AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **93(2)** (2025), Pages 398–407

Incomplete tournaments without disjoint cycles of different lengths

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Abstract

We show in this paper that every strong incomplete tournament with minimum out-degree 3, except for a well-defined digraph on seven vertices, contains two vertex-disjoint cycles of different lengths.

1 Introduction

We consider here only a *finite simple digraph*, i.e., a digraph that has a finite number of vertices, no loops and no multiple arcs. Unless otherwise indicated, our graph-theoretic terminology will follow [2]. We will also adopt notation and basic definitions that are used in [16].

Thomassen in [18] has proved that every digraph with minimum out-degree at least 3 contains two vertex-disjoint cycles. Recently, in connection with 2-coloring of hypergraphs, Henning and Yeo have begun to study in [6] the existence of vertex-disjoint cycles of different lengths in digraphs and they have posed there several conjectures. One of these conjectures has been solved by Lichiardopol in [10], which asserts that every digraph with minimum out-degree at least 4 contains two vertex-disjoint cycles of different lengths. From the results obtained by Thomassen [18] and Lichiardopol [10], the investigation of structure for digraphs without vertex-disjoint cycles of different lengths can be restricted to digraphs with minimum out-degree 3. Further, from the result obtained by Tan in [17], the investigation of this problem can be reduced to the investigation of such a problem for strong digraphs.

A digraph $D = (V, A)$ is called *k-regular* if $d^+_D(v) = d^-_D(v) = k$ for every vertex $v \in V$. A digraph $D = (V, A)$ is called *acyclic* if it has no cycles. A digraph $D = (V, A)$ is called *strong* if for every pair x, y of distinct vertices in D, there exist both a path from x to y and a path from y to x. A digraph with only one vertex is considered to be strong.

An *oriented graph* is a digraph with no cycle of length 2. An *incomplete tournament* is an oriented graph $D = (V, A)$ with a partition $V = \{w\} \cup K$ such that $D[K]$

ISSN: 2202-3518

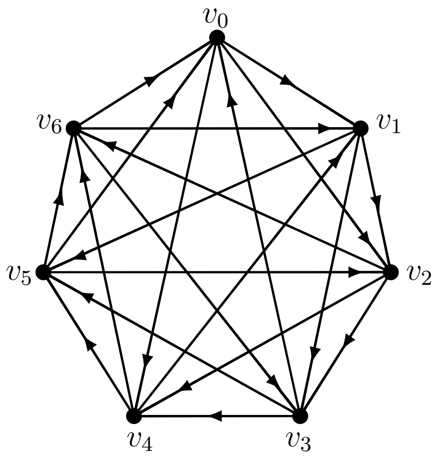

Figure 1: The digraph D_7^3

is a tournament. We will denote such a digraph by $D = T(\{w\} \cup K, A)$. Inspired by this result, we study the existence of vertex-disjoint cycles of different lengths in strong incomplete tournaments with minimum out-degree 3.

A 3-regular digraph having no vertex-disjoint cycles of different lengths has been given in [1] and is denoted here by D_7^3. This digraph has the vertex set $V(D_7^3) = \{v_0, v_1, \ldots, v_6\}$ and the arc set $A(D_7^3) = \{v_i v_j \mid (j - i)(\mathrm{mod}\ 7) \in \{1, 2, 4\}\}$. The digraph D_7^3 is illustrated in Figure 1.

We note that the digraph D_7^3 is a strong incomplete tournament with minimum out-degree 3. So, it is natural to ask whether there are other strong incomplete tournaments with minimum out-degree 3 without vertex-disjoint cycles of different lengths. In this paper, we will give the answer to this question by proving the following main result.

Theorem 1.1. *Every strong incomplete tournament D with minimum out-degree at least* 3*, except the digraph D_7^3, contains two vertex-disjoint cycles of different lengths.*

Further information can be found for vertex-disjoint cycles of different lengths in digraphs in the papers [3, 4, 5, 7, 8, 9, 11, 12, 13, 14, 15].

