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On link-irregular labelings of graphs

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Abstract

We introduce the concept of link-irregular labelings for graphs, extending the notion of link-irregular graphs through edge labeling with positive integers. A labeling is link-irregular if every vertex has a uniquely labeled subgraph induced by its neighbors. We establish necessary and sufficient conditions for the existence of such labelings and define the link-irregular labeling number $\eta(G)$ as the minimum number of distinct labels required. Our main results include necessary and sufficient conditions for the existence of link-irregular labelings. We show that certain families of graphs, such as bipartite graphs, trees, cycles, hypercubes, and complete multipartite graphs, do not admit link-irregular labelings, while complete graphs and wheel graphs do. Specifically, we prove that $\eta(K_n) = 2$ for $n \geq 6$ and $\eta(K_n) = 3$ for $n \in \{3, 4, 5\}$. For wheel graphs W_n, we establish that $\eta(W_n) \approx \sqrt{2n}$ asymptotically. Finally, we prove that for every positive integer n, there exists a graph with a link-irregular labeling number n, and provide several results on graph operations that preserve labeling numbers.

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We follow standard terminology and notation in graph theory as presented in West's textbook [4]. The number of vertices and edges in a graph G are denoted by $n(G)$ and $e(G)$, respectively. The number of edges incident with a vertex $u \in V(G)$ is the degree of a vertex u, denoted $d(u)$. And the set of degrees of the vertices of graph G is denoted by $D(G)$. For graphs G and H, the join $G \vee H$ is the graph formed by taking both graphs and adding edges from each vertex of G to every vertex of H. We use $G + H$ to denote the disjoint union of G and H, meaning the graph consisting of all vertices

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and edges of G and H with no additional edges and no shared vertices. A graph is irregular if no two vertices in the graph have the same degree. It is known that no such graph exists because there is no simple graph in which all vertex degrees are distinct. Ali, Chartrand, and Zhang [2] introduced the notions of the *link* of a vertex, *link-regular* graphs, and *link-irregular* graphs. The *link* $L(v)$ of a vertex v in a graph G is defined as the subgraph induced by the neighborhood of v; that is, $L(v) = G[N(v)]$. A graph G is called *link-regular* if all of its vertices have isomorphic links—that is, $L(u) \cong L(v)$ for all $u, v \in V(G)$. Conversely, G is *link-irregular* if every pair of distinct vertices has non-isomorphic links; that is, $L(u) \ncong L(v)$ for all distinct $u, v \in V(G)$. One of their results is that there exists a link-irregular graph of order n if and only if $n \geq 6$. The authors provided the unique link-irregular graph with 6 vertices in [1]. This graph is shown in Figure 1.

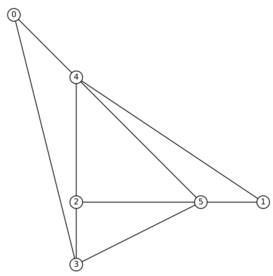


Figure 1: The unique link-irregular graph with 6 vertices [1].

Recently, Bastien and Khormali in [3] showed that there are no bipartite link-irregular graphs, and for $n \leq 9$, there are no regular link-irregular graphs on n vertices. Additionally, they showed that finitely many link-irregular graphs are planar. They also disproved a conjecture that no regular link-irregular graphs exist, through explicit and probabilistic constructions. Link-irregularity is defined for simple graphs, and a natural generalization arises by extending the definition to loopless multigraphs where multiple edges are allowed between any pair of vertices. Instead of working directly with loopless multigraphs, we consider the edge-labeling of underlying simple graphs, where each edge is assigned a positive integer label indicating its multiplicity. This labeling helps us to work with simple graphs while still having the complexity of loopless multigraphs. Motivated by this, we extend the concept of link-irregularity to labeled simple graphs. By this extension, we are able to study the structural uniqueness in cases where the simple graph alone is not link-irregular, but edge labeling can create link distinctions. Throughout the remainder of this paper, we consider only simple graphs that have no loops or multiple edges, and the labeling of edges is given by positive integers unless stated otherwise.

Let G be a graph and let $l : E(G) \to \mathbb{Z}^+$ be an edge-labeling of G, assigning a positive integer to each edge. For a vertex $v \in V(G)$, the *labeled link* of v, denoted $L_l(v)$, is the induced subgraph of G on the neighborhood $N(v)$, with edge labels coming from labeling l.

In this paper, an isomorphism of labeled graphs is an isomorphism of their underlying graphs that maps edges with a given label onto edges with that same label.

