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On links between G-graphs and lifts

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Abstract

In 2005, Bretto and Faisant introduced a construction of G-graphs, that is, graphs with the vertex set consisting of the right cosets of a cyclic subgroup of a group, while the non-empty intersections of these cosets determine the vertex adjacency. It is natural to compare this construction with the well-known lifting construction, introduced in 1987. In this paper we investigate under which circumstances the G-graph can be obtained as a lift of a dipole, since such a comparison has not been done yet. Using the derived results we generalize constructions of two families of small regular graphs related to extremal graph theory to broader families.

1 Introduction

Constructions of regular graphs with given properties are of special interest in the problems in extremal graph theory, e.g., in the well-known degree-diameter and the

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degree-girth problems. The degree-diameter problem is about finding the largest possible number of vertices of a graph with given degree d and diameter k. On the other hand, the degree-girth problem deals with finding the smallest possible order $n(d, g)$ of a regular graph of degree d and girth g. Such a regular graph of degree d and girth g is called a (d, g)-*graph*, and a (d, g)-graph of minimal order is a (d, g)-*cage*. The number $n(d, g)$ is bounded below by the so-called *Moore bound* [10], which arises naturally by counting the number of vertices in the distance partition from a chosen vertex (for odd g), or an edge (for even g):

$$n(d,g) \geq \begin{cases} 1 + d + d(d-1) + \ldots + d(d-1)^{(g-3)/2} & \text{if } g \text{ is odd;} \\ 2\left(1 + (d-1) + (d-1)^2 + \ldots + (d-1)^{g/2-1}\right) & \text{if } g \text{ is even.} \end{cases}$$

One of the approaches of finding small d-regular graphs of given girth are constructions based on groups which are quite popular, since some of the graph invariants can be computed algebraically simply from the knowledge of the used group and its basic properties. The most famous such construction of graphs from groups are known as Cayley graphs, first used by Cayley in [5]. The vertex-transitivity of these graphs makes them good candidates in the degree-girth problem; moreover, their existence for any pair (d, g) with $d \geq 2$ and $g \geq 3$ was established in [11]. Gross and Tucker in [9] provided another construction called the technique of voltage assignments, or the lifting graph construction, which can be regarded, in a sense, as a generalisation of the concept of Cayley graphs. Another type of graph construction exhibiting highly regular properties is the class of G-graphs, introduced by Bretto and Faisant in [2].

The purpose of this paper is to show relations between G-graphs and lifting construction, motivated by an easy observation showing close links between them (see Section 3). It is worth mentioning that both methods can produce graphs that are not regular, providing an advantage in comparison with the Cayley graphs. However, in this paper we compare these two constructions from the point of view when they are giving regular graphs with prescribed girth.

Our paper is organised as follows. In Section 2, we give basic definitions and fundamental properties of all the mentioned constructions. In Section 3, as the main result of this paper, we provide a sufficient condition under which a G-graph can be obtained as a lift. In addition, we generalize two families of G-graphs appearing in the degree-girth problem for new values of degrees, simply using their descriptions as voltage assignments. The last section is devoted to concluding remarks, where we point out the existence of isomorphism between the known families of graphs and those we constructed, using the results obtained in Section 3.

2 Preliminaries

In this section we recall a few basic facts of the Cayley graphs, G-graphs and lifting graph construction. For more details on these methods in correspondence with the degree-girth problem, we refer the reader to [6].

Let G be a group and let S be a generating set of G, for short $G = \langle S \rangle$. Then the *Cayley graph*, denoted by $X(G, S)$, has as vertex set $V(X) = G$ and two distinct vertices $g, h \in G$ are joined by an edge directed from g to h if and only if there exists $s \in S$ such that $g = sh$. If S is closed under taking inverses the graph $X(G, S)$ can be regarded as undirected after identifying two oppositely oriented darts into a single undirected edge. It is known that the Cayley graphs are vertex transitive, hence regular. Note that if S does not contain the identity of G then $X(G, S)$ has no loops.