2 Preliminary results

From now on, we always assume that $D = T(\{w\} \cup K, A)$ is a strong incomplete tournament with minimum out-degree 3, having no vertex-disjoint cycles of different lengths. In further discussions, we will use this assumption implicitly. Then by the result obtained by Thomassen in [18], D has two vertex-disjoint cycles A^0 and A^1. By our assumption, it is clear that A^0 and A^1 must have the same length $t \geq 3$. Let

$$\begin{aligned} A^0 &= (a_0^0, a_1^0, \ldots, a_{t-1}^0, a_0^0), \\ A^1 &= (a_0^1, a_1^1, \ldots, a_{t-1}^1, a_0^1), \text{ and} \\ V' &= V(D) \setminus (V(A^0) \cup V(A^1)). \end{aligned}$$

Further, when we talk about the cycles A^j and A^{j+1} (respectively, the vertices $a_i^j, a_{i+1}^j, a_{i+2}^j, \ldots$, of A^j), we implicitly understand that $j \in \{0,1\}$ and $j+1$ is taken modulo 2 (, $j \in \{0,1\}$, $i \in \{0,1,\ldots,t-1\}$ and $i+1, i+2, \ldots$, are taken modulo t).

Then we have the following trivial observations.

Observation 2.1. *Neither of A^0 nor A^1 has a chord.*

Observation 2.2. *Let $V' \neq \emptyset$ and $P = (v_1, \ldots, v_\ell)$ be a path in $D[V']$. If v_ℓ has out-neighbors in $V(P)$, then $\ell \geq t$ and $v_{\ell-t+1}$ is the unique out-neighbor of v_ℓ in $V(P)$.*

Observation 2.3. *Let $V' \neq \emptyset$ and $P = (v_1, \ldots, v_\ell)$ be a path in $D[V']$. Then*

1. *if v_1 has an in-neighbor in $V(A^j)$, then v_ℓ has at most one out-neighbor in $V(A^j)$;*
2. *if v_1 has two in-neighbors in $V(A^j)$, then v_ℓ has no out-neighbor in $V(A^j)$;*
3. *if v_ℓ has an out-neighbor in $V(A^j)$, then v_1 has at most one in-neighbor in $V(A^j)$;*
4. *if v_ℓ has two out-neighbors in $V(A^j)$, then v_1 has no in-neighbor in $V(A^j)$.*

Observation 2.4. *Let $V' \neq \emptyset$ and $u \in V'$. Then*

1. *if $a_i^j \to u$ then $u \not\to a_{i+1}^j$;*
2. *if $u \to a_i^j$ then $a_{i+t-1}^j \not\to u$.*

Based on an idea in the paper [7], we prove some claims below that serve to prove the main result.

Claim 2.5. *Let $u \in V'$ and*

$$X_u = \{v \in V' \mid v \text{ is reachable from } u \text{ in } D[V']\}.$$

If u has two in-neighbors in $V(A^j)$, then any vertex in X_u has no out-neighbor in $V(A^j)$ and no in-neighbor in $V(A^{j+1})$.

Proof. Clearly, $u \in X_u$. Let v be a vertex in X_u and P be a path from u to v in $D[V']$. By Observation 2.3, v has no out-neighbor in $V(A^j)$.

For a contradiction, assume that v has an in-neighbor in $V(A^{j+1})$. Let $Q = (v_1, \ldots, v_\ell)$ be a longest path in $D[V']$ with the initial vertex $v_1 = v$. Then v_ℓ has at most one out-neighbor in X_u by Observation 2.2, no out-neighbor in $V' \setminus X_u$ by the definition of X_u and no out-neighbor in $V(A^j)$ by the assertion just proved before. Therefore, v_ℓ has at least two out-neighbors in $V(A^{j+1})$. This contradicts Observation 2.3. Thus, v has no in-neighbor in $V(A^{j+1})$, and Claim 2.5 is proved. □

Claim 2.6. *The length t of A^0 and A^1 is equal to* 3.

Proof. For a contradiction, assume that $t \geq 4$. If both a_0^j and a_2^j are in K then a_0^j is adjacent to a_2^j, a contradiction to Observation 2.1. So, $a_0^j = w$ or $a_2^j = w$. Without loss of generality, we may assume that $a_0^j = w$. Then both a_1^j and a_{t-1}^j are in K. So, a_1^j is adjacent to a_{t-1}^j, a contradiction to Observation 2.1. Thus, $t = 3$. □