We say that the labeling l is *link-irregular* if for all distinct vertices $u, v \in V(G)$, the labeled links are pairwise non-isomorphic:

$$L_l(u) \not\cong L_l(v) \quad \text{for all } u \neq v.$$

A graph G together with a link-irregular labeling l is called a *labeled link-irregular graph*. We now provide some examples to illustrate these concepts.

Example 1.1. Consider the labeled graph below. We have shown alongside it the labeled links of its vertices. Since $L_l(y)$ and $L_l(z)$ are both K_2 with the same label for their edges, $L_l(y) \cong L_l(z)$. On the other hand, while $L_l(x)$ and $L_l(w)$ are both P_3, the edges of $L_l(x)$ are both labeled 1 while those of $L_l(w)$ are both labeled 2. Hence these labeled links are not isomorphic.
Since $L_l(y) \cong L_l(z)$, this is not a link-irregular labeling. In fact, as we will see from Observation 2.1, this graph admits no link-irregular labeling.

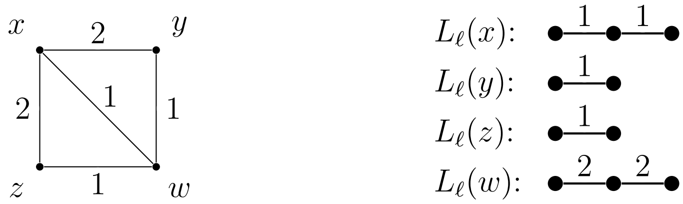


Figure 2: A labled graph on four vertices. We have $L_\ell(y) \cong L_\ell(z)$ and $L_\ell(x) \not\cong L_\ell(w)$.

We now provide an example of a link-irregular labeling of a graph.

Example 1.2. Consider the labeling of K_3 shown below. While $L_l(a)$, $L_l(b)$ and $L_l(c)$ are all labeled K_2 graphs, they each have a different label for their only edge. Since there is no isomorphism of the unlabeled graphs that maps like labels into like labels, none of these links are isomorphic to the others. Hence, this is a link-irregular labeling.

Figure 3: A labeled K_3.

Note that if our set of labels for a graph consists of only one element, then labeled link-irregular graphs coincide with link-irregular graphs.

In the following, we investigate the structural properties of the labeled link-irregular graphs.

2 Results

The following observation provides a complete characterization of graphs that admit link-irregular labeling.

Observation 2.1. *Let G be a graph. Then G has a link-irregular labeling if and only if given any distinct $x, y \in V(G)$, either $L(x) \ncong L(y)$, or $E(L(x)) \neq E(L(y))$.*

This follows directly from the definitions: if the condition holds, assigning distinct labels to all edges yields the desired labeling; conversely, if some vertices have isomorphic links with identical edge sets, no labeling can distinguish their labeled links.

Definition 2.2. The *linked neighborhood* of a vertex x in a graph G, denoted $NL(x)$, is the set of neighbors of x that are adjacent to at least one other neighbor of x, that is, neighbors that are not isolated in the link $L(x)$. Formally,

$$NL(x) = \{u \in N(x) : \deg_{L(x)}(u) \geq 1\}.$$

The following result is a direct consequence of Observation 2.1 and the definition of linked neighborhood.

Corollary 2.3. *A graph G has a link-irregular labeling if and only if for all distinct $x, y \in V(G)$, at least one of the following holds:*

(a) $L(x) \ncong L(y)$, or

(b) both of the following:

  (i) $NL(x) \neq NL(y)$, and

  (ii) For each integer $n \geq 0$, at most one vertex $x \in V(G)$ satisfies $L(x) \cong \bar{K}_n$.

This follows by noting that when $L(x) \cong L(y)$, the condition $E(L(x)) \neq E(L(y))$ translates to either different neighborhoods (when the links are non-empty) or the restriction on isolated vertices of the same degree (when $L(x) \cong L(y) \cong \bar{K}_n$).

The following observation is similar to Corollary 2.3, but states only the necessary condition.

Observation 2.4. *If a graph G admits a link-irregular labeling, then:*

1. *$N(x) \neq N(y)$ for all distinct $x, y \in V(G)$, and*
2. *For each $n \geq 0$, at most one vertex x satisfies $L(x) \cong \bar{K}_n$.*

These conditions are necessary but not sufficient for the existence of a link-irregular labeling. To illustrate this, consider a graph G in which two vertices x and y have isomorphic links of the form $K_2 + 2K_1$ (a disjoint union of an edge and two isolated vertices), and $G[N(x) \cap N(y)] = K_2$. In this case, although $N(x) \neq N(y)$

and at most one vertex may have an empty link, we have $L_l(x) \ncong L_l(y)$, and G does not admit a link-irregular labeling, despite satisfying conditions (1) and (2).

