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Runner locating with multiple probes

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Abstract

In the runner locating variation of cops and robbers on a graph, a chaser attempts to locate an invisible runner by probing a single vertex v each turn, from which the chaser learns the runner's distance. The runner is then permitted to stay at his current vertex or move to an adjacent vertex other than v. A graph is locatable if the chaser is able to locate the runner in a finite number of turns, and the location number of a graph is the minimum number of turns necessary to determine the runner's location regardless of the runner's evasion strategy. In this paper, we allow the chaser to use multiple probes per turn; this is related to the metric dimension of a graph, which is equivalent to the number of chaser probes needed to locate the runner in one turn. We explore the number of turns required for the chaser to locate the runner when the number of probes ranges from the minimum needed to locate the runner, up to the graph's metric dimension.

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1 Introduction

Motivated by the metric dimension of a graph, Seager introduced a variation of cops and robbers in 2012 [16], which we will refer to as the runner locating game.[1] In this variation, a chaser pursues an invisible runner on the vertices of a connected graph by *probing* a vertex on each turn. From this probe, the chaser is given the distance from the probed vertex to the vertex the runner is currently occupying. On the runner's turn, he may remain at his current vertex or move to an adjacent vertex, as long as it is not the vertex that was just probed. Prohibiting the runner from moving to a vertex that was just probed by the chaser is referred to as the *no-backtrack condition*. The chaser and runner alternate turns; a *round* is a chaser probe and a subsequent runner move.

A graph G is called *locatable* if there is a chaser strategy that guarantees that the chaser will locate the runner in a finite number of turns. The minimum number of turns needed to locate the runner in a locatable graph is called the *location number* of the graph, denoted $\mathrm{loc}(G)$. Seager proved that paths, $K_{2,3}, C_4$, and cycles of length at least 6 are all locatable, and that C_5 and any graphs containing $K_{3,3}$ or K_4 as induced subgraphs are not locatable. She also showed that trees are locatable, and gave upper and lower bounds for the location number of a tree. Brandt et al. [6] gave an explicit strategy to locate the runner on a tree, and improved the upper bound on the location number for trees.

Carraher et al. [8] dropped the no-backtrack condition and studied the game on subdivisions of graphs. Since then, the game without the no-backtrack condition has become known as the *localization game*. Haslegrave et al. [11, 12] continued the study of the localization game on subdivisions of graphs, and also introduced the idea of allowing the chaser to probe multiple vertices each turn [13]. Simultaneously, Bosek et al. [4] began to study the multiple-probe version and introduced the parameter $\zeta(G)$, known as the *localization number* of G, which is the minimum number of chaser probes needed to guarantee that the runner can be located on a graph. Further work on the localization variation with multiple probes has been done in [1, 2, 3] and [5]. Particularly relevant to our discussion will be the results of Boshoff and Roux [5] on the localization number of Cartesian products of graphs, and the 2022 paper of Behague et al. [1], which introduced the *k-localization capture time* of a graph, which is the minimum number of rounds needed to locate a runner using k probes each turn.

In this paper, we focus on the variation wherein the chasers use multiple probes each turn and the no-backtrack condition is required. We will continue to call this the *runner locating game*, to distinguish the no-backtrack version from the localization game. Under these conditions, we say that a graph G is *k-locatable* if there is a strategy using k probes on each turn under which the chaser is guaranteed to

[1] The authors do not wish to trivialize interactions with police by referring to them as a game, and have chosen to avoid the terminology of the "game of cops and robbers" on graphs. We will instead discuss "chasers" and "runners"; this does not change the mathematical questions being explored, which are based on those introduced by Seager.

determine the location of the runner. Let $\zeta^-(G)$ denote the *runner-locating number*, the minimum integer k such that G is k-locatable. For a k-locatable graph G, the *k-location number*, denoted $\mathrm{loc}_k(G)$ is the minimum number of chaser turns necessary to determine the runner's location using k probes each turn.