Claim 2.7. *$V' \neq \emptyset$.*

Proof. If $V' = \emptyset$, then

$$18 \leq \sum_{v \in V(D)} d^+(v) = |A(D)| \leq \binom{6}{2} = 15,$$

a contradiction. Thus, we must have $V' \neq \emptyset$. □

Claim 2.8. *Let $u \in V'$ and suppose u is adjacent to all vertices in $V(A^j)$. Then*

1. *if u has an in-neighbor in $V(A^j)$ then $A^j \to u$;*
2. *if u has an out-neighbor in $V(A^j)$ then $u \to A^j$.*

Proof. 1. By renaming the vertices of $V(A^j)$, we may assume that $a_0^j \to u$. If $u \to a_1^j$ then $(u, a_1^j, a_2^j, a_0^j, u)$ and A^{j+1} are two disjoint cycles of different lengths in D, a contradiction. So, $a_1^j \to u$. If $u \to a_2^j$ then $(u, a_2^j, a_0^j, a_1^j, u)$ and A^{j+1} form a bad collection of cycles in D, a contradiction. Thus, $A^j \to u$.

2. By arguments similar to those used in Assertion 1, we can prove Assertion 2. □

Claim 2.9. *Let $u \in V' \cap K$ and suppose u has two in-neighbors in $V(A^j)$. Then any vertex in $V(A^{j+1})$ has no out-neighbor in $V' \cap K$. Moreover, if $w \in V'$ and w has an in-neighbor in $V(A^{j+1})$ then w has no out-neighbor in V'.*

Proof. For a contradiction, assume that $v \in V' \cap K$ has an in-neighbor in $V(A^{j+1})$. Since u and v are in K, u is adjacent to v. So $u \to v$ or $v \to u$. By Claim 2.5, $u \not\to v$. So $v \to u$. Since $u \in K$ and u has no in-neighbor in $V(A^{j+1})$, u has at least two out-neighbors in $V(A^{j+1})$, a contradiction to Observation 2.3. Thus, any vertex in $V(A^{j+1})$ has no out-neighbor in $V' \cap K$. Now, assume that $w \in V'$ and w has an in-neighbor in $V(A^{j+1})$ and w has an out-neighbor v in V'. It is clear that $v \in K$. By renaming the vertices of $V(A^{j+1})$, we may assume that $a_0^{j+1} \to w$. Because all vertices of $V(A^{j+1})$ belong to K and v has no in-neighbor in $V(A^{j+1})$, v has three out-neighbors in $V(A^{j+1})$, a contradiction to Observation 2.3. □

Claim 2.10. *Let $u \in V'$ and let u be adjacent to all vertices in $V(A^0) \cup V(A^1)$. Then*

1. *if u has an in-neighbor in $V(A^j)$ then $A^j \to u$ and $u \to A^{j+1}$;*
2. *if u has an out-neighbor in $V(A^j)$ then $u \to A^j$ and $A^{j+1} \to u$.*

Proof. 1. If u has an in-neighbor in $V(A^j)$ then by Claim 2.8, $A^j \to u$. For a contradiction, assume that u has an in-neighbor in $V(A^{j+1})$. Again by Claim 2.8, $A^{j+1} \to u$. Since D is strong, we can find in D a path $(v_1, \ldots, v_\ell, v_{\ell+1})$ such that

$v_1 = u$, $v_{\ell+1} \in V(A^0) \cup V(A^1)$. Then, we get a contradiction to Observation 2.3. So, u has no in-neighbor in $V(A^{j+1})$. Thus, $A^j \to u$ and $u \to A^{j+1}$.

2. By arguments similar to those used in Assertion 1 we can prove Assertion 2. □

Claim 2.11. *Let $u \in V'$ and suppose that u is not adjacent to exactly one vertex in A^j and is adjacent to all vertices in A^{j+1}. Then*

1. *$u \to A^{j+1}$;*
2. *if u is not adjacent to a^j_r then $\{a^j_{r+1}, a^j_{r+2}\} \to u$, where $r \in \{0, 1, 2\}$, $r + 1$ and $r + 2$ are taken modulo 3.*

Proof. By renaming the cycles A^0 and A^1, we may assume that $j = 0$. By renaming the vertices of $V(A^0)$, we may assume that u is not adjacent to a^0_0. So, $u = w$ or $a^0_0 = w$.

