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A direct approach to the Gallai-Schur numbers

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Abstract

This paper characterises the structure of every maximal, weak or strong, Gallai-Schur partition, as well as describing a family of minimal partitions. The results for maximal partitions confirm the exact values of Gallai-Schur numbers provided by Budden (2020) in the strong case, and provide corresponding values for weak Gallai-Schur numbers. The main proofs are elementary and standalone.

I dedicate this paper to the memory of my late father,
Gordon Rowley MA, formerly of Queens' College, Cambridge,
who took on some memorable and fulfilling challenges, late in life.

1 Introduction

In [1], Budden deduced the exact values of the Gallai-Schur numbers (in the strong case, as defined here) from an earlier result of Chung and Graham [3].

These numbers represent upper bounds on the size of colourings (or partitions) of integer intervals $[1, n]$ which avoid both monochromatic and rainbow sums of the form $a+b = c$. We distinguish the *strong case*, where we seek to avoid monochromatic sums where $a = b$, from the *weak case*, where such sums are permitted.

The result of Chung and Graham in [3] is graph-theoretic, and supported Budden's approach in the strong case very well, but it does not appear to lend itself easily to the weak case. Budden mentioned the weak case, but left its investigation for later work. Robertson *et al.* (2024) derived upper and lower bounds for the weak numbers [5].

In this paper, we provide a self-contained proof of Budden's result, and a parallel result for the weak case. We prove that the weak Gallai-Schur numbers are in fact equal to the lower bound derived in [5].

We also apply recent work on rainbow numbers to establish a **lower** bound on the possible size of Gallai-Schur partitions.

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2 Terminology and Definitions

In concept, the Gallai-Schur partition follows on from the Schur partition, so we begin there.

If an integer interval $U = [1, n]$ can be partitioned into r disjoint subsets S_i for $i = 1, 2, \ldots, r$, where no subset contains any set of integers a, b, c, such that $a+b = c$, then each such subset is said to be *sum-free* and that partition is called a *Schur (r-)partition.* The order of the set U is clearly n, and is also referred to as the order of the partition. We may refer to this type of partition as a *strong* partition, to distinguish it from the weak case defined next below.

If we limit the definition above so that it excludes only sets of three **distinct** integers a, b, c, such that $a+b = c$, then each such subset is *weakly* sum-free and that partition is a *weak Schur (r-)partition.* We define a *weak pair* of integers as a pair of the form $a, 2a$. The strong partition excludes such pairs from each subset, whereas the weak partition does not.

A Schur partition is a *Gallai-Schur partition*, if in addition there is no triple a, b, c, such that $a + b = c$, and each of a, b, c is in a different subset, noting that the partition must contain exactly r non-empty subsets. We define a *weak Gallai-Schur partition* correspondingly.

For any r, the *Gallai-Schur number* $GS(r)$ is the least integer n such that no Gallai-Schur r-partition of $[1, m]$ exists for any[1] integer $m \geq n$. We may define the *weak Gallai-Schur number* $WGS(r)$ in an exactly parallel manner.

A Gallai-Schur partition P_r (strong or weak) is *maximal* if it partitions $[1, m]$ into r non-empty subsets, where $m = GS(r)$ or $WGS(r)$ respectively.

In Section 3, we define some mappings that we will use in Section 4 to determine the structure of maximal (strong or weak) Gallai-Schur partitions, and in Section 5 we deduce $GS(r)$ and $WGS(r)$ directly. Section 6 serves to tie in a related result for the so-called rainbow numbers. In Section 7, we set out some brief conclusions.

3 Gallai-Schur Partitions

We begin by defining two very simple constructions, which we will use iteratively. These are derived by generalising and extending the construction used by Budden [1] to demonstrate lower bounds. They are used here, along with their inverses, to derive both upper and lower bounds.

Theorem 3.1. *(2-fold / 5-fold Construction)*
If there is a (weak) Gallai-Schur partition of the integers $[1, m]$ into r non-empty subsets, then there is a (weak) Gallai-Schur partition of $[1, 2m + 1]$ into $r + 1$ non-empty subsets; and a (weak) Gallai-Schur partition of $[1, 5m+4]$ into $r+2$ non-empty subsets.

[1] This definition may seem slightly longer than necessary, but it must deal with the fact that choosing $m = r$ forces at least $1, 2, 3$ to be a rainbow triple when $r \geq 3$.