Another family of graphs we are interested in are called G-graphs, introduced in [2]. Again, let G be a group and $S = \{s_1, s_2, \ldots, s_n\}$ be a set of elements such that $G = \langle S \rangle$. The *G-graph* $\Phi(G, S)$ is a graph with the vertex set $V(\Phi) = \{\langle s \rangle g; s \in S, g \in G\}$, where $\langle s \rangle g$ is the right coset of the cyclic subgroup $\langle s \rangle$ generated by s, containing g. Two different vertices $\langle s_1 \rangle g$ and $\langle s_2 \rangle h$ are adjacent if and only if the intersection $\langle s_1 \rangle g \cap \langle s_2 \rangle h$ is non-empty. Our definition is slightly simplified in comparison with the original definition presented in [2]; however, for our purposes of obtaining simple undirected graphs, it is sufficient. On the other hand, it directly follows from the nature of the definition that the G-graph construction belongs to a much bigger family of coset incidence systems and coset geometries, whose origins go back to the so-called Tits buildings and related structures.

Moreover, from this definition it immediately follows that $\Phi(G, S)$ is an n-partite graph with $n = |S|$. It is worth mentioning the following connection between Cayley graphs and G-graphs, proved in [3]:

Lemma 2.1 *Let $L(\Phi(G, S))$ be the line graph associated with Φ, that is a graph having vertices the edges of Φ and two vertices in L are adjacent if and only if the corresponding edges in Φ are incident. Then for generating set $S = \{s_1, s_2\}$ such that $\langle s_1 \rangle \cap \langle s_2 \rangle = \{e\}$ it holds that $L(\Phi(G, S)) \cong X(G, A)$, where $A = (\langle s_1 \rangle \cup \langle s_2 \rangle) \setminus \{e\}$.*

For more properties of G-graphs and their connections to Cayley graphs we refer the reader to [1, 2, 3, 4].

The last part of this section is devoted to the theory of graph lifting by voltage assignments, introduced in [9]. Let Γ be a graph (possibly with loops and parallel edges), and let $V(\Gamma)$ and $E(\Gamma)$ be the vertex set and the edge set of Γ. In order to describe the graph lifting construction we will consider that every edge h of Γ can be viewed as consisting of two oppositely directed darts x and x^{-1}. Thus we can obtain the dart set $D(\Gamma)$ associated to the edge set $E(\Gamma)$. Given a finite group G, a *voltage assignment* on Γ in G is a mapping $\alpha : D(\Gamma) \to G$ such that $\alpha(x^{-1}) = (\alpha(x))^{-1}$ for every $x \in D(\Gamma)$. We call the group G the *voltage group*. Then the *lift* Γ^α of Γ with respect to the voltage assignment α has vertex set $V(\Gamma^\alpha) = V(\Gamma) \times G$ and dart set $D(\Gamma^\alpha) = D(\Gamma) \times G$, and for any dart x of $D(\Gamma)$ directed from a vertex u to a vertex v and for any $g \in G$ there is a dart (x, g) in Γ^α from the vertex (u, g) to the vertex $(v, g\alpha(x))$; the darts (x, g) and $(x^{-1}, g\alpha(x))$ form an edge of Γ^α, and the edges obtained this way give us the edge set $E(\Gamma^\alpha)$. The original graph Γ is called the *base graph* of the lift. For each vertex u of Γ the set $\{(u, g), g \in G\}$ of vertices of the lift is the *fibre* above u.

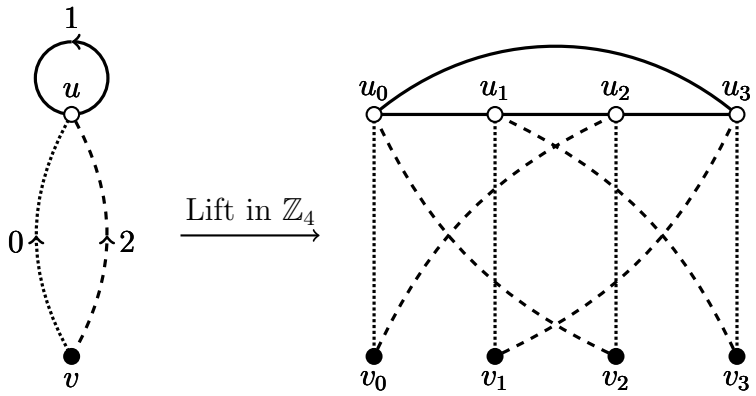

Figure 1: The base graph and its lift using group $\mathbb{Z}_4$.