1. Since u is adjacent to all vertices in $V(A^1)$ and by Claim 2.8, $u \to A^1$ or $A^1 \to u$. For a contradiction, assume that $A^1 \to u$. By Claim 2.5, $u \to \{a^0_1, a^0_2\}$. So, u has an out-neighbor $v \in V'$.

First, assume that $a^0_0 = w$. Then, u, v, a^0_1, a^0_2 and all vertices of $V(A^1)$ belong to K. By Observation 2.3 and Claim 2.9, $A^1 \to v$, $v \to \{a^0_1, a^0_2\}$ and each vertex of $V(A^0)$ has no out-neighbor in V'. By renaming the vertices of $V(A^1)$, we may assume that $a^0_0 \to \{a^1_0, a^1_1\}$. Since a^0_1 has two out-neighbors in $V(A^1)$, $a^0_1 \to a^1_0$ or $a^0_1 \to a^1_1$. If $a^0_1 \to a^1_0$ (respectively, $a^0_1 \to a^1_1$) then (a^0_1, a^1_0, v, a^0_1) and $(a^0_0, a^1_1, u, a^0_2, a^0_0)$ (respectively, (a^0_1, a^1_1, v, a^0_1) and $(a^0_0, a^1_0, u, a^0_2, a^0_0)$) are two disjoint cycles of different lengths in D, a contradiction.

Now, assume that $u = w$. Then, v and all vertices of $V(A^0) \cup V(A^1)$ belong to K. By Observation 2.3 and Claim 2.10, $A^1 \to v$ and $v \to A^0$. By Claim 2.5, each vertex of $V(A^0)$ has no out-neighbor in V'. So, each vertex of $V(A^0)$ has two out-neighbors in $V(A^1)$. By renaming the vertices of $V(A^1)$, we may assume that $a^0_0 \to \{a^1_0, a^1_1\}$. Since a^0_1 has two out-neighbors in $V(A^1)$, $a^0_1 \to a^1_0$ or $a^0_1 \to a^1_1$. If $a^0_1 \to a^1_0$ (respectively, $a^0_1 \to a^1_1$) then (a^0_1, a^1_0, v, a^0_1) and $(a^0_0, a^1_1, u, a^0_2, a^0_0)$ (respectively, (a^0_1, a^1_1, v, a^0_1) and $(a^0_0, a^1_0, u, a^0_2, a^0_0)$) are two disjoint cycles of different lengths in D, a contradiction.

2. Suppose that $u \to \{a^0_1, a^0_2\}$. Since D is strong, there exists a path $(v_1, \ldots, v_\ell, v_{\ell+1})$ such that $v_1 \in V(A^0) \cup V(A^1)$, $v_{\ell+1} = u$ and $v_2, \ldots, v_\ell \in V'$, giving a contradiction to Observation 2.3. Thus, $a^0_1 \to u$ or $a^0_2 \to u$.

First, assume that $u \to a^0_2$. Then $a^0_1 \to u$, a contradiction to Observation 2.4. Thus $a^0_2 \to u$.

Next, assume that $u \to a^0_1$ and $u = w$. Set $(A^0)' = (u, a^0_1, a^0_2, u)$ and A^1. Then a^0_0 is not adjacent to exactly one vertex in $(A^0)'$, and is adjacent to all vertices in A^1. By Assertion 1, $a^0_0 \to A^1$. Suppose there exists a vertex of $V(A^1)$ having an out-neighbor in V'. Without loss of generality, we may assume that a^1_0 has an out-neighbor $v \in V'$. By Claim 2.10, $v \to A^0$ and $A^1 \to v$. So, (a^0_1, a^0_2, u, a^0_1) and $(v, a^0_0, a^1_2, a^1_0, v)$ are two disjoint cycles of different lengths in D, a contradiction. Now, suppose that any vertex of $V(A^1)$ has no out-neighbor in V'. Therefore every vertex of $V(A^1)$ has at least two out-neighbors in $V(A^0)$. It follows that $A^1 \to \{a_1^0, a_2^0\}$. So, the vertex a_1^0 has only one out-neighbor in $V(A^0) \cup V(A^1)$, and therefore it must have an out-neighbor $v \in V'$. It is clear that $v \neq u$. By Claim 2.10, $A^0 \to v$ and $v \to A^1$. So (a_1^0, v, a_0^1, a_1^0) and $(a_2^0, u, a_1^1, a_2^1, a_2^0)$ are two disjoint cycles of different lengths in D, a contradiction.