Proof. Let P_r be such a partition of $[1, m]$. We can identify the colour of an integer with the index number of the unique subset $S_i \subset P_r$ that it belongs to. Thus for an integer $a \in S_i$, we write the colour as $\chi(a) = i$.

The first part of the theorem can be demonstrated by considering a mapping, which we call ${}^2\Theta$, from P_r to a new partition Q_{r+1} of $[1, 2m+1]$. In Q_{r+1} all the odd numbers belong to a single new subset S'_1. The members of each of the other new subsets $S'_{t+1} \subset Q_{r+1}$ can be determined by multiplying the members of $S_t \subset P_r$ by 2, for $1 \leq t \leq r$. The mapping ${}^2\Theta$ is clearly well-defined, and has an obvious inverse. There is no 'weak pair' $(a, 2a)$ in S'_1, so we can deduce that if P_r is a strong or weak partition, Q_{r+1} has the same property.

In this way, for example, a strong partition P_2 of $[1, 4]$ into $S_1 = \{1, 4\}$, $S_2 = \{2, 3\}$ gives rise to Q_3 with $S'_1 = \{1, 3, 5, 7, 9\}, S'_2 = \{2, 8\}, S'_3 = \{4, 6\}$ and $2m + 1 = 9$. The subsets are clearly distinct and fully cover the interval $[1, 9]$. There is no 'weak pair' $(a, 2a)$ in S'_1, so Q_3 is strong, like P_2.

For any triple with $a + b = c$, either (i) exactly two of them are odd, or (ii) none of them is odd. In case (i), it is easy to check that the sum or difference of two members of S'_1 cannot be of the same colour, since the sum or difference is even: and of course a, b, c is not a rainbow triple if two of them are members of S'_1. In case (ii), if there is either a monochromatic or a rainbow triple in the other subsets, then we would know there was such a triple in the partition P_r, which is a contradiction. This proves the first part of the theorem.

The second part can be demonstrated by considering a mapping, which we call ${}^5\Theta$, from P_r to a new partition R_{r+2} of $[1, 5m+4]$. In R_{r+2} all integers x in the range $[1, 5m + 4]$ are assigned to a subset of R_{r+2} as follows. If $x \equiv 1$ or 4 modulo 5, then $x \in S''_1$. If $x \equiv 2$ or 3 modulo 5, then $x \in S''_2$. The members of each of the other subsets S''_{t+2} can be determined by multiplying the members of S_t by 5, for $1 \leq t \leq r$. Once more, the subsets are clearly distinct and fully cover the interval $[1, 5m + 4]$. The mapping from P_r to R_{r+2} is clearly well-defined, and has an obvious inverse. There is no 'weak pair' $(a, 2a)$ in S''_1 or S''_2, so if P_r is a strong or weak partition, R_{r+2} has the same property.

It is easy to see that the sum of two members of S''_1 must be in S''_2, and *vice versa*, and the resulting sum is obviously not rainbow in either case. Adding a member of S''_1 to a member of S''_2 clearly cannot form a monochromatic triple, but that triple is never rainbow either, since the sum cannot be a multiple of 5. Thus we have dealt with the case where no member of the triple is divisible by 5.

If any member of a triple with $a + b = c$ is divisible by 5, then either (i) all of them are divisible by 5, or (ii) exactly one of them is.

In case (i), we note that if there is either a monochromatic or a rainbow triple contained wholly in the subsets with index greater than 2, then we would know there was such a triple in the partition P_r, which is a contradiction.

In case (ii), we must have two members of either S''_1 or S''_2 whose sum or absolute difference is a multiple of 5, and hence they are in the same subset. Thus, the triple is neither monochromatic nor rainbow.

This proves the second part of the theorem. □

4 Structure of Maximal Partitions

The result in the previous section provides a basis for lower bounds on $GS(r)$ and $WGS(r)$. In this section, we explore the structure of **maximal** examples of Gallai-Schur partitions.

The assumption that a partition is maximal – i.e. that there is no larger Gallai-Schur partition into the same number of colours – is remarkably powerful in this instance.

We aim to prove the theorem below. To do so, we assume here that the colours of any partition occur in the natural order – so that if we have two colours i, j, and $i < j$, then the smallest member of subset S_i is less than the smallest member of S_j. We note that the main arguments in this section apply with the same force whether the strong or the weak criterion applies.