Example 2.1 Let us consider a multigraph with two vertices u and v as a base graph, such that there is a loop at vertex u, and there are two parallel edges between u and v. Take the cyclic group $\mathbb{Z}_4$ of order 4 as the voltage group. If we assign $1 \in \mathbb{Z}_4$ as a voltage on the loop in one direction, then 3 is the voltage on the loop in the opposite direction. Assign voltages 0 and 2 on the darts from v to u, thus in the opposite direction the voltages are also 0 and 2, since these are the inverses of 0 and 2 in $\mathbb{Z}_4$, respectively. See Figure 1 (the voltages are displayed only in one direction for each edge). In the lifted graph, the fiber corresponding to the vertex u is represented by empty circles, and the fiber over v by full circles. In order to make the understanding easier, the three edges in the base graph are displayed by continuous, dashed and dotted edges, respectively. Their fiber in the lifted graph uses the corresponding style.

In the graph lifting construction we can easily count the girth of a lift in terms of the base graph and the voltage assignment. Given a base graph Γ and a voltage assignment α on Γ in a group G, let $W = x_0x_1 \dots x_t$ be a walk in Γ described as a sequence of darts such that the terminal vertex of x_{i-1} is the initial vertex of x_i for $i \in \{1, \dots, t\}$. Then the *net voltage* $\alpha(W)$ is simply the product $\alpha(x_0)\alpha(x_1) \dots \alpha(x_t)$. A walk W is called *non-reversing* if it contains no pair of consecutive oppositely directed darts. Now we are ready to formulate Lemma 2.2 about the relation between non-reversing walks and girth given by Exoo and Jajcay [7].

Lemma 2.2 *Let Γ be a finite graph and $\alpha : D(\Gamma) \to G$ be a voltage assignment of Γ in a group G. The girth of the lift Γ^α is equal to the length of a shortest closed non-reversing walk W in Γ of net voltage 1_G.*

3 Main result

We recall that Lemma 2.1 described a connection between Cayley graphs and G-graphs. It is also natural to consider connections between G-graphs and lifts. In

order to do so in the first step we declare in clear terms a basic property of G-graphs in the following lemma, which, according to our best knowledge, has not been mentioned earlier, though it is straightforward.

Lemma 3.1 *Let $\Phi(G,S)$ be a G-graph for an arbitrary group G and its subset S. Then the following holds:*

- *If $|S|=1$, then the graph $\Phi(G,S)$ is an empty graph.*
- *If $|S|=2$, then the graph $\Phi(G,S)$ is bipartite, hence its girth is even.*
- *If $|S|\geq 3$, then $\Phi(G,S)$ has girth 3.*

Proof. When $|S|=1$, i.e. $S=\{s\}$ and $G=\langle s\rangle$, the resulting G-graph $\Phi(G,S)$ is trivial having one vertex and no edges. For $|S|=2$ the graph $\Phi(G,S)$ is bipartite by the definition of G-graphs, hence its girth is even. Finally, in the case of $|S|\geq 3$ we get an n-partite graph where $n\geq 3$. Considering the identity element e of G, for any $s,t\in S$ there exists an edge between each two vertices $\langle s\rangle e$ and $\langle t\rangle e$ in different partitions, thus for $n\geq 3$ the resulting graph contains a cycle of length 3. □

A direct consequence of this lemma is that for obtaining a regular graph of girth $g\geq 4$ as a G-graph, we need to consider only such G-graphs $\Phi(G,S)$ for which $S=\{a,b\}$ and the group elements a and b are of the same order. This inspires us to compare the G-graph construction, where the generating set has precisely two elements, with the lifting construction with the base graph having two vertices.