The following observation identifies several graph families that do not admit the link-irregular labeling conditions.

Observation 2.5. *The following families of graphs do not admit a link-irregular labeling: all bipartite graphs (hence trees and hypercubes Q_n for $n \geq 2$), cycles C_n for $n \geq 4$, and complete multipartite graphs with at least one part containing two or more vertices.*

Proof. Each of these graph families violates at least one of the necessary conditions stated in Corollary 2.3. In a cycle C_n for $n \geq 4$, every vertex has $\bar{K}_2$ as its link. By Corollary 2.3, this means that C_n has no link-irregular labeling. Bipartite graphs have the property that for every vertex x, the link $L(x)$ is edgeless. And multiple vertices in the bipartite graphs have the same degree since the set of degrees $D(G)$ has size less than n. This leads to multiple identical edgeless links, violating the uniqueness condition for link-irregular labeling. Trees and hypercubes are subclasses of bipartite graphs, and the above argument applies. If two vertices x, y of a complete multipartite graph lie in the same part, then they have the same neighborhood, hence $L(x) \cong L(y)$ regardless of any possible edge labeling. □

We use W_n as a wheel graph, which is a graph formed by connecting a single universal vertex to all vertices of a cycle C_n. Note that W_4 does not admit the link-irregular labeling. It can be seen that two non-adjacent vertices on the circle have isomorphic labeled links.

The following observation identifies several graph families that admit the link-irregular labeling conditions.

Observation 2.6. *The following families of graphs admit a link-irregular labeling:*

- *Complete graphs K_n, $n \geq 3$,*
- *Wheels W_n, $n = 3$, and $n \geq 5$.*

Proof. When we assign a distinct label to each edge of the graphs in both families, all labeled links are pairwise non-isomorphic. Hence, the desired result is concluded. □

Motivated by the above examples, we define a new graph invariant that quantifies the minimum number of labels required for a graph to admit a link-irregular labeling.

Definition 2.7. Let G be a graph. The link-irregular labeling number of G, denoted by $\eta(G)$, is the minimum number of edge labels required to assign to $E(G)$ such that the resulting labeled graph is labeled link-irregular . If no such labeling exists, we define $\eta(G) = \infty$.

Observation 2.8. *A graph G is link-irregular if and only if $\eta(G) = 1$.*

In the following, we find the link-irregular labeling number of complete graphs and wheels.

Theorem 2.9. *Let $i_j \geq 5$ for $j \geq 3$, and define the intervals for i_j based on the parity of j as follows:*

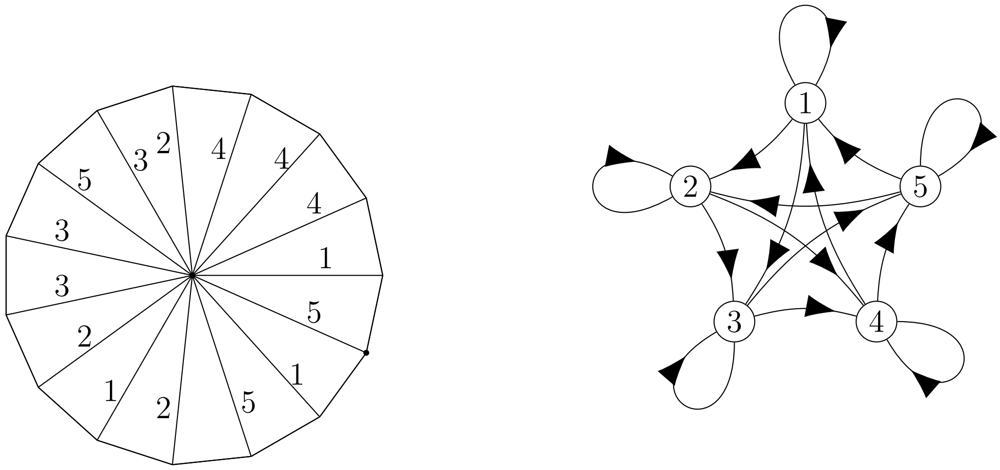
$$5 \leq i_3 \leq \binom{3+1}{2} < i_4 \leq \binom{4+1}{2} - \frac{4}{2} < \cdots < i_j \leq \begin{cases} \binom{j+1}{2} & \text{if } j \text{ is odd,} \\ \binom{j+1}{2} - \frac{j}{2} & \text{if } j \text{ is even.} \end{cases} < \cdots$$

Then for each such $i_j \neq 6, 8$, we have $\eta(W_{i_j}) = j$. Moreover, for large enough n, $\eta(W_n) \approx \sqrt{2n}$. In addition, $\eta(W_3) = 3$, $\eta(W_6) = 5$, $\eta(W_8) = 5$ and $\eta(W_4) = \infty$.