Theorem 4.1. *(5-fold / 2-fold Image)*
Let the (strong or weak) Gallai-Schur partition P_r be a maximal partition of order m, with $r > 3$. The first three integers must have colours 1, 2, 2 or 1, 2, 1 respectively. In the first case, m has residue 4 modulo 5 and P_r is an image, under the mapping ${}^5\Theta$, of a Gallai-Schur partition (strong or weak, respectively) of order $(m-4)/5$. In the second case, m is odd, and P_r is an image, under the mapping ${}^2\Theta$, of a Gallai-Schur partition (strong or weak, respectively) of order $(m-1)/2$.

Proof. Let P_r be a partition of $[1, m]$ into r colours, where $r > 3$. We assume there is no larger Gallai-Schur partition into r colours, since it is maximal.

If we look at the colours of P_r in increasing sequence, we can derive some very useful results.

We recall first that the colour of an integer i is written $\chi(i)$. So by our assumption above, $\chi(1) = 1$.

We must consider whether we can have $\chi(2) = 1$. If that were so, then with $r > 3$, there would be a contradiction, since we know that for some $s > 2$ we would have $\chi(s) = 2$, and for some $t > s$ we must have $\chi(t) = 3$. We can assume without loss of generality that these are the smallest integers of their respective colours. We can immediately deduce that $\chi(t-1) = \chi(t-2) = 1$, since otherwise we would have rainbow triples $(1, t-1, t)$ and $(2, t-2, t)$. However, we now have a monochromatic triple $(1, t-1, t)$. Thus we must take $\chi(2) = 2$.

It is worth pausing to note that the reasoning above ensures a full reflection symmetry. That is, if $t > 2$ is the smallest integer of a given colour, then all smaller integers must conform to a symmetrical colour pattern within the interval $[1, t-1]$, if there are no rainbow sums. Every pair of integers i and $t - i$ must belong to the same colour.

If $\chi(1) = 1$ and $\chi(2) = 2$, then we must have $\chi(3) \in \{1, 2\}$, since there are no rainbow colourings.

We note here that since $r > 3$, we cannot have $\chi(4) = 2$ by the same symmetry argument as above. That is, for some minimal $t > 4$ we would have $\chi(t) = 4$ and therefore $\chi(t-2) = \chi(t-4) = 2$, but this must produce a monochromatic triple $(2, t-4, t-2)$. We note that this argument applies to both strong and weak partitions.

Case: $\chi(3) = 2$

We first assume that $\chi(3) = 2$. Since there are no rainbow colourings, $\chi(4) \in \{1, 2\}$, and we know (from the above) that this means $\chi(4) = 1$. We now have the pattern "1221" for the first four integers, and we can see this implies $\chi(5) \notin \{1, 2\}$. We must then have $\chi(5) = 3$, and clearly this does not create a rainbow colouring, since the preceding sequence is symmetrical. We have thus proved that the first five integers are coloured "12213".

The fact that $\chi(5) = 3$ has interesting consequences for values of $s \in [6, 9]$. We have assumed there are no rainbow sums, so we can deduce that $\chi(s) \in \{3, \chi(s-5)\}$. However, we also know that $\chi(6) \in \{\chi(2), \chi(4)\}$, and $\chi(7) \in \{\chi(3), \chi(4)\}$. Thus $\chi(6) = 1$, and $\chi(7) = 2$. Similarly, we deduce that $\chi(8) = 2$, and $\chi(9) = 1$. It is easy to see that $\chi(10) \notin 1, 2$.

So far, based on our assumptions, we have proved that colours of the first ten integers must be "122131221x" in either a strong or a weak maximal Gallai-Schur partition, with $x \notin \{1, 2\}$.

This pattern of colouring continues indefinitely for the integers that are not multiples of 5. The fact that no multiple of 5 can be of colour 1 or 2, forces repetition of the "1221[$\chi(5k)$]" pattern iteratively for every subsequent set of 5 consecutive integers.

This can easily be formalised inductively for all integers in P_r. We assume that, for any $t \geq 2$, every integer of the form $5s$ with $1 \leq s \leq t$ is not of colour 1 or 2; that any integer of the form $5s-1$ or $5s-4$ is colour 1; and that any integer of the form $5s-2$ or $5s-3$ is of colour 2. We have already proved this for $t \leq 2$.

Since $\chi(6) = \chi(5t-1) = 1$ we deduce that $\chi(5(t+1)) \neq 1$.

Since $\chi(7) = \chi(5t-2) = 2$ we deduce that $\chi(5(t+1)) \neq 2$.

Let $x = \chi(5t)$. The absence of rainbow triples means that

$\chi(4) = 1$ implies $\chi(5(t+1)-1) \in \{1, x\}$, since $5t+4 = 5(t+1)-1$.