The following theorem provides a sufficient condition under which a G-graph can be described as a lift of a dipole.

Theorem 3.1 *Let $G=\langle a,b\rangle$ be a group generated by two elements a and b of the same order k such that $\langle a\rangle\cap\langle b\rangle=\{e\}$. Further, let H be such a subgroup of G for which $H\cap\langle a\rangle=H\cap\langle b\rangle=\{e\}$ and $\langle a\rangle H=\langle b\rangle H=G$. Let Θ be a dipole with two vertices u and v with k parallel edges between them, where the vertices u and v correspond to the subgroups $\langle a\rangle$ and $\langle b\rangle$, respectively. Then the G-graph $\Phi(G,\{a,b\})$ can be described as a lift of dipole Θ with voltage group H and voltage assignment such that $h\in H$ is a voltage on the dart from u to v if and only if $h^{-1}\in\langle a\rangle\langle b\rangle$.*

Proof. First of all, we mention that the vertices in V_a corresponding to the right cosets of $\langle a\rangle$ can be indexed by elements of H, since for all $x\in G$ we have $|H\cap\langle a\rangle x|=1$, i.e., each right coset of $\langle a\rangle$ contains precisely one element of H. The same argumentation holds also for V_b. Let $H=\{e,h_2,h_3,\ldots,h_m\}$, thus the set of right cosets of $\langle a\rangle$ and $\langle b\rangle$ can be denoted as $\langle a\rangle e$, $\langle a\rangle h_2$, $\langle a\rangle h_3$, …, $\langle a\rangle h_m$, and $\langle b\rangle e$, $\langle b\rangle h_1$, $\langle b\rangle h_2$, …, $\langle b\rangle h_m$, respectively.

In the first step we prove that right-multiplication by any element $h\in H$ induces an automorphism θ_h of the graph Φ. Let $\{\langle a\rangle h_i,\langle b\rangle h_j\}$ be an edge in Φ for some $h_i,h_j\in H$. Then there exists $g\in G$, and $r,s\in\{0,1,\ldots,k-1\}$ such that $g=a^rh_i=b^sh_j$. Thus, for all $h\in H$ we have that $gh=a^rh_ih=b^sh_jh$, therefore $\{\langle a\rangle h_ih,\langle b\rangle h_jh\}$ is an edge in Φ.

As a consequence of [9, Theorem 2.2.2] we have that a graph Γ is isomorphic to a voltage lift of a dipole if and only if the group of automorphisms $\mathrm{Aut}(\Gamma)$ contains a semiregular subgroup with two orbits. (A nontrivial permutation group is

semiregular, if all of its orbits have equal size and the stabilizer of all the elements is trivial.)

We have shown that H acts regularly on V_a and V_b, respectively, thus H acts semiregularly on $V(\Phi)$, hence Φ can be obtained as a lift of a dipole with voltage group H.

Now, we derive the set of voltages on the darts directed from the vertex u to the vertex v, where u and v correspond to the fiber of right cosets of $\langle a\rangle$ and $\langle b\rangle$, respectively. (This is sufficient since the voltages over the opposite darts directed from v to u will be their inverses.) The vertices in the fibers can be indexed by the elements of the voltage group H. For any $h \in H$ we can identify the vertex u_h and v_h by the right cosets $\langle a\rangle h$ and $\langle b\rangle h$, respectively. Since H acts semiregularly, it is enough to consider the neighbourhood of one vertex u_e. The vertex u_e is joined with v_h if and only if $h \in H$ appears as a voltage on a dart from u to v in Θ. So let us take an element $h \in H$ such that there is an edge between u_e and v_h in Φ, or equivalently, $\langle a\rangle e$ and $\langle b\rangle h$ are adjacent in Φ. It easily follows from the definition of G-graphs that $\{\langle a\rangle x, \langle b\rangle y\}$ is an edge in Φ if and only if there exists g such that $g \in \langle a\rangle x \cap \langle b\rangle y$, i.e., when $g = a^i x = b^j y$ for some $i, j \in \{0, 1, \ldots, k-1\}$, what is equivalent to the condition $xy^{-1} \in \langle a\rangle\langle b\rangle$. In our case $x = e$ and $y = h$, hence there is an edge in Φ from $\langle a\rangle e$ to $\langle b\rangle h$ if and only if $h^{-1} \in \langle a\rangle\langle b\rangle$. This means that h is a voltage on a dart from u to v in Θ precisely when $h^{-1} \in \langle a\rangle\langle b\rangle$. □

We apply the previous theorem on a family of $(p, 6)$-graphs that was constructed as G-graphs from the Heisenberg group modulo p in [4, Section 4], where p is a prime.