Proof. In the wheel graph W_n, the link of the universal vertex is C_n, which is unique. All other vertices lie on the cycle and have links isomorphic to a path of length two, where the central vertex of the path is the universal vertex. Thus, to ensure labeled link-irregularity, the labels of the edges from the universal vertex to the cycle vertices must yield non-isomorphic labeled links.

Let A denote the set of n edges incident to the universal vertex in the wheel graph W_n. These edges connect the universal vertex to each of the n vertices on the cycle. Each such edge $e \in A$ is involved in exactly two vertex links: the link of one cycle vertex adjacent to e and the link of the cycle vertex which is in distance 2 in the clockwise or counterclockwise direction on the cycle. Therefore, every link of a cycle vertex involves two edges from the set A. When we assign labels to the edges in A using r distinct labels, each cycle vertex's link corresponds to a multiset of two labels. Since there are n such links and each labeled link is formed from a pair of labels, we obtain n such labeled multisets that should be non-isomorphic.

Now consider K_r^*, the complete graph on r labeled vertices with a loop at each vertex (but no multiple edges). A labeling of A corresponds to a closed trail in K_r^*: each labeled link corresponds to an edge in the trail connecting the two labels used in that link. Since each edge in A belongs to exactly two links, and links must be distinct, this process generates a trail that covers the edges of A.



(a) W_{15} (b) K_5^*

Figure 4: A closed directed trail with arcs 11, 12, 22, 23, 33, 34, 44, 45, 55, 51, 13, 35, 52, 24, 41 in K_5^* corresponds to a link-irregular labeling of the wheel graph W_{15}. The labeling begins at the vertex marked in black on the cycle. Each subsequent label is assigned to a vertex located two positions clockwise from the previous one, continuing this pattern until all cycle vertices are labeled.

When n is odd, all edges in A can be traversed in a single closed trail. When n is even, they are split into two disjoint closed trails of length $n/2$. In either case, the labels used form the vertices of the trail(s), and the trail edges correspond to labeled links.

Thus, finding the smallest number of labels r such that K_r^* contains a closed trail (or two edge-disjoint trails) covering n edges is equivalent to determining $\eta(W_n)$.

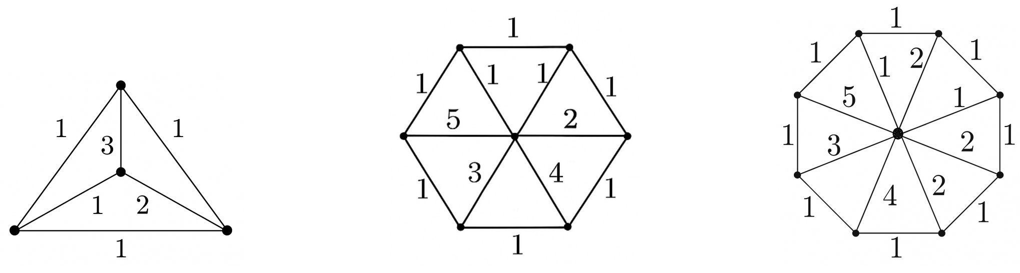
When r is odd, K_r^* has all vertices of even degree, and an Eulerian circuit (closed trail) exists that traverses all $\binom{r+1}{2}$ edges. When r is even, the degrees are odd, and removing a perfect matching ($r/2$ edges) yields a graph with even degrees, supporting one or two equal-length edge-disjoint Eulerian circuits. Hence, for:

$$\text{even } r\text{: we need } n \leq \binom{r+1}{2} - \frac{r}{2}, \quad \text{odd } r\text{: we need } n \leq \binom{r+1}{2}.$$

This establishes the result.

Also, based on the bounds for n and r, we have one of the two inequalities $\binom{r}{2} < n \leq \frac{r^2}{2}$ and $\frac{r^2}{2} < n \leq \binom{r+1}{2}$. Solving each inequality for r yields an approximation $r \approx \sqrt{2n}$ for large enough n. This confirms that $\eta(W_n) \approx \sqrt{2n}$ asymptotically.

In addition, the following graph is labeled link-irregular W_3, and shows $\eta(W_3) = 3$.