Finally, assume that $u \to a_1^0$ and $a_0^0 = w$. First, suppose that a_0^0 has an out-neighbor $v \in V'$. By Claim 2.10, $A^0 \to v$ and $v \to A^1$. By Claim 2.9, each vertex of $V(A^1)$ has no out-neighbor in V'. So, each vertex of $V(A^1)$ has two out-neighbors in $V(A^0)$. If there exists $i \in \{0, 1, 2\}$ such that $a_i^1 \to a_0^0$ then (a_i^1, a_0^0, v, a_i^1) and $(a_{i+2}^1, a_1^0, a_2^0, u, a_{i+2}^1)$, or (a_i^1, a_0^0, v, a_i^1) and $(a_{i+2}^1, a_2^0, u, a_1^1, a_{i+2}^1)$, are two disjoint cycles of different lengths in D, a contradiction. So, $a_i^1 \not\to a_0^0$ for every $i \in \{0, 1, 2\}$. It follows that $A^1 \to \{a_1^0, a_2^0\}$. Then, (a_0^1, a_1^0, v, a_0^1) and $(a_2^1, a_2^0, u, a_1^1, a_2^1)$ are two disjoint cycles of different lengths in D, a contradiction. Now, suppose that a_0^0 has no out-neighbor in V'. So, a_0^0 has two out-neighbors in $V(A^1)$. By renaming the vertices of $V(A^1)$, we may assume that $a_0^0 \to \{a_0^1, a_1^1\}$. If there exists $i \in \{0, 1, 2\}$ such that a_i^1 has an out-neighbor $v \in V'$ then by Claim 2.8, $A^1 \to v$. By Claim 2.9, each vertex of $V(A^0)$ has no out-neighbor in V', a contradiction to $a_2^0 \to u$. Thus, each vertex of $V(A^1)$ has two out-neighbors in $V(A^0)$. It follows that $\{a_0^1, a_1^1\} \to \{a_1^0, a_2^0\}$, and a_1^0 has an out-neighbor $v \in V'$. By Claim 2.9, $v \to A^1$. Then, (a_1^1, a_2^0, u, a_1^1) and $(a_1^0, v, a_2^1, a_0^1, a_1^0)$ are two disjoint cycles of different lengths in D, a contradiction.

Thus we have proved $a_1^0 \to u$. □

Claim 2.12. *If $|V'| = 1$ then D is isomorphic to D_7^3.*

Proof. Let there exist $x, y \in V(D)$ such that x is not adjacent to y. Then

$$21 \leq \sum_{v \in V(D)} d^+(v) = |A(D)| \leq \binom{7}{2} - 1 = 20,$$

a contradiction. So, each pair of vertices $x, y \in V(D)$ is adjacent.

Let $V' = \{u\}$. By renaming the cycles A^0 and A^1, we may assume that $A^0 \to u$, $u \to A^1$. By renaming the vertices of $V(A^0)$, we may assume that $a_0^1 \to \{a_0^0, a_1^0\}$.

Suppose that $a_1^1 \to a_1^0$; it follows that $a_1^0 \to a_2^1$, $a_2^1 \to \{a_0^0, a_2^0\}$, $a_0^0 \to a_1^1$, $a_1^1 \to a_2^0$, $a_2^0 \to a_0^1$. So, $(a_2^0, a_0^1, a_1^1, a_2^0)$ and $(u, a_2^1, a_0^0, a_1^0, u)$ are two disjoint cycles of different lengths in D. Thus, $a_1^0 \to a_1^1$. It follows that $a_1^1 \to \{a_0^0, a_2^0\}$, $a_0^0 \to a_2^1$, $a_2^1 \to \{a_1^0, a_2^0\}$, $a_2^0 \to a_0^1$. Now let φ be the following mapping from V to $V(D_7^3)$: $u \mapsto v_0$, $a_0^1 \mapsto v_1$, $a_1^1 \mapsto v_2$, $a_0^0 \mapsto v_3$, $a_2^1 \mapsto v_4$, $a_1^0 \mapsto v_5$ and $a_2^0 \mapsto v_6$. Then it is not difficult to verify that φ is an isomorphism between D and D_7^3. □

3 Proof of Theorem 1.1

Now we continue to prove Theorem 1.1. If $|V'| = 1$ then by Claim 2.12, D is isomorphic to D_7^3. So, $|V'| \geq 2$. Note that if V' contains exactly one vertex in K then it contains w. We consider the following cases separately.