Similarly,

$\chi(3) = 2$ implies $\chi(5(t+1)-2) \in \{2, x\}$

$\chi(2) = 2$ implies $\chi(5(t+1)-3) \in \{2, x\}$ and

$\chi(1) = 1$ implies $\chi(5(t+1)-4) \in \{1, x\}$.

We also see that:

$\chi(6) = 1$ and $\chi(5t-2) = 2$ implies $\chi(5(t+1)-1) \in \{1, 2\}$, so $\chi(5(t+1)-1) = 1$.

$\chi(6) = 1$ and $\chi(5t-3) = 2$ implies $\chi(5(t+1)-2) \in \{1, 2\}$, so $\chi(5(t+1)-2) = 2$.

$\chi(3) = 2$ and $\chi(5t-1) = 1$ implies $\chi(5(t+1)-3) \in \{1, 2\}$, so $\chi(5(t+1)-3) = 2$.

$\chi(2) = 2$ and $\chi(5t-1) = 1$ implies $\chi(5(t+1)-4) \in \{1, 2\}$, so $\chi(5(t+1)-4) = 1$.

This completes the induction. We conclude that for all $t > 0$,

$\chi(5t-1) = \chi(5t-4) = 1$, $\chi(5t-2) = \chi(5t-3) = 2$ and $\chi(5t) \notin \{1, 2\}$.

We see that the integers in P_r that are divisible by 5 are not of colour 1 or 2. Those integers must therefore be coloured in such a way that if they are all divided by 5, the resulting sets will themselves form a strong or weak Gallai-Schur partition, according to the same property of P_r, using two fewer colours than P_r.

We now also see that m must have residue 4 modulo 5: if not, we could extend the interval by adding members to S_1 or S_2 to make it so – which would contradict the assumed maximality of P_r. (It should be clear from the above induction that the extension of this pattern of colouring integers not divisible by 5 in this way can never give rise to monochromatic or rainbow sums.)

Summarising the above, the members of S_1 consist exactly of all the members of $[1, m]$ with residue 1 or 4 modulo 5, and the members of S_2 consist exactly of all the members of $[1, m]$ with residue 2 or 3 modulo 5. All members of all the other subsets must be divisible by 5.

The multiples of 5 in that pattern form a subset of the interval $[1, m]$, such that if any two members of that subset form part of an additive triple within the interval, then the other member of that triple is a member of the same subset.

Thus we can form a new partition of $[1, (m-4)/5]$ by taking the subsets S_t of P_r for $t > 2$, dividing their members by 5, and re-labelling them with indices reduced by 2. We have clearly created the inverse of the mapping ${}^5\Theta$. Since there are no weak pairs in S_1 or S_2, the new partition is strong or weak according to the status of P_r. This proves the first part of the theorem.

Case: $\chi(3) = 1$

Moving to the second part of the theorem, we recall that the only alternative assumption for $\chi(3)$ which avoids a rainbow colouring is that $\chi(3) = 1$, and hence we cannot have $\chi(4) = 1$. Since $r > 3$, by the argument of reflection used above, we cannot have $\chi(4) = 2$. Therefore $\chi(4) = 3$.

We have assumed there are no rainbow sums, so we can further deduce that $\chi(5) \in \{1, 3\}$ and $\chi(5) \in \{1, 2\}$. Thus $\chi(5) = 1$, and we have the initial colour sequence "12131". Since there are no monochromatic sums, we can now see that no even integer up to 8 can be of colour 1. We also know that $\chi(6)$ must be either 2 or 3. Whichever it is, we know that $\chi(7) \in \{1, 3\}$ and $\chi(7) \in \{1, 2\}$. Thus $\chi(7) = 1$. This pattern can also be seen to repeat indefinitely: all the odd integers are members of S_1 and all the even integers are in other subsets. A full formal proof is omitted, since only a simpler version of the proof above is needed. We observe that the order m must be odd, since otherwise we could add the odd integer $(m + 1)$ to S_1, which would be a contradiction of our assumption of maximality.

As in the first part, we can form a new partition of $[1, (m-1)/2]$ by taking the subsets S_t of P_r for $t > 1$, dividing their members by 2, and re-labelling them with indices reduced by 1. We have clearly created the inverse of the mapping ${}^2\Theta$. Since there are no weak pairs in S_1, the new partition is again strong or weak according to the status of P_r. This proves the second part of the theorem, which now includes all possible colourings of a maximal partition. □

5 Size of Maximal Partitions

To examine the size of a maximal Gallai-Schur partition, we start with any such partition Q_r of $[1, m]$ into exactly r colours. To better manage the arithmetic, we define a simple function $g(Q_r) = |Q_r| + 1 = m + 1$.