The metric dimension of a graph G, introduced by Slater [17] and Harary and Melter [10] and denoted $\dim(G)$, is defined as the minimum k such that $\mathrm{loc}_k(G) = 1$. Since a backtracking runner can always follow the same strategy as a non-backtracking runner, we see that $\zeta^-(G) \leq \zeta(G) \leq \dim(G)$ for all graphs G. Equality holds for some, but not all, graph classes. For example, $\zeta^-(P_n) = \dim(P_n) = 1$ for $n \geq 2$, but we will show that for $n \geq 3$, $\zeta^-(K_n) = \left\lceil \frac{n-1}{2} \right\rceil$ while $\dim(K_n) = n - 1$. Similarly, it is possible for $\zeta^-(G)$ to be strictly smaller than $\zeta(G)$, and the k-location number and k-localization capture time can also differ significantly. For example, Boshoff and Roux [5] proved that $\zeta(C_3 \,\square\, C_3) = 3$, but we will show in Section 4 that a runner who is not allowed to backtrack can be caught with only 2 probes, meaning $\zeta^-(C_3 \,\square\, C_3) = 2$.

This paper has the following structure. In Section 1.1, we introduce further notation and terminology that will be used. Section 2 gives results on the runner-locating number and k-location number for complete graphs and complete bipartite graphs, and Section 3 does the same for trees. In Section 4 we explore these parameters for Cartesian products of paths, cycles, and complete bipartite graphs. Throughout, we will demonstrate how not allowing the runner to backtrack can reduce the number of probes or the number of turns needed for the chaser to locate him.

1.1 Notation

We will use the definitions and terminology in [19] unless otherwise noted.

When the chaser probes a vertex $v \in V(G)$, we let $P(v) = \mathrm{dist}(v, r)$, where r is the vertex the runner is currently occupying. If the chaser probes a set of k vertices, say $L = \{v_1, v_2, \ldots, v_k\}$, we let $P(L)$ be the list $(P(v_1), P(v_2), \ldots, P(v_k))$. Given a vertex r, we also define $\mathrm{dist}(r, L) = (\mathrm{dist}(r, v_1), \ldots, \mathrm{dist}(r, v_k))$. Note that when the chaser probes a set of vertices, she receives the distance from each probed vertex to the runner simultaneously.

If the runner is not immediately located after a set L has been probed, then there is a set $R(L)$ of vertices such that $\mathrm{dist}(r, L) = \mathrm{dist}(r', L) = P(L)$ for some pair $r, r' \in R(L)$. That is, the set $R(L)$ is the set of vertices identified by $P(L)$ that are not uniquely identifiable, and we say that the vertices in $R(L)$ are *indistinguishable*. Vertices uniquely identifiable by $P(L)$ are called *distinguishable*. Note that if $R(L) = \emptyset$, then the runner is immediately located, and if $R(L) \neq \emptyset$, then $|R(L)| \geq 2$. Finally, we let $R'(L) = N[R(L)] - L$ denote the set of vertices the runner could occupy in the next round; that is, all vertices in $R(L)$ and its neighborhood, excluding those that were just probed. When the context is clear we will simply use R and R' to denote these sets.

2 Complete and Complete Bipartite Graphs

In this section, we explore the runner-locating number and k-location number for complete graphs and complete bipartite graphs.

2.1 Complete graphs

The results in Seager [16] show that $\zeta^-(K_3) = 1$ and $\zeta^-(K_n) > 1$ if $n \geq 4$. Chartrand, Eroh, Johnson, and Oellermann [9] showed that $\dim(K_n) = n - 1$ for all $n \geq 2$, and it is straightforward to show that $\zeta(K_n) = \dim(K_n) = n - 1$. (Recall that $\zeta(G)$ is the minimum number of probes needed to locate a runner when backtracking is allowed.) However, when backtracking is not allowed, we see that far fewer chaser probes are necessary, and the runner can be located in two turns.

Theorem 2.1. *For $n \geq 4$, $\zeta^-(K_n) = \lceil \frac{n-1}{2} \rceil$ and $\mathrm{loc}_{\lceil \frac{n-1}{2} \rceil}(K_n) = 2$.*

Proof. Let $k = \lceil \frac{n-1}{2} \rceil$, and $V(K_n) = \{v_1, \ldots, v_n\}$. Without loss of generality, suppose the chaser probes vertices in the set $L = \{v_1, v_2, \ldots, v_k\}$. Note that for all $v_i \in \{v_{k+1}, \ldots, v_n\}$, $\mathrm{dist}(v_i, L) = (1, 1, \ldots, 1)$ so if the runner is not located, he is at a vertex in $R = \{v_{k+1}, \ldots, v_n\}$. Due to the no-backtrack condition, the runner cannot move to a vertex in L, so R', the set of vertices to which the runner may move on his turn, is equal to R. Since $k \leq |R'| \leq k + 1$, the chaser can probe k of the vertices in R' on her next turn. At worst, there is 1 vertex that has not been probed on either turn; if the runner is not at that vertex, he is at one of the vertices probed during the chaser's second turn. If he is at that vertex, then he is located immediately. Therefore, the runner is located in two turns.