Recall that the *continuous Heisenberg group* is a group of 3×3 upper triangular matrices over the field of real numbers, having 1's on the diagonal, and the operation is the usual matrix multiplication. This group is well-known and used mainly in quantum physics. When we change the basic field of the real numbers to the finite field of order p, for p prime, we obtain the *Heisenberg group modulo p*.

Example 3.1 Let $p \geq 3$ be a prime number. Denote by G the Heisenberg group modulo p, which has representation:

$$G = \langle x, y, z \mid x^p = y^p = z^p = 1, xz = zx, yz = zy, zyx = xy\rangle.$$

This group is of order p^3 and it can be also identified with the set of 3×3 matrices

$$G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_p \right\}$$

over the finite field $\mathbb{F}_p$, together with the usual multiplication of matrices. Moreover, the elements x, y, z from the presentation can be chosen as

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In fact, $z = xyx^{-1}y^{-1}$, so G can be generated by two elements $G = \langle x, y\rangle$, although sometimes it is more natural to work with three generators related to the three variable entries in the matrix representation.

Let us consider the subgroup H of G defined by

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \mid a, c \in \mathbb{F}_p \right\}.$$

It is easy to check that H is of order p^2 and it is closed under multiplication. Further, let us consider the cyclic subgroups $M_1 = \langle x\rangle$ and $M_2 = \langle y\rangle$ generated by matrices x and y, respectively. Clearly, $|M_1| = |M_2| = p$, $H \cap M_1 = H \cap M_2 = M_1 \cap M_2 = \{e\}$. Moreover, we already know that $G = \langle x, y\rangle$, thus the assumptions of Theorem 3.1 are satisfied. Hence, the $(p, 6)$-graphs of order $2p^2$ constructed as G-graphs from the Heisenberg group modulo p in [4, Section 4], can be constructed also as lifts of a dipole with voltage group H and voltages as inverses of the elements in the set

$$H \cap M_1M_2 = \left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_p \right\}.$$

This is the same $(p, 6)$-graph of order $2p^2$, which was constructed first in [8].

Using the obtained voltage assignment and the voltage group H, we can prove that the same voltage assignment works not only over the finite field $\mathbb{F}_p$ of prime order p, but also over any finite field $\mathbb{F}_q$ of prime power order q, independently on the existence on the overgroup G and the two specific generators of the same order which were necessary for the G-graph construction. Thus we can state the following theorem.

Theorem 3.2 *Let q be a prime power and consider the group H of order q^2, where*

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{F}_q \right\}.$$

Let Γ be a dipole consisting from two vertices u and v joined by q parallel darts $\{e_i, i \in \mathbb{F}_q\}$ indexed by the elements of $\mathbb{F}_q$. Define the voltage assignment $\alpha : D(\Gamma) \to H$ by the following rule: place the voltage $\alpha(e_i)$ on the dart e_i oriented from u to v and its inverse on the opposite dart e_i^{-1}, where

$$\alpha(e_i) = \begin{pmatrix} 1 & i & 0 \\ 0 & 1 & i \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the lift Γ^α is a $(q, 6)$-graph of order $2q^2$.