(a) W_3 (b) W_6 (c) W_8

Figure 5: A link-irregular labeling of W_3, W_6, and W_8

Earlier, we mentioned that W_4 does not admit the link-irregular labeling. Then $\eta(W_4) = \infty$. $\square$

We now aim to determine the link-irregular number of K_n for $n \geq 3$. Observe that the labeled link of any vertex v in an edge-labeled K_n is the same as the graph $K_n - v$, which is a labeled K_{n-1}. Suppose that K_n admits a link-irregular 2-labeling, and assume the edge labels used are "red" and "blue". Then all the key structural information of the labeled K_n is contained in the graph G_{red} that consists of all the vertices of K_n and only the red edges. Note that G_{red} may have vertices with degree zero. The labeled link of a vertex $v \in V(K_n)$ can be reconstructed from $G_{red} - v$ by adding any missing edges as blue. Consequently, we will be able to conclude that K_n has a link-irregular 2-labeling if and only if there exists a graph G on n vertices such that for any two distinct vertices $u, v \in G$, $G - u \ncong G - v$. This motivates the following definition.

Definition 2.10. An unlabeled graph G is said to be cut-irregular if for all distinct vertices $u, v \in V(G)$, the vertex-deleted subgraphs $G - u$ and $G - v$ are non-isomorphic.

We now explicitly state and prove the correspondence between the link-irregular 2-labelings of K_n and cut-irregular graphs on n vertices.

Proposition 2.11. *Let l be a link-irregular labeling of K_n using the labels "red" and "blue". Then the subgraph G_{red} consisting of all red edges in K_n is a cut-irregular graph on n vertices.*

Conversely, if G is a cut-irregular graph on n vertices, then assigning the label "red" to the edges of a subgraph of K_n isomorphic to G, and labeling all remaining edges "blue", yields a link-irregular 2-labeling of K_n.

Proof. We first prove the first statement. Suppose l is a link-irregular labeling of K_n. For any vertex v of K_n, we have $L(v) = K_n - v$. Now assume there exist vertices u and v of K_n such that $G_{\text{red}} - u \cong G_{\text{red}} - v$. Adding back all the blue edges to $G_{\text{red}} - u$ and $G_{\text{red}} - v$, we see that the labeled graphs $K_n - u$ and $K_n - v$ are isomorphic; equivalently, $L_l(u) \cong L_l(v)$. Since l is link-irregular, this is impossible. Hence $G_{\text{red}} - u \ncong G_{\text{red}} - v$, which shows that G_{red} is cut-irregular.

Now we prove the second statement. Suppose G is cut-irregular. Label all edges of G red, and add blue edges on all non-edges of G to obtain a labeling l of K_n. If $L_l(u) \cong L_l(v)$ in K_n, then $K_n - u \cong K_n - v$. By removing the blue edges from both sides, we would obtain $G - u \cong G - v$, contradicting the cut-irregularity of G. Therefore, the labeling l is a link-irregular 2-labeling.

□

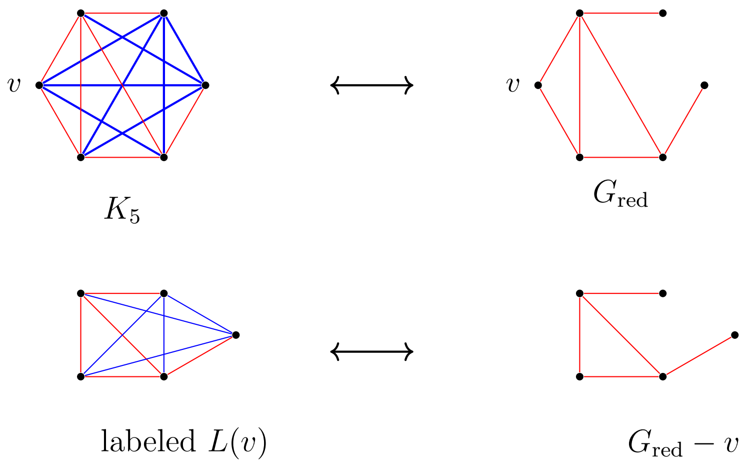


Figure 6: Transformation between a labeled complete graph K_6 and its link graph $L(v)$, compared to G_{red} and $G_{\text{red}} - v$.

We have thus reduced the problem of determining whether K_n admits a link-irregular 2-labeling to the problem of finding a cut-irregular graph on n vertices. As we now show, such a graph exists if and only if $n \geq 6$. The proof is analogous to the one presented in [2].