Case 1. V' contains two vertices $u, v \in K$.

Without loss of generality, we may assume that $u \to v$. We again divide Case 1 into several subcases.

Subcase 1.1. There exists $j \in \{0, 1\}$ such that $w \in V(A^j)$.

By renaming the cycles A^0 and A^1, we may assume that $j = 0$. By renaming the vertices of $V(A^0)$, we may assume that $a_0^0 = w$.

First, assume that a_0^0 is not adjacent to u or a_0^0 is adjacent to u but $a_0^0 \to u$. By Claim 2.11 and Claim 2.10, $\{a_1^0, a_2^0\} \to u$ and $u \to A^1$. By Claim 2.10, Claim 2.11 and Observation 2.3, it is easy to see that $\{a_1^0, a_2^0\} \to v$ and $v \to A^1$. By Claim 2.9, each vertex of $V(A^1)$ has no out-neighbor in V'. So, each vertex of $V(A^1)$ has two out-neighbors in $V(A^0)$. It follows that $a_0^1 \to a_0^0$ or $a_0^1 \to a_1^0$. If $a_1^1 \to a_2^0$ and $a_0^1 \to a_0^0$ (respectively, $a_0^1 \to a_1^0$) then (a_1^1, a_2^0, u, a_1^1) and $(a_0^1, a_0^0, a_1^0, v, a_0^1)$ (respectively, $(a_0^1, a_1^0, v, a_2^1, a_0^1)$) are two disjoint cycles of different lengths in D, a contradiction. So, $a_2^0 \to a_1^1$. It follows that $a_1^1 \to \{a_0^0, a_1^0\}$. If $a_0^1 \to a_2^0$ then (a_0^1, a_2^0, u, a_0^1) and $(a_1^1, a_0^0, a_1^0, v, a_1^1)$ are two disjoint cycles of different lengths in D, a contradiction. So, $a_2^0 \to a_0^1$ and $a_0^1 \to \{a_0^0, a_1^0\}$. It follows that a_0^0 has an out-neighbor $x \in V'$. It is not difficult to verify that $A^0 \to x$ and $x \to A^1$. Then, $(a_1^1, a_1^0, a_2^0, a_1^1)$ and $(a_0^0, x, a_2^1, a_0^1, a_0^0)$ are two disjoint cycles of different lengths in D, a contradiction.

Now, assume that a_0^0 is adjacent to u but $u \to a_0^0$. By Claim 2.10, $u \to A^0$ and $A^1 \to u$. If a_0^0 is not adjacent to v then by Claim 2.11, $v \to A^1$, a contradiction to Observation 2.3. So, a_0^0 is adjacent to v. By Claim 2.10 and Observation 2.3, $v \to A^0$ and $A^1 \to v$. By Claim 2.9, each vertex of $V(A^0)$ has no out-neighbor in V'. So, each vertex of $V(A^0)$ has at least two out-neighbors in $V(A^1)$. By renaming the vertices of $V(A^1)$, we may assume that $a_0^0 \to \{a_0^1, a_1^1\}$. Since a_2^0 has two out-neighbors in $V(A^1)$, $a_2^0 \to a_1^1$ or $a_2^0 \to a_2^1$. If $a_2^0 \to a_1^1$ (respectively, $a_2^0 \to a_2^1$) then (a_0^0, a_0^1, u, a_0^0) and $(a_2^0, a_1^1, v, a_1^0, a_2^0)$ (respectively, $(a_2^0, a_2^1, v, a_1^0, a_2^0)$ are two disjoint cycles of different lengths in D, a contradiction.

Subcase 1.2. $w \notin V(A^j)$ for every $j = 0, 1$.

By Claim 2.10 and by renaming the cycles A^0 and A^1, we may assume that $A^0 \to u$ and $u \to A^1$. By Claim 2.10 and Observation 2.3, $A^0 \to v$ and $v \to A^1$.