Proof. It is clear from the construction that the resulting graph is bipartite of order $2q^2$ and is regular of valency q. From the Moore bound it immediately follows that the girth of this graph cannot be 8 or more. In order to prove that its girth is 6, it is enough to exclude the existence of the 4-cycles, since it is bipartite. By contradiction, suppose that there is a cycle of length 4. Thus, there are two different paths of length 2 from vertex (u, x) to another vertex, say (u, y), giving the same net voltage. Thus there exists $a, b, c, d \in \mathbb{F}_q$ such that $a \neq b$, $c \neq d$ and

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & d & 0 \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix}^{-1}.$$

Equivalently,

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -b & b^2 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -d & d^2 \\ 0 & 1 & -d \\ 0 & 0 & 1 \end{pmatrix},$$

thus

$$\begin{pmatrix} 1 & a-b & b^2-ba \\ 0 & 1 & a-b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c-d & d^2-dc \\ 0 & 1 & c-d \\ 0 & 0 & 1 \end{pmatrix}.$$

The latter equation holds if and only if $a - b = c - d$ and $b^2 - ab = d^2 - cd$. Thus $b^2 - ab = d^2 - cd = d(d - c) = d(b - a)$, and since $a - b \neq 0$, after cancellation we obtain $b = d$ and, finally, $a = c$. This means that there is just one path of length 2 from (u, x) to (u, y), a contradiction. Hence, there are no 4-cycles in the considered graph proving that its girth is 6. Thus, we obtained a $(q, 6)$-graph of order $2q^2$, as stated. □

For the sake of brevity, we establish the following notation: under symbol $2 \in \mathbb{F}_q$ we mean the element $1 + 1 \in \mathbb{F}_q$, while $\binom{n}{2}$ means the element $n \cdot (n-1) \cdot 2^{-1} \in \mathbb{F}_q$.

In a similar way as above, we can extend the construction of $(p, 8)$-graphs presented in [4, Section 5] using our Theorem 3.1.

Example 3.2 The construction of a $(p, 8)$-graph in [4, Section 5] can be described as voltage assignment in the following manner. Let

$$H = \left\{ \begin{pmatrix} 1 & a & \binom{a}{2} & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_p \right\}$$

be the voltage group of order p^3 and let the base graph be a dipole with vertices u and v. Then the voltage $\alpha(e_i)$ on the dart e_i oriented from vertex u to vertex v is

$$\alpha(e_i) = \begin{pmatrix} 1 & i & \binom{i}{2} & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We formulate the extension of this observation to any finite field $\mathbb{F}_q$ as a theorem.

Theorem 3.3 *Let q be a prime power and consider the group H of order q^3, where*

$$H = \left\{ \begin{pmatrix} 1 & a & \binom{a}{2} & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_q \right\}.$$

Let Γ be a dipole consisting from two vertices u and v joined by q parallel edges $\{e_i, i \in \mathbb{F}_q\}$ indexed by the elements of $\mathbb{F}_q$. Define the voltage assignment $\alpha : D(\Gamma) \to H$ by the following rule: place the voltage $\alpha(e_i)$ on the dart e_i oriented from u to v and its inverse on the opposite dart e_i^{-1}, where

$$\alpha(e_i) = \begin{pmatrix} 1 & i & \binom{i}{2} & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the lift Γ^α is a $(q, 8)$-graph of order $2q^3$.

Proof. By the feature of the construction, Γ^α is bipartite, hence in order to prove that its girth is 8, we need to exclude the presence of 4-cycles and 6-cycles, since from the Moore bound we have that its girth cannot be larger than 8.

By contradiction, suppose that there is a cycle of length 4 in Γ^α. This is equivalent to the existence of two different paths of length 2 between two different vertices in the same fiber. Without loss of generality, we can consider vertex $u_{0,0,0}$ to be the initial vertex for those paths, and $u_{a,b,c}$ as their terminal vertex, for suitable $a, b, c \in \mathbb{F}_q$. Thus, due to the correspondence between the paths and net voltages, we have at least two solutions for the equation

$$\begin{pmatrix} 1 & i & \binom{i}{2} & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & j & \binom{j}{2} & 0 \\ 0 & 1 & j & 0 \\ 0 & 0 & 1 & j \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & a & \binom{a}{2} & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with unknowns $i, j \in \mathbb{F}_q$ such that $i \neq j$. This equation takes the following equivalent form after some computations:

$$\begin{pmatrix} 1 & i-j & \binom{i-j}{2} & -j\binom{i-j}{2} \\ 0 & 1 & i-j & -j(i-j) \\ 0 & 0 & 1 & i-j \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & \binom{a}{2} & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

giving us a system of three equations:

$$\begin{aligned} a &= i - j \\ b &= -j(i-j) \\ c &= -j\binom{i-j}{2}. \end{aligned}$$