Theorem 2.12. *There exists a cut-irregular graph on n vertices if and only if $n \geq 6$.*

Proof. That there is no cut-irregular graph on fewer than six vertices is readily verified by examining each graph on five or fewer vertices. Below is a cut-irregular graph on six vertices. Call this graph G_6 This is our base step.

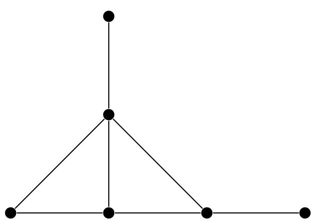


Figure 7: A cut-irregular graph on 6 vertices.

From this, we will construct an infinite set $\{G_6, G_7, G_8, \cdots\}$ such that G_n, $n \geq 6$, is a cut-irregular graph on n vertices. For $n > 6$, we construct G_n as follows

- If n is odd, we put $G_n = G_{n-1} \vee K_1$. That is, G_n is the join of G_{n-1} and K_1.
- If n is even, we obtain G_n by adding a new vertex and connecting it to a vertex of minimum degree in G_{n-1}.

We claim that each G_n is cut-irregular. We first notice that G_6 has no vertex of degree 5. Hence G_7 has only one vertex of degree 6. G_7 also has no vertex of degree 1 since G_6 has no isolated vertices and joining K_1 to G_6 increases the degree of each vertex by 1. Then G_8 has only one vertex of degree 1 and no vertex of degree 7 (see Figure 8). More generally, we have:

- If n is odd, then G_n has exactly one vertex of degree $n-1$ and no vertex of degree 1.
- If n is even, then G_n has exactly one vertex of degree 1 and no vertex of degree $n-1$.

Suppose that G_{n-1} is cut-irregular for $n-1 \geq 6$. Then we examine two cases.

Case 1. n is odd. Then, $n-1$ is even, and by construction, G_{n-1} has exactly one vertex of degree 1 and no vertex of degree $n-1$. Then, to obtain G_n, we join a new vertex w to all the vertices of G_{n-1}. Suppose for contradiction that $G_n - u \cong G_n - v$ for some $u, v \in V(G_n)$. First, consider the case where both u and v are in G_{n-1}. Then, We have $G_n - u = (G_{n-1} \vee K_1) - u \cong (G_{n-1} - u) \vee K_1$. Likewise, $G_n - v \cong (G_{n-1} - v) \vee K_1$. Therefore, $G_n - u \cong G_n - v$ implies $(G_{n-1} - u) \vee K_1 \cong (G_{n-1} - v) \vee K_1$. Removing the common K_1 from both sides, we get $G_{n-1} - u \cong G_{n-1} - v$, contradicting the assumption that G_{n-1} is cut-irregular. Hence, $G_n - u \ncong G_n - v$ when $u, v \in V(G_{n-1})$.

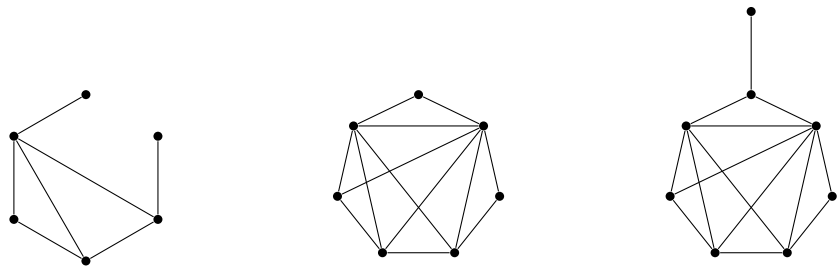
Now consider the case where $v = w$. Observe that if $G_n - u \cong G_n - w$, then u and w must have the same degree in G_n; otherwise, the graphs would have different numbers of edges and could not be isomorphic. Since w is the only vertex of degree $n-1$ in G_n, no other vertex can have the same degree. Then, $G_n - u \ncong G_n - w$ for any $u \in V(G_{n-1})$. Therefore, in all cases, $G_n - u \ncong G_n - v$ for distinct $u, v \in V(G_n)$, and thus G_n is cut-irregular in this case.