Now, suppose $k < \lceil \frac{n-1}{2} \rceil$, and the chaser probes k vertices each turn. Let L_i be the set of k vertices probed in turn i. In this case, when the chaser probes L_i, she prohibits the runner from moving to one of the vertices in L_i on his next turn. Then, when she probes L_{i+1}, she can locate the runner if he is at any vertex in L_{i+1}. However, $|L_i| + |L_{i+1}| \leq 2k \leq n - 2$, which means there are always at least two vertices that are not blocked and not probed, and the runner could be at any of these vertices. For any vertex w not in L_i, $\mathrm{dist}(w, L_i) = (1, 1, \ldots, 1)$, so none of these vertices is distinguishable. Thus, the runner cannot be located using fewer than $\lceil \frac{n-1}{2} \rceil$ probes, and $\zeta^-(K_n) = \lceil \frac{n-1}{2} \rceil$. □

2.2 Complete Bipartite Graphs

Bosek et al. [4] proved that for $n \leq m$, $\zeta(K_{n,m}) = n$, and Behague et al. [1] showed that it takes at most $n + m - 1$ turns to locate the (backtracking) runner with n probes. Again, we will see that applying the no-backtrack condition allows the runner to be located with approximately half as many probes per turn. The strategy is to use probes to "block" half of the smaller partite set, while methodically probing a single vertex of the other partite set until the runner is located.

Theorem 2.2. *For $3 \leq n \leq m$, $\zeta^-(K_{n,m}) = \lfloor \frac{n+1}{2} \rfloor$ and*

$$\mathrm{loc}_{\lfloor \frac{n+1}{2} \rfloor}(K_{n,m}) \leq \begin{cases} m-1 & \text{if } n \text{ is odd} \\ 2m-3 & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ be the partite sets of $K_{n,m}$, and let $k = \lfloor \frac{n+1}{2} \rfloor$.

If n is odd, we define sets L_i, which the chaser will probe on turn i, as

$$L_i = \begin{cases} \{x_1, \ldots, x_{k-1}\} \cup \{y_i\} & \text{if } i \text{ is odd} \\ \{x_k, \ldots, x_{n-1}\} \cup \{y_i\} & \text{if } i \text{ is even.} \end{cases}$$

For $i > 1$, if $P(L_i) = (2, 2, \ldots, 2, 1)$, then the runner is located at x_n, because this distance list implies he is in $X - L_i$, but he cannot be at a vertex in L_{i-1} because of the no-backtrack condition. Thus, if the runner ever moves to X on his turn he will be located on the next turn, which means he must remain at a vertex in Y. Since vertices of Y are probed sequentially, if the runner is at y_j for $j < m$, he will be located in turn j. If he is at y_m, he will be located on turn $m-1$, because if he is not located at y_{m-1} at that point, he must be at y_m by process of elimination. Thus, $\zeta^-(K_{n,m}) \leq \frac{n+1}{2}$ and $\mathrm{loc}_{\frac{n+1}{2}}(K_{n,m}) \leq m-1$.

If n is even, we define the sets L_i as

$$L_i = \begin{cases} \{x_1, \ldots, x_{k-1}\} \cup \{y_{\frac{i+1}{2}}\} & \text{if } i \text{ is odd} \\ \{x_k, \ldots, x_{n-1}\} & \text{if } i \text{ is even.} \end{cases}$$

Again we see that vertex x_n will never be probed, yet if the runner ever moves to X he will be located on the chaser's next turn. However, in order to maintain this property, the chaser can only probe a vertex of Y on every other turn. Thus, since y_1 is probed in the first turn and the runner will be located at y_m when y_{m-1} is probed, it will take at most $2m-3$ turns to locate the runner.

If the chaser uses fewer than $\lfloor \frac{n+1}{2} \rfloor$ probes per turn, the runner can avoid location indefinitely. To see this, note that if at any point L_i and L_{i+1} are entirely contained in one partite set, then the runner can escape detection by moving to or remaining in the other partite set on turn $i+1$. Otherwise, $X - (L_i \cup L_{i+1})$ and $Y - (L_i \cup L_{i+1})$ both contain at least two vertices, so there are at least two vertices in each partite set that the runner can move to on turn $i+1$ that will be indistinguishable by probing L_{i+1}. Thus, $\zeta^-(K_{n,m}) = \lfloor \frac{n+1}{2} \rfloor$. □

A simple case analysis as shown in Figure 1 shows that with two probes on $K_{4,4}$, the runner can be located in at most 4 turns; thus, our bound on $\mathrm{loc}_{\lfloor \frac{n+1}{2} \rfloor}(K_{n,m})$ is not optimal.

