Excess Graphs and Bicoverings

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Abstract. The classical bicovering problem seeks to cover all pairs from a v-set by a family F of k-sets so that every pair occurs at least twice and the cardinality of F is minimal. A weight function is introduced for blocks in such a design, and its use in constructing bicoverings is illustrated.

1. Introduction.

A covering is a collection of k-sets (blocks) chosen from elements of a v-set so that each pair from the v elements occurs at least once. The cardinality of a minimal covering is written N_{\lambda}(2,k,v), or simply N(2,k,v). A \lambda-covering is a covering where each pair appears at least \lambda times; if \lambda = 2, we have a bicovering. The cardinality of a minimal bicovering is denoted by N_{\lambda}(2,k,v).

It is well known that

\[ N_{\lambda}(2,k,v) \geq v \frac{N_{\lambda}(1,k-1,v-1)}{k} = v\lceil \frac{\lambda(v-1)}{(k-1)} \rceil /k. \]

Henceforth, we shall use the term bicovering to denote a minimal bicovering.

2. The Weight Function

For any \lambda-covering consisting of b blocks, take a block B. Let x_i be the number of blocks meeting B in exactly i elements (0 \leq i \leq k). Then the number of blocks in the covering, excluding block B, is:

\[ \Sigma x_i = b - 1. \]

If r_i is the frequency of element i, and \lambda_{ij} is the number of pairs (i,j), then counting the number of other occurrences of elements from B and the other occurrences of element pairs from B gives:

\[ \Sigma ix_i = \Sigma (r_i - 1), \]
\[ \Sigma i(i-1)x_i/2 = \Sigma (\lambda_{ij} - 1). \]

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We now define a weight function \( w(B) \) for the block \( B \). Clearly, this function should be non-negative, and we choose the definition

\[
w(B) = \sum a_i x_i
\]

where the \( a_i \) are non-negative integers. We may select the \( a_i \) in any way that we choose; however, for applications to the case \( k = 5 \), we shall use the definition

\[
w(B) = x_0 + x_3 + 3x_4 + 6x_5.
\]

Direct computation then establishes

**Lemma 1.** \( w(B) = (b-1) - \sum (r_i - 1) + \sum (\lambda_{ij} - 1) \).

3. Excess Graphs and Excess Pairs

In a bicovering, each pair of elements occurs at least twice. If we represent pairs by edges of a graph on \( v \) nodes, then each edge occurs with multiplicity at least 2. Removing 2 copies of \( K_v \) from this graph leaves the *excess graph* of the bicovering. For example, consider a (2,5,19) bicovering; then \( N_2(2,5,19) \geq \lceil 171/5 \rceil = 35 \). Suppose that there is a bicovering in 35 blocks; these 35 blocks contain 350 pairs, and the bicovering requires 342 pairs. Hence there are 8 excess pairs and the excess graph contains 8 edges.

The frequency of each element in this bicovering is at least \( N_2(1,4,18) = 9 \). Hence, in the excess graph, any node of frequency 9 is an isolated point; any node of frequency 10 has degree 4; any node of frequency 11 has degree 8; etc. Since there are 175 elements in the bicovering, and at least 171 are required, there is an excess of 4 in the frequency counts. We refer to the points of frequency greater than 9 as excess points; they are the only points of positive degree in the excess graph (henceforth, we shall delete the isolated points).

If only one excess point appears in a (2,5,19) bicovering in 35 blocks, then 8 excess pairs are formed. This requires 8 loops in the excess graph (Figure 1). Since loops are not permitted, this case is trivially excluded.

![Figure 1: One Point in the Excess Graph](image-url)
If two points A and B appear in the excess graph (Figure 2), then each must have valence 8. Now, the pair AB appears 8 times in the excess graph and twice normally; so AB occurs 10 times in the bicovering. However, A and B each appear exactly 11 times. Thus, there are 10 blocks of the form ABxxx, a block Axxxx, and a block Bxxxx. Since each of A and B appears twice with the other 17 elements, it follows that we must take these blocks as A1234 and B1234. From Lemma 1, we have $w(A1234) = 34 - (8 \times 4 + 10) + 10 = 2$. But $w(B) = x_0 + x_3 + 3x_4 + 6x_5 \geq 3$, since A1234 has a quadruple intersection with B1234. This contradiction rules out the case of two exceptional points.

![Figure 2: Two Points in the Excess Graph](image)

Figure 3 illustrates the excess graph on three points; element A has frequency 11, while B and C each have frequency 10.

![Figure 3: Three Points in the Excess Graph](image)

Figure 4 illustrates the four excess graphs on four points. All four points in these graphs would have a frequency of 10 in a bicovering.

![Figure 4: Excess Graphs with Four Excess Points](image)
4. Upper Bounds.

There is little point in discussing the excess graphs for a possible bicoverying in 35 blocks unless a good upper bound on the number of blocks in a $(2,5,19)$ bicoverying is known. Now $N(2,5,19) \geq 19$, and the bound 19 is easily achieved by cycling on the block $(1 \ 2 \ 3 \ 7 \ 10)$. Hence, duplication of this covering results in a $(2,5,19)$ bicoverying in 38 blocks. (If we want a bicoverying without repeated blocks, we could cycle the blocks $(0 \ 1 \ 5 \ 11 \ 13)$ and $(0 \ 1 \ 4 \ 7 \ 9)$.)

However, an upper bound of 36 can be achieved. Take ordinary elements 1 through 9, barred elements 1 through 9, and a fixed point $P$. Take four initial blocks and cycle on the 9 ordinary and 9 barred elements (leaving $P$ fixed). Since $P$ must appear twice with each element, $P$ occurs in one initial block with two ordinary and two barred elements. Since there are ordinary (and barred) differences $1,2,3,4$, and 16 mixed differences, we see that the other three initial blocks contribute at least 7 ordinary differences, 7 barred differences, and 14 mixed differences. There are six possible block types $(x,y)$, where $x$ is the number of ordinary elements, $y$ the number of barred elements.

A: $(5,0)$ B: $(4,1)$ C: $(3,2)$ D: $(2,3)$ E: $(1,4)$ F: $(0,5)$

The contributions to the ordinary, barred, and mixed differences (in that order) are as follows.

A: $(10,0,0)$ B: $(6,0,4)$ C: $(3,1,6)$ D: $(1,3,6)$ E: $(0,6,4)$ F: $(0,10,0)$

If there are no type C or D blocks, then at most 12 mixed differences appear. Since at least 14 mixed differences are required, we may, by symmetry, take a block of type D. The two other blocks must give at least 6 ordinary differences, 4 barred differences, and 8 mixed differences; hence there must one block of type B and one of type E. Once the patterns of ordinary and barred elements in the four blocks have been determined, it is easy to write down a set of four initial blocks as $(1467)$, $(15689)$, $(12381)$, $(12571)$.

It is easily verified that these 4 initial blocks generate a bicoverying in 36 blocks; thus $N_2(2,5,19) \leq 36$. This establishes

**Lemma 2.** $N_2(2,5,19)$ is either 35 or 36.

Determination of the exact value of $N_2(2,5,19)$ will require discussion of the excess graphs shown in Figures 3 and 4.