Case 2. n is even. Then $n-1$ is odd, and by construction, G_{n-1} has exactly one vertex of degree $n-1$ and no vertex of degree 1. We obtain G_n by adding a new vertex w and connecting it to a vertex z of minimum degree in G_{n-1}, making w the only vertex of degree 1 in G_n. Let u and v be arbitrary vertices in $G_{n-1} \setminus \{z\}$. We aim to show that $G_n - u \ncong G_n - v$. First, observe that w remains the unique vertex of degree 1 in both $G_n - u$ and $G_n - v$. Then, removing w from both graphs gives

$$(G_n - u) - w \cong G_{n-1} - u \quad \text{and} \quad (G_n - v) - w \cong G_{n-1} - v.$$

Therefore, if $G_n - u \cong G_n - v$, then $(G_n - u) - w \cong (G_n - v) - w$, and $G_{n-1} - u \cong G_{n-1} - v$, which contradicts the assumption that G_{n-1} is cut-irregular. Next, we claim that $G_n - z \ncong G_n - w$. Indeed, $G_n - z$ contains an isolated vertex (namely w), while $G_n - w$ is connected. Similarly, $G_n - w \ncong G_n - u$ for any $u \in V(G_{n-1})$ since

w is the only vertex of degree 1, and $G_n - u$ has no vertex of degree 1. Finally, to show that $G_n - z \ncong G_n - u$, note that $G_n - z$ contains an isolated vertex (w), while $G_n - u$ does not, because w is only adjacent to z and $z \neq u$. Hence, in all cases, $G_n - u \ncong G_n - v$ for distinct $u, v \in V(G_n)$, and then G_n is cut-irregular in this case as well. This completes the proof. $\square$



G_6 G_7 G_8

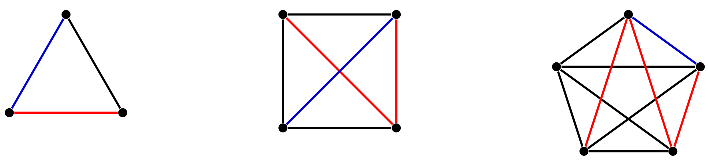
Figure 8: The graphs G_6, G_7, and G_8.

We now have the following result.

Theorem 2.13. *The link-irregular numbers of the K_n are as follows*

1. $\eta(K_1) = 0$ *and* $\eta(K_2) = \infty$
2. $\eta(K_n) = 3$ *if* $n = 3, 4, 5$
3. $\eta(K_n) = 2$ *if* $n \geq 6$

Proof. It is straightforward to verify that $\eta(K_1) = 0$ and that $\eta(K_2) = \infty$. That $\eta(K_n) = 2$ if $n \geq 6$ follows directly from Theorem 2.12 and Proposition 2.11.
It remains to show that $\eta(K_n) = 3$ if $n = 3, 4, 5$. By Theorem 2.12, there is no cut-irregular graph on 3, 4, or 5 vertices. Then, $\eta(K_n) \geq 3$ for $n = 3, 4, 5$. In Figure 9, the link-irregular 3-labelings of K_3, K_4, and K_5 are shown. So, $\eta(K_n) \leq 3$. Hence, $\eta(K_n) = 3$ for $n = 3, 4, 5$. $\square$



K_3 K_4 K_5

Figure 9: Examples of complete graphs K_3, K_4, and K_5 with selected colored edges.

Note that K_n is link-regular for all $n \geq 2$, as the link of every vertex is isomorphic to K_{n-1}. Thus, K_n provides an example of a link-regular graph that admits a link-irregular labeling.

Observation 2.14. *There exist link-regular graphs that admit link-irregular labelings.*

This highlights that link-regularity and the existence of link-irregular labelings are distinct properties. While K_n is link-regular and admits a link-irregular labeling, not all link-regular graphs share this property. For instance, cycles C_n with $n \geq 4$ are link-regular but do not admit any link-irregular labeling.

This demonstrates that while link-irregularity generalizes the classical notion of irregularity, link-irregular labelings further expand the link-irregularity.

At this point, we answer the question: for which values of n does there exist a graph H_n such that $\eta(H_n) = n$? Using the facts that $\eta(K_3) = 3$ and $\eta(G) = 1$ for any link-irregular graph G, we show that such a graph H_n exists for every integer $n \geq 1$. Note that while we claim that such a graph exists, we are not claiming that there is only one such graph.

Theorem 2.15. *For every positive integer n, there exists a graph H_n such that $\eta(H_n) = n$.*

Proof. We construct the graph H_n in three cases based on the congruence of n modulo 3: $n \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$, and $n \equiv 2 \pmod 3$. Throughout the proof, we denote by G the unique link-irregular graph on six vertices in Figure 1. We now consider our three cases.

Case 1. $n \equiv 0 \pmod 3$. In this case, we can write n in the form $n = 3k$. Then consider the graph $H_n = kK_3$. Each edge of H_n is in one and only one link, end each link is a K_2. Thus assigning every edge of H_n a unique label provides us with a link-irregular labeling of H_n using $n = 3k$ labels. If we use any fewer labels, then two edges must have the same label. Since each edge constitutes the link of one vertex, this means two vertices have isomorphic labeled-links. Then, a labeling with fewer labels is not link-irregular. Hence $\eta(H_n) = n$.