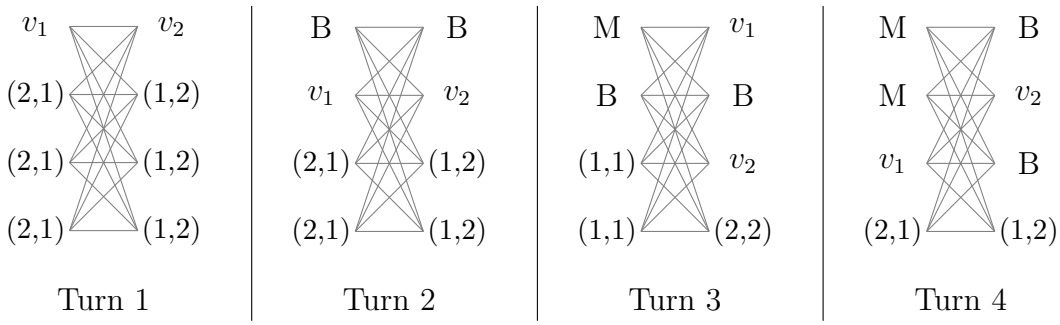


Figure 1: A four-turn strategy to locate a runner on $K_{4,4}$ by probing vertices v_1 and v_2. This strategy assumes that probes return (2,1) in turns 1 and 2; other cases are symmetric. Vertices that the runner could not be occupying due to the no-backtrack condition or because he could not have been occupying an adjacent vertex on the previous turn without being located are identified with B and M respectively.

3 Trees

Since Seager showed in [16] that all trees are locatable, we know that $\zeta^-(T) = 1$ for any tree T. The explicit strategy given in [6] gave an upper bound on the location number for trees that depends on the number of vertices, leaves, and radius of the tree. In this section, we give a strategy to locate a runner on a caterpillar using two or more probes, and see that the upper bound on the k-location number for $k \geq 2$ is significantly lower than the upper bound given for the location number in [6]. This gives an example of the significant advantage the chaser receives when using more than one probe.

First, we need the following lemma and its corollary, which generalize the strategy for locating a runner on a spider or star graph to the situation with multiple probes. Recall that a *spider* is a star in which each edge may be subdivided into a longer path. Each path from a leaf to the center vertex of maximum degree is called a *thread*. We will call the leaf endpoint of a thread the *terminal endpoint*.

Lemma 3.1. *Let Q be a spider with maximum degree $\Delta(Q) \geq 3$. Then* $\mathrm{loc}_k(Q) \leq \left\lceil \frac{\Delta(Q)-1}{k} \right\rceil$.

Proof. Since Q is a tree, we know that Q is k-locatable for all k. Let $\Delta = \Delta(Q)$, and let w be the vertex of maximum degree. Additionally, let $\{v_1, \ldots, v_\Delta\}$ be the set of vertices of degree one in Q. On the first turn, the chaser probes the leaves in the set $L = \{v_1, \ldots, v_k\}$. If $P(v_i) \leq \mathrm{dist}(w, v_i)$ for some $v_i \in L$, then the runner is on the thread whose terminal endpoint is v_i and is immediately located. (This includes the case that the runner is at w.) Otherwise, $P(v_i) > \mathrm{dist}(w, v_i)$ for all $v_i \in L$, so the runner is on a thread whose endpoint was not probed. Next, the chaser probes the leaves in the set $\{v_{k+1}, \ldots, v_{2k}\}$. As before, if the runner is on a thread whose endpoint is in this set, he is located. If not, the chaser probes the next set of k vertices. The runner will be located if he is ever at w or on a thread whose endpoint

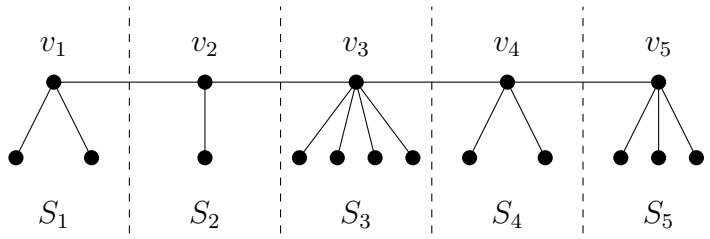


Figure 2: An example of a caterpillar; the *spine* is the set $\{v_1, v_2, \ldots, v_5\}$.