Clearly, $a \neq 0$, since $i \neq j$. Thus, from the second equation $j = -b/a$, and consequently $i = a - (b/a)$ from the first one. However, this means that the system of equations has at most one solution with the required property $i \neq j$, a contradiction with the existence of two different paths of length 2 between the vertices $u_{0,0,0}$ and $u_{a,b,c}$.

In order to exclude 6-cycles in Γ^α we follow a similar approach as above. By contradiction, suppose that there exist vertices $u_{0,0,0}$ and $v_{a,b,c}$ such that there exist two different paths of length 3 between them in Γ^α. Thus, there exist $i, j, k, I, J, K \in \mathbb{F}_q$ such that $i \neq j \neq k$; $I \neq J \neq K$ and $(i, j, k) \neq (I, J, K)$, satisfying

$$\begin{pmatrix} 1 & i & \binom{i}{2} & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & j & \binom{j}{2} & 0 \\ 0 & 1 & j & 0 \\ 0 & 0 & 1 & j \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & k & \binom{k}{2} & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & I & \binom{I}{2} & 0 \\ 0 & 1 & I & 0 \\ 0 & 0 & 1 & I \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & J & \binom{J}{2} & 0 \\ 0 & 1 & J & 0 \\ 0 & 0 & 1 & J \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & K & \binom{K}{2} & 0 \\ 0 & 1 & K & 0 \\ 0 & 0 & 1 & K \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Equivalently,

$$\begin{pmatrix} 1 & i-j+k & \binom{i-j+k}{2} & (k-j)\binom{i-j}{2} \\ 0 & 1 & i-j+k & (k-j)(i-j) \\ 0 & 0 & 1 & i-j+k \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & I-J+K & \binom{I-J+K}{2} & (K-J)\binom{I-J}{2} \\ 0 & 1 & I-J+K & (K-J)(I-J) \\ 0 & 0 & 1 & I-J+K \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This gives us a system of three equations

$$\begin{aligned} i-j+k &= I-J+K \\ (i-j)(k-j) &= (I-J)(K-J) \\ \binom{i-j}{2}(k-j) &= \binom{I-J}{2}(K-J). \end{aligned}$$

The last two equations give us $i - j = I - J$, which after plugging into the first equation implies $k = K$. After that from the second equation we obtain that $j = J$ and finally, from the first one: $i = I$. Thus $(i, j, k) = (I, J, K)$, a contradiction. □

4 Concluding remarks

Several $(q, 6)$-graphs of order $2q^2$ are known. In this section we prove that the $(q, 6)$-graph Γ^α constructed in Theorem 3.2 is isomorphic to the $(q, 6)$-graph Υ^β constructed in [12], which was also described in the terms of voltage assignments.

Proposition 4.1 *([12]) Let q be a power of an odd prime and let $\mathbb{F}_q$ be the Galois field of order q. Define a voltage assignment β on each dart $x \in \mathbb{F}_q$ of a dipole Υ of degree q in the additive group $\mathbb{F}_q^+ \times \mathbb{F}_q^+$ by letting $\beta(x) = (x, x^2)$, $x \in \mathbb{F}_q$. The corresponding lift denoted by Υ^β is regular of valency q having girth 6 and order $2q^2$.*

Theorem 4.2 *For any odd prime power q the $(q, 6)$-graphs Γ^α and Υ^β are isomorphic.*