Theorem 4.1 tells us that provided $r > 3$, Q_r is the image of another Gallai-Schur partition under either ${}^5\Theta$ or ${}^2\Theta$, and that we can determine which is the mapping concerned and apply the inverse mapping. If we apply the inverse of ${}^5\Theta$, we will get a new partition Q_{r-2} of order $(m-4)/5$ and we therefore have $g(Q_{r-2}) = g(Q_r)/5$. If we apply the inverse of ${}^2\Theta$, we will get a new partition Q_{r-1} of order $(m-1)/2$ and we therefore have $g(Q_{r-1}) = g(Q_r)/2$.

We can continue this process until the number of colours remaining falls below 4. We will then have a partition Q_{r-t} where $r - t = 2$ or 3.

At that point we may consider how many times we have applied ${}^2\Theta^{-1}$. We find it cannot be more than once. If it were more, we could take Q_{r-t} and apply a different sequence of ${}^i\Theta$ (replacing two instances of ${}^2\Theta$ by one instance of ${}^5\Theta$) to generate a Q'_r with $g(Q'_r) > g(Q_r)$, which is a contradiction of maximality.

Clearly, the partition Q_{r-t} must also be maximal, otherwise we could easily generate a larger Q'_r.

The maximal Gallai-Schur partitions for s colours, $s < 4$ are highly constrained and are easily proven to be as follows. We use two forms of notation which are equivalent, the first specifying subsets and the second listing colours of the integers in increasing order.

Strong case:

$B_1 = \{1\} \equiv$ "1". $|B_1| = 1$.

$B_2 = \{1, 4\}, \{2, 3\} \equiv$ "1221". $|B_2| = 4$.

$B_{3A} = B_1 * {}^5\Theta = \{1, 4, 6, 9\}, \{2, 3, 7, 8\}, \{5\} \equiv$ "122131221". $|B_{3A}| = 9$.

$B_{3B} = B_2 * {}^2\Theta = \{1, 3, 5, 7, 9\}, \{2, 8\}, \{4, 6\} \equiv$ "121313121". $|B_{3B}| = 9$.

Weak case:

$C_1 = \{1, 2\} \equiv$ "11". $|C_1| = 2$.

$C_2 = \{1, 2, 4, 8\}, \{3, 5, 6, 7\} \equiv$ "11212221". $|C_2| = 8$.

$C_3 = C_2 * {}^2\Theta = \{1, 3, 5, 7, 9, 11, 13, 15, 17\}, \{2, 4, 8, 16\}, \{6, 10, 12, 14\}$

$C_3 \equiv$ "12121312131313121". $|C_3| = 17$.

To be certain, these have been verified as being the only maximal cases, up to re-labelling, using a SAT-solver (Penelope).

Returning now to the main argument, we have several sub-cases to consider.

Strong case:

If $r - t = 2$ we must have $Q_{r-t} = B_2$. If r is even we therefore have $g(Q_r) = g(B_2) \cdot 5^{(r-2)/2} = 5 \cdot (5^{(r-2)/2})$. If r is odd, we must have $g(Q_r) = 2 \cdot g(B_2) \cdot 5^{(r-3)/2} = 10 \cdot (5^{(r-3)/2})$.

If $r - t = 3$ we must have $Q_{r-t} = B_{3A}$ or B_{3B}. If r is odd, we therefore have $g(Q_{r-t}) = g(B_{3A}) = g(B_{3B}) = 10$ and $g(Q_r) = 10 \cdot (5^{(r-3)/2})$ as above. If r is even, we can produce a larger partition Q'_r by setting $Q_{r-t} = B_{3B}$ and replacing two applications of ${}^2\Theta$ with one application of ${}^5\Theta$, showing that Q_r is not maximal. This contradiction shows we cannot have $r - t = 3$ in a partition when r is even and Q_r is maximal.

Thus in the strong case, for $r \geq 1$ we have:
$g(Q_r) = 5^{(r/2)}$ for even r, and
$g(Q_r) = 10 \cdot 5^{(r-3)/2}$ for odd r.

