 28$.

We now illustrate the work involved in establishing the existence of d-sets, by consider the case $d = 5$.

Lemma 2.3 implies that 5-sets do not exist in $K_{2n}$ for $2n \leq 20$. So suppose $2n \geq 22$. We will exhibit 5-sets in $K_{22}$, $K_{24}$ and $K_{26}$. Then the corollary to Theorem 3.1 implies the existence of 5-sets in $K_{2n}$ for every $2n \geq 22$. 

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Consider $K_{22}$ with vertices labelled $1, 2, \ldots, 9, A, B, \ldots, M$. Take the 16 1-factors:

\begin{align*}
T_1 &= 18 \ 25 \ 3D \ 4L \ JC \ 6H \ 7A \ 9I \ BF \ EK \ MG \\
T_2 &= 15 \ 2G \ 3E \ 4I \ 8H \ 6A \ 7J \ 9F \ BK \ CM \ DL \\
T_3 &= 19 \ 2E \ 3L \ 4H \ 5D \ 6J \ 7G \ 8K \ AF \ BM \ CI \\
T_4 &= 1A \ 2H \ 3F \ 4M \ 5C \ 6I \ 7K \ 8L \ 9D \ BG \ EJ \\
T_5 &= 1B \ 2F \ 3K \ 4C \ 5I \ 6D \ 7L \ 8J \ 9M \ AH \ GE \\
T_6 &= 1C \ 29 \ 3H \ 4J \ 5K \ 6F \ 7E \ 8G \ DI \ AM \ BL \\
T_7 &= 1D \ 2I \ 3M \ 4F \ 5A \ 6G \ 7H \ 8C \ 9K \ BJ \ EL \\
T_8 &= 1E \ 2C \ 3J \ 48 \ 5B \ 69 \ 7D \ AI \ LH \ MF \ KG \\
T_9 &= 1F \ 2L \ 3G \ 4E \ 59 \ 6M \ 7C \ 8I \ AJ \ DK \ BH \\
T_{10} &= 1G \ 2M \ 38 \ 47 \ 5E \ 6B \ LI \ AK \ CH \ DF \ 9J \\
T_{11} &= 1H \ 2J \ 3A \ 4K \ 5G \ 6C \ 7M \ 8F \ 9L \ BI \ DE \\
T_{12} &= 1I \ 2A \ 36 \ 4B \ 58 \ HE \ 7F \ LC \ SG \ DJ \ KM \\
T_{13} &= 1J \ 28 \ 3C \ 49 \ 5L \ DG \ 7I \ 8A \ FE \ HM \ K5 \\
T_{14} &= 1K \ 2B \ 79 \ 4A \ 5F \ 6L \ 3I \ 8E \ CG \ DH \ JM \\
T_{15} &= 1L \ 2K \ 3B \ 4D \ 5J \ 6E \ 78 \ 9H \ AG \ FC \ MI \\
T_{16} &= 1M \ 2D \ 39 \ 4G \ 5H \ 68 \ 7B \ CK \ AL \ FJ \ EI \\
\end{align*}

The leave of this set of 1-factors is given in Figure 4.1. Thus we have a 5-set in $K_{22}$.

![Figure 4.1](image-url)
Consider $K_{24}$ with vertices labelled $1, 2, \ldots, 9, A, B, \ldots, 0$. Take the 18 1-factors:

$R_1 = 14 \ 2J \ 36 \ DI \ 5G \ 8F \ 7E \ NO \ SH \ AL \ BM \ CK$

$R_2 = 16 \ 2D \ 30 \ 4B \ 58 \ 9L \ 7C \ JN \ AE \ FG \ HK \ MI$

$R_3 = 17 \ 25 \ 3L \ 48 \ 6M \ 9K \ AN \ BF \ CD \ EH \ GI \ JO$

$R_4 = 1A \ 2N \ 3M \ 47 \ 59 \ 6J \ 8C \ BL \ DH \ EQ \ FI \ GK$

$R_5 = 1B \ 20 \ 3N \ 4M \ 5L \ 69 \ 7G \ 8A \ CH \ DJ \ EI \ FK$

$R_6 = 10 \ 2K \ 3I \ 4L \ 5B \ 6C \ 7M \ 8D \ 9A \ CJ \ EN \ FH$

$R_7 = 1D \ 2C \ 3E \ 4F \ 5I \ 6N \ 7H \ 8M \ 9C \ AK \ BJ \ LO$

$R_8 = 1C \ 2F \ 3H \ 4G \ 5O \ 6K \ 7N \ 8I \ 9B \ AM \ EJ \ DL$

$R_9 = 1E \ 2H \ 3F \ 40 \ 5J \ 6G \ 7K \ 8L \ 9D \ AI \ BN \ CM$

$R_{10} = 1L \ 2A \ 37 \ 4K \ 5C \ 6E \ 8J \ 9F \ BH \ DM \ GN \ IO$

$R_{11} = 1G \ 2B \ 3A \ 4J \ 5H \ 6D \ 7I \ 8K \ 9E \ CL \ MO \ FN$

$R_{12} = 1H \ 2L \ 3J \ 4A \ 5E \ 6B \ 7D \ 8G \ 9N \ CJ \ FM \ KO$

$R_{13} = 1I \ 26 \ 39 \ 4C \ 5D \ 70 \ 8N \ AH \ BK \ EL \ FJ \ GM$

$R_{14} = 1J \ 28 \ 3K \ 4I \ 5F \ 6A \ 7B \ 9G \ CO \ DN \ EM \ HL$

$R_{15} = 1K \ 2M \ CN \ 4E \ 5A \ 6H \ 7F \ 80 \ 9J \ GL \ 3D \ BI$

$R_{16} = 1F \ 2C \ 3B \ DO \ 6L \ 7J \ 9I \ 5M \ AG \ EK \ 4N \ 8H$

$R_{17} = 1M \ 2E \ 3G \ DK \ 5N \ FL \ 7A \ 8B \ 90 \ 4H \ 6I \ CJ$

$R_{18} = 1N \ 21 \ 3C \ 4D \ 5K \ 6F \ 7L \ 8E \ 9M \ HG \ AJ \ BO$

The leaf of this set of 1-factors is given in Figure 4.2. We thus have a 5-set in $K_{24}$.
Finally, consider $K_{26}$ with vertices labelled $1,2,\ldots,9$, $A,B,\ldots,Q$. A suitable 5-set is:

$$
\begin{align*}
T_1 &= 14 \quad 2G \quad 3Q \quad 5L \quad 6J \quad 7M \quad 8C \quad 9H \quad AO \quad BF \quad DP \quad EK \quad NI \\
T_2 &= 15 \quad 27 \quad 39 \quad 4C \quad 6B \quad 8A \quad DK \quad EQ \quad FN \quad GM \quad HL \quad IO \quad JP \\
T_3 &= 16 \quad 2A \quad 3C \quad 49 \quad 50 \quad 7D \quad 8B \quad EP \quad FI \quad GK \quad HN \quad JM \quad LQ \\
T_4 &= 17 \quad 28 \quad 36 \quad 4B \quad 5A \quad 9C \quad DE \quad FH \quad GJ \quad IL \quad KN \quad MP \quad OQ \\
T_5 &= 1A \quad 2D \quad 3B \quad 47 \quad 5I \quad 6N \quad 8F \quad 9P \quad CO \quad EJ \quad GL \quad HK \quad MQ \\
T_6 &= 1B \quad 2M \quad 3H \quad 4L \quad 5K \quad 6P \quad 7A \quad 8E \quad 9J \quad CQ \quad DI \quad FO \quad GN \\
T_7 &= 1C \quad 2L \quad 3J \quad 4G \quad 5P \quad 6M \quad 7H \quad 8I \quad 9B \quad AK \quad DN \quad EO \quad FQ \\
T_8 &= 1D \quad 2F \quad 3N \quad 4K \quad 5E \quad 6H \quad 7L \quad 8O \quad 9A \quad BI \quad CM \quad GP \quad JQ \\
T_9 &= 1E \quad 2B \quad 38 \quad 4A \quad 5N \quad 6L \quad 7C \quad DM \quad 9K \quad FP \quad GH \quad IQ \quad JO \\
T_{10} &= 1F \quad 2Q \quad 3D \quad 4P \quad 5J \quad 60 \quad 7G \quad 8K \quad 9I \quad AB \quad CL \quad EN \quad HM \\
T_{11} &= 1G \quad 20 \quad 3P \quad 4D \quad 5C \quad 6F \quad 7J \quad 8M \quad 9N \quad BL \quad AH \quad EI \quad KQ \\
T_{12} &= 1H \quad 2E \quad 3A \quad 4N \quad 5Q \quad 6G \quad 70 \quad 8P \quad 9M \quad BK \quad CI \quad DL \quad FJ \\
T_{13} &= 1I \quad 2N \quad 3M \quad 40 \quad 58 \quad 6A \quad 7E \quad 9G \quad BH \quad CJ \quad DQ \quad FL \quad KP \\
T_{14} &= 1J \quad 2K \quad 3G \quad 4E \quad 5B \quad 6I \quad 7P \quad 8H \quad 9Q \quad AL \quad CN \quad DO \quad FM \\
T_{15} &= 1K \quad 2J \quad 30 \quad 4Q \quad 5F \quad 6D \quad 7N \quad 8L \quad 9E \quad AP \quad BM \quad CH \quad GI \\
T_{16} &= 1L \quad 2H \quad 3E \quad 4I \quad 5M \quad 6K \quad 7F \quad 8Q \quad 9D \quad AN \quad CP \quad BJ \quad GO \\
T_{17} &= 1M \quad 2P \quad 3L \quad 4F \quad 5D \quad 6Q \quad 7I \quad 8G \quad 9O \quad AJ \quad BN \quad CK \quad EH \\
T_{18} &= 1N \quad 2I \quad 3K \quad 4J \quad 5H \quad 6C \quad 7Q \quad 8D \quad 9F \quad AG \quad BP \quad EM \quad LO \\
T_{19} &= 1O \quad 2C \quad 3I \quad 4H \quad 5G \quad 69 \quad 7B \quad 8N \quad AM \quad EL \quad DJ \quad FK \quad PQ \\
T_{20} &= 1P \quad 2S \quad 3F \quad 4M \quad DH \quad 6E \quad 7K \quad 8J \quad 9L \quad AI \quad BO \quad CG \quad NQ
\end{align*}
$$

The leave of this set is given in Figure 4.3.

![Figure 4.3](image_url)
We have proved:

Theorem 4.1. There exists a 5-set in $K_{2n}$ for every $2n \geq 22$. 

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