First, we may assume that there exists vertex of $V(A^1)$ has an out-neighbor in $V(A^0)$. Without loss of generality, we may assume that $a_0^1 \to a_0^0$. If $a_2^1 \to a_1^0$ (respectively, $a_2^1 \to a_2^0$) then (a_0^1, a_0^0, v, a_0^1) and $(a_2^1, a_1^0, u, a_1^1, a_2^1)$ (respectively, $(a_2^1, a_2^0, u, a_1^1, a_2^1)$) are two disjoint cycles of different lengths in D, a contradiction. So, $\{a_1^0, a_2^0\} \to a_2^1$. It follows that a_2^1 has an out-neighbor $z \in V'$. By Claim 2.9, $z = w$ and z has no out-neighbor in V'. So, z has at least two out-neighbors in $V(A^0)$. It follows that $z \to a_1^0$ or $z \to a_2^0$. If $z \to a_1^0$ (respectively, $z \to a_2^0$) then (a_2^1, z, a_1^0, a_2^1) (respectively, (a_2^1, z, a_2^0, a_2^1)) and $(a_0^1, a_0^0, u, v, a_0^1)$ are two disjoint cycles of different lengths in D, a contradiction.

Now, we may assume that any vertex of $V(A^1)$ has no out-neighbor in $V(A^0)$. So, $A^0 \to A^1$. It follows that a_0^1 has at least two out-neighbors $z_1, z_2 \in V'$ with $z_1 \neq z_2$. It is easy to see that $z_1 \in K$ or $z_2 \in K$, a contradiction to Claim 2.9.

Case 2. V' contains exactly one vertex $u \in K$.

In this case, all vertices of $V(A^0) \cup V(A^1)$ belong to K. We again divide Case 2 into several subcases.

Subcase 2.1. $u \to w$.

By Claim 2.10 and by renaming the cycles A^0 and A^1, we may assume that $A^0 \to u$, $u \to A^1$. By Observation 2.3, $w \to A^1$. So, each vertex of $V(A^1)$ has at least two out-neighbors in $V(A^0)$ and each vertex of $V(A^0)$ has at least one out-neighbor in $V(A^1)$. By renaming the vertices of $V(A^0)$, we may assume that $a_0^1 \to \{a_0^0, a_1^0\}$. If $a_2^1 \to a_1^0$ then $a_1^0 \to a_1^1$. So, $(a_2^1, a_1^0, a_1^1, a_2^1)$ and $(a_0^0, u, w, a_0^1, a_0^0)$ are two disjoint cycles of different lengths in D, a contradiction. Thus, $a_1^0 \to a_2^1$. So, $a_2^1 \to \{a_0^0, a_2^0\}$ and $a_2^0 \to a_1^1$. Then, $(a_2^0, a_1^1, a_2^1, a_2^0)$ and $(a_0^0, u, w, a_0^1, a_0^0)$ are two disjoint cycles of different lengths in D, a contradiction.

Subcase 2.2. $w \to u$.

By Claim 2.10 and by renaming the cycles A^0 and A^1, we may assume that $A^0 \to u$, $u \to A^1$. Since D is strong, there exists $x \in V(A^0) \cup V(A^1)$ sao cho $x \to w$. By Observation 2.3, $x \in V(A^0)$. Without loss of generality, we may assume that $x = a_0^0$. By Observation 2.4, $w \not\to a_1^0$.

First, assume that $v \to a_2^0$. By Claim 2.8, w is not adjacent to a_1^0. Set $(A^0)' = (w, a_2^0, a_0^0, v)$ and A^1, it is not difficult to see that a_1^0 is not adjacent to exactly one vertex in $(A^0)'$ and is adjacent to all vertices in A^1. By Claim 2.11, $a_1^0 \to A^1$ and $\{a_0^0, a_2^0\} \to a_1^0$, a contradiction to $a_1^0 \to a_2^0$.

Now, assume that $w \not\to a_2^0$. Then, w has two out-neighbors in $V(A^1)$. Without loss of generality, we may assume that $w \to \{a_0^1, a_1^1\}$. It is not difficult to see that each vertex of $V(A^1)$ has at least two out-neighbors in $V(A^0)$. Let $a_0^1 \to a_0^0$. If $a_2^1 \to a_1^0$ (respectively, $a_2^1 \to a_2^0$) then (a_0^1, a_0^0, w, a_0^1) and $(a_2^1, a_1^0, u, a_1^1, a_2^1)$ (respectively, $(a_2^1, a_2^0, u, a_1^1, a_2^1)$) are two disjoint cycles of different lengths in D, a contradiction. So, $a_0^0 \to a_0^1$ and $a_0^1 \to \{a_1^0, a_2^0\}$. If $a_1^1 \to a_1^0$ (respectively, $a_1^1 \to a_2^0$) then (a_1^1, a_1^0, u, a_1^1) and $(a_0^1, a_2^0, a_0^0, w, a_0^1)$ (respectively, (a_0^1, a_1^0, u, a_0^1) and $(a_1^1, a_2^0, a_0^0, w, a_1^1)$) are two disjoint cycles of different lengths in D, a contradiction.