Case 2. $n \equiv 1 \pmod 3$. In this case, we write $n = 3k + 1$. Consider the graph $H_n = kK_3 + G$, which has $3k + 6$ vertices. This graph includes exactly $3k + 1 = n$ vertices whose links are isomorphic to K_2. Since each such link must receive a distinct label, we have $\eta(H_n) \geq n$. To construct a link-irregular labeling with n labels, assign a unique label to each edge of the kK_3 component, and assign a single, unused label to all edges of G. Note that this is valid because G is link-irregular, and every vertex outside of G has link K_2. Although one vertex of G also has K_2 as its link, labeling the edges of G with a unique label ensures that all K_2 links remain distinguishable. Therefore, we conclude that $\eta(H_n) = n$.

Case 3. $n \equiv 2 \pmod 3$. In this case, write $n = 3k + 2$. Let $H_n = kK_3 + 2G$, which has $3k + 12$ vertices. In this graph, exactly $3k + 2 = n$ vertices have K_2 as their link. Then, we have $\eta(H_n) \geq n$. To construct a link-irregular labeling using exactly n labels, we assign a unique label to each edge of the kK_3 component. For the two copies of G, assign all edges in the first copy the same new label, and do likewise for the second copy using a different new label not used previously. Since G is link-irregular and none of the links are isomorphic, this labeling ensures link-irregularity. Thus, we obtain a link-irregular labeling of H_n using $n = 3k + 2$ labels. Hence, $\eta(H_n) = n$. This completes the proof. □

In the following propositions, we investigate the link-irregular labeling number of graphs of the form $G \vee K_n$ under certain conditions.

Proposition 2.16. *Suppose $\Delta(G) < n(G) - 1$ and $\eta(K_n) \leq \eta(G) < \infty$ for some positive integer n. Then $\eta(G \vee K_n) = \eta(G)$.*

Proof. It is clear that $\eta(G \vee K_n) \geq \eta(G)$. Then, we only need to show that $\eta(G \vee K_n) \leq \eta(G)$. We do this by providing a link-irregular $\eta(G)$-labeling of $G \vee K_n$. Suppose we have a link-irregular labeling of G using labels from $[\eta(G)]$ and using the same labeling-set, we may choose a link-irregular labeling for K_n too. We define the labeling l of $G \vee K_n$ as follows:

Label G and K_n according to the labelings we have chosen for them.

Label the edges between G and K_n all with the same label, say 1.

We claim this is a link-irregular labeling. Clearly no two vertices in G have the same labeled link, and no two vertices in K_n have the same link. Suppose $u \in V(G)$ and $v \in V(K_n)$. Then the link of u has n points of degree $n(G) + n - 2$ while the link of v only has $n - 1$ points of degree $n(G) + n - 2$ hence the links cannot be isomorphic. □

Proposition 2.17. *Let G be a graph such that $\eta(G) < \infty$, and let H be the subgraph of G induced by all vertices of degree $n(G) - 1$. If $3 \leq \eta(G - H) < \infty$ then $\eta(G) = \eta(G - H)$.*

Proof. In this proof, we use Proposition 2.16. If H is empty, the result is trivial. Then, assume $n(H) > 0$. Since H contains all vertices of degree $n(G) - 1$, $G - H$ has no vertices of degree $n(G - H) - 1$. By Proposition 2.16, $\eta((G - H) \vee K_{n(H)}) = \eta(G - H)$, but $(G - H) \vee K_{n(H)} \cong G$. Hence $\eta(G) = \eta(G - H)$. □

Proposition 2.18. *Suppose $\eta(K_n) \leq \eta(G) < \infty$ for some value n, and $3 \leq \eta(G - H) < \infty$ where H is the subgraph induced by the vertices of degree $n(G) - 1$ in G. Then $\eta(G \vee K_n) = \eta(G)$.*

Proof. If G has no vertex of degree $n(G) - 1$, this is just Proposition 2.16. If G does have a vertex of degree $n(G) - 1$, then we let H be as in the proof of Proposition 2.17. We have $G \vee K_n \cong (G - H) \vee K_{n+n(H)}$. Then, by both Proposition 2.16 and Proposition 2.17, we obtain $\eta(G \vee K_n) = \eta((G - H) \vee K_{n+n(H)}) = \eta(G - H) = \eta(G)$. □

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