Weak case:

If $r - t = 2$, we must have $Q_{r-t} = C_2$. Since we know there cannot be more than one occurrence of ${}^2\Theta^{-1}$ in the sequence of inverse mappings applied, we must have $g(Q_r) = g(C_2) \cdot 5^{(r-2)/2} = 9 \cdot (5^{(r-2)/2)})$ if r is even. If r is odd, $g(Q_r) = 2 \cdot g(C_2) \cdot 5^{(r-3)/2} = 18 \cdot 5^{(r-3)/2)}$.

If $r - t = 3$, we must have $Q_{r-t} = C_3$. Since we know there cannot be more than one occurrence of ${}^2\Theta^{-1}$ in the sequence of inverse mappings applied, we must again have $g(Q_r) = 2 \cdot g(C_3) \cdot 5^{(r-2)/2} = 18 \cdot 5^{(r-3)/2})$ if r is odd. If r is even, we have one or more occurrences of ${}^2\Theta^{-1}$ in the sequence of inverse mappings, so the value of $g(Q_r)$ will be less than $g(C_2) \cdot 5^{(r-2)/2}$, hence Q_r is not maximal. This contradiction shows we cannot have $r - t = 3$ in the weak case when r is even and Q_r is maximal.

It should be clear that since each of the Q_r is maximal, $g(Q_r)$ corresponds to the (strong or weak) Gallai-Schur number. Thus we have proved the following:

Theorem 5.1. *(Strong and weak Gallai-Schur numbers)*
In the strong case, for $r \geq 1$ we have:
$GS(r) = 5^{(r/2)}$ *for even r, and*
$GS(r) = 10 \cdot (5^{(r-3)/2})$ *for odd r.*
In the weak case, we have the exceptional value $WGS(r) = 3$ for $r = 1$, and for $r > 1$,
$WGS(r) = 9 \cdot (5^{(r-2)/2})$ *for even r, and*
$WGS(r) = 18 \cdot (5^{(r-3)/2})$ *for odd r.*

Alternatively, $WGS(r) = (9/5) \cdot GS(r)$ for $r > 1$.

6 Further Observation — a Lower Gallai-Schur Number

In formalising the definition of Gallai-Schur partitions used above, it was apparent that the size of Gallai-Schur partitions is bounded below as well as above. Just as the maximum size of a strong or weak Gallai-Schur partition into r colours is one less then the corresponding Gallai-Schur number, we can identify a lower bound LGS(r) as the maximum number which is less than the order of any Gallai-Schur partition into exactly r colours. We may refer to this as the *lower Gallai-Schur number*.

The author is not aware that the existence of these numbers has been discussed previously. The concept of rainbow numbers is clearly related, although rainbow numbers take no account of monochromatic sums. Rainbow numbers have been addressed recently in [2] and [4].

The paper by Fallon *et al.* states precise values for the rainbow number rb(n) of a partition $[1, n]$, being the minimum number of colours required in order that every colouring of that partition must contain a rainbow sum. This is an anti-Ramsey concept, and to be of most use here we must re-express their result as follows. In view of the context, we will assume that the number of colours is always three or more.

In [4] (Theorem 2.4) the authors prove that for all $r \geq 3$, the interval $[1, 2^{r-1} - 1]$ cannot be coloured in exactly r colours without producing a rainbow sum. To illustrate, if $r = 3$, we cannot avoid a rainbow sum if we have to partition $[1, 3]$, (since "123" is mandated), but clearly "1213" avoids rainbow sums over $[1, 4]$.

They also demonstrate a colouring of the positive integers that induces a colouring of $[1, 2^{r-1}]$ in exactly r colours avoiding rainbow sums. Their proof in fact shows that the number of colours in every sum of the form $a + b = c$ is always two, even when $a = b$. Therefore this induced colouring is a Gallai-Schur colouring, and there can be none smaller. So $\mathrm{LGS}(r) = 2^{r-1} - 1$ for all $r \geq 3$. The same lower bound applies in both the strong and the weak case.

7 Conclusions

We have precisely characterised maximal Gallai-Schur partitions in both the strong and the weak case. This allows us to derive exact values for the corresponding Gallai-Schur numbers, replicating those derived by Budden [1] for the strong case. This is possible because the combination of constraints that require a triple to be free of both rainbow and monochromatic sums is very strong, so that maximal partitions take a restricted range of forms.

We see that the weak and strong maximal Gallai-Schur partitions are unique for even r, although not for odd r.

By addressing the lower Gallai-Schur number, we have also established the full range of m for which a strong or weak Gallai-Schur colouring can be applied to $[1, m]$, as well as describing minimal examples for any number of colours.

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