Subcase 2.3. u is not adjacent to w.

By Claim 2.10 and by renaming the cycles A^0 and A^1, we may assume that $A^0 \to u$, $u \to A^1$. Since D is strong, there exists $x \in V(A^0) \cup V(A^1)$ such that $x \to w$.

First, assume that $x \in V(A^0)$. By renaming the vertices of $V(A^0)$, we may assume that $x = a_0^0$. Easy to see $w \not\to a_1^0$. First, suppose that $w \to a_2^0$. Easy to see w is not adjacent to a_1^0. Then, a_1^0 is not adjacent to exactly one vertex in $(A^0)' = (w, a_2^0, a_0^0, w)$ and is adjacent to all vertices in A^1, from which we deduce the contradiction with the Claim 2.11. Now, suppose that $w \not\to a_2^0$. Then, $w \to A^1$. It is easy to see that each vertex of $V(A^1)$ has two out-neighbors in $V(A^0)$. Let $a_0^1 \to a_0^0$. If $a_2^1 \to a_1^0$ (respectively, $a_2^1 \to a_2^0$) then (a_0^1, a_0^0, w, a_0^1) and $(a_2^1, a_1^0, u, a_1^1, a_2^1)$ (respectively, $(a_2^1, a_2^0, u, a_1^1, a_2^1)$) are two disjoint cycles of different lengths in D, a contradiction. So, $a_0^0 \to a_0^1$ and $a_0^1 \to \{a_1^0, a_2^0\}$. If $a_1^1 \to a_1^0$ (respectively, $a_1^1 \to a_2^0$) then (a_1^1, a_1^0, u, a_1^1) and

$(a_0^1, a_2^0, a_0^0, w, a_0^1)$ (respectively, (a_0^1, a_1^0, u, a_0^1) and $(a_1^1, a_2^0, a_0^0, w, a_1^1)$) are two disjoint cycles of different lengths in D, a contradiction.

Now, assume that $x \in V(A^1)$. Without loss of generality, we may assume that $x = a_0^1$. It is easy to see that $w \not\to a_1^1$. If $w \to a_2^1$, then it is easy to see that w is not adjacent to a_1^1; it then follows that a_1^1 is not adjacent to exactly one vertex in $(A^1)' = (w, a_2^1, a_0^1, w)$ and is adjacent to all vertices in A^0, from which we deduce the contradiction with the Claim 2.11. So, $w \not\to a_2^1$. Then, $w \to A^0$. First, let a_1^1 has an out-neighbor in $V(A^0)$. Without loss of generality, we may assume that $a_1^1 \to a_0^0$. If $a_2^0 \to a_0^1$ (respectively, $a_2^0 \to a_2^1$) then (a_1^1, a_0^0, u, a_1^1) and $(a_2^0, a_0^1, w, a_1^0, a_2^0)$ (respectively, $(a_2^0, a_2^1, a_0^1, w, a_2^0)$) are two disjoint cycles of different lengths in D, a contradiction. So, $\{a_0^1, a_2^1\} \to a_2^0$ and $a_2^0 \to a_1^1$. Then, $(a_2^0, a_1^1, a_2^1, a_2^0)$ and $(a_0^0, u, a_0^1, w, a_0^0)$ are two disjoint cycles of different lengths in D, a contradiction. Thus, a_1^1 has no out-neighbor in $V(A^0)$, that is $A^0 \to a_1^1$. So, a_1^1 has at most two out-neighbors in D, a contradiction.

The proof of Theorem 1.1 is now complete.

Acknowledgements

The author of the article would like to express his sincere thanks to the Reviewers for their comments. These contributions have helped to make the article more complete and scientific.

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(Received 23 Apr 2025; revised 9 Sep 2025)