Proof. In order to shorten the proof, let us denote the element $\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{smallmatrix}\right)$ of the group H from Theorem 3.2 by $h(a, b)$. Firstly, one may easily check that the voltage groups $\mathbb{F}_q^+ \times \mathbb{F}_q^+$ for Υ^β and H for Γ^α are isomorphic, where an isomorphism $\theta : \mathbb{F}_q^+ \times \mathbb{F}_q^+ \to H$ can be given by $(a, b) \mapsto h(a, b + \frac{1}{2}a^2)$ for any $a, b \in \mathbb{F}_q$. Secondly, we invoke a result of Šiagiová [13, discussion after Theorem 1] by which the voltage assignment $\beta(x) = (x, x^2)$ from Proposition 4.1 is equivalent to $\beta'(x) = (x, p(x))$ for any quadratic polynomial over $\mathbb{F}_q$. For our purposes we can take $p(x) = -x^2/2$. Putting the pieces together one sees that, in terms of the edges $\{e_i;\ i \in \mathbb{F}_q\}$ from Theorem 3.2, the voltage assignment β' given now by $\beta'(e_i) = (i, -i^2/2) \in \mathbb{F}_q^+ \times \mathbb{F}_q^+$ is, by way of the above isomorphism θ, equivalent to $\theta(i, -i^2/2) = h(i, 0)$, which is the voltage assignment from the statement of Theorem 3.2. The claim follows. □

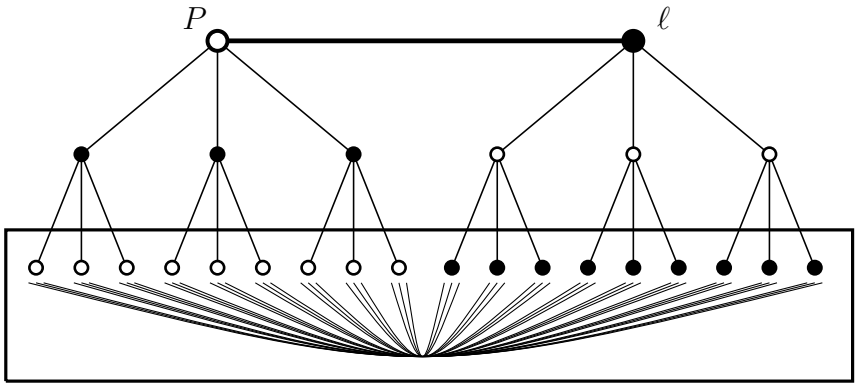

Figure 2: The incidence graph of the projective plane of order 3.

For completeness, and in view of the isomorphism just established, we note that by the constructions described in [8] both the above $(q, 6)$-near-cages and $(q, 8)$-near-cages arise geometrically from finite projective planes and generalized quadrangles of order q (and hence degree $q + 1$), respectively, by removing a point-line incident pair (P, ℓ) together with all the incident lines to P and points on ℓ. The next example shows the construction of a $(3, 6)$-near-cage in this manner.

Example 4.1 Let us consider the projective plane of order 3. Its incidence graph can be depicted using layers: on the top layer there is an incident point-line pair (P, ℓ); on the subsequent layers, the neighbors of the chosen point P and line ℓ are placed; and finally, on the bottom layer, all the remaining vertices. See Figure 2. By removing the incident point-line pair (P, ℓ) and their neighbours from the incidence

graph, only the $2 \cdot 3^2 = 18$ vertices on the marked bottom layer remain. The induced subgraph on this layer is regular of degree 3 and girth 6, hence it is a $(3, 6)$-graph. It is isomorphic to the well-known Pappus graph, depicted in Figure 3. For comparison, the $(3, 6)$-cage is the Heawood graph, having order 14.

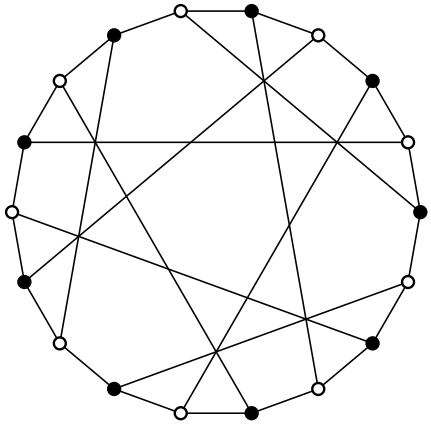

Figure 3: The Pappus graph.

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