has just been probed, so he is restricted to the thread he started on. The chaser continues to sequentially probe new sets of k leaves until fewer than k leaves remain that have not yet been probed. If there is more than one such leaf, the chaser probes all of the remaining leaves and an arbitrary set of other points, and the runner must be located on one of the remaining threads. If the only leaf that has not yet been probed is v_Δ, then when the set $\{v_{\Delta-k-1}, \ldots, v_{\Delta-1}\}$ was probed, the chaser would have learned that $P(v_i) \geq \text{dist}(w, v_i)$ for $\Delta - k - 1 \leq i \leq \Delta - 1$, meaning the runner was not on any of those threads. Thus, the chaser can determine that the runner is on the thread ending at v_Δ, and using $P(v_{\Delta-1})$, she locates the runner. Thus, it takes at most $\lceil \frac{\Delta-1}{k} \rceil$ turns to locate the runner with k probes. □

We have the following immediate corollary giving the k-location number for stars.

Corollary 3.1. *For $n \geq 3$, $\text{loc}_k(K_{1,n-1}) = \lceil \frac{n-2}{k} \rceil$.*

Recall that a *caterpillar* is a tree in which all leaves are adjacent to a central path. We will refer to the non-leaf vertices on the caterpillar as the *spine*, and let the order of the spine be the number of vertices of degree greater than one. Note that a caterpillar with a spine of order m can be thought of as a series of m stars connected by a path through their center vertices. Suppose the spine of a caterpillar T_m is the path $v_1 v_2 \cdots v_m$ and $d_i := d_{T_m}(v_i)$. Let the star subgraph induced by vertex v_i and its leaf-neighbors be called S_i. For $2 \leq i \leq m-1$, we see that $S_i = K_{1,d_i-2}$, and the endpoints v_1 and v_m and their leaf-neighbors induce the stars K_{1,d_1-1} and K_{1,d_m-1}. (If $d_i = 2$ for some $i, 2 \leq i \leq m-1$, then S_i is simply the vertex v_i.)

Since the number of turns it takes to locate the runner on a star depends on how large the star is, the idea of the multiple-probe strategy to locate the runner on T_m is to use the first turn to identify which star subgraph S_i the runner is on and also effectively reduce the maximum degree of each star. Then the remaining turns use the strategy from Lemma 3.1 to locate the runner on the identified star.

Theorem 3.1. *Let T_m be a caterpillar with maximum degree Δ and a spine of order m, and let $k \geq 2$. For each i with $0 \leq i \leq \Delta - 2$, let M_i be the set of vertices of degree at least $\Delta - 2 - i$. If $|M_0| > k-2$, let $q = -1$; otherwise, let q be the maximum integer such that*

$$\sum_{i=0}^{q} |M_i| \leq k - 2.$$

Then,

$$\mathrm{loc}_k(T_m) \leq \left\lceil \frac{\Delta - q - 4}{k} \right\rceil + 1.$$

Proof. Let $v_1 v_2 \cdots v_m$ be the spine of T_m, and let x and y be leaves adjacent to v_1 and v_m, respectively. For $2 \leq i \leq m-1$, define the stars S_i as before. Let S_1 be the subgraph induced by v_1 and all its degree-one neighbors except for x, and define S_m similarly. Notice that each star subgraph defined in this way has at most $\Delta - 2$ leaves.

The set of vertices at a distance i from the leaf x consists of v_i and the leaves in S_{i-1}, and the vertices at distance j from the leaf y consists of v_{m-j+1} and the leaves in S_{m-j+2}. Thus, if the chaser probes x and y and the runner is at any spine vertex, the runner is immediately located. If the runner is not at a spine vertex, the chaser can identify which star subgraph he is in. Unfortunately, at this point no additional information can necessarily be gained by probing other vertices on the caterpillar, so we use the remaining probes (if $k > 2$) to effectively reduce the number of leaves in the star subgraphs. This may reduce the number of turns it takes to search any particular star in the next round. Now we will formally describe the process for doing that.

If $q = -1$, the first turn for the chaser will be to probe x, y, and one leaf on $k-2$ of the vertices in M_0. If the runner is identified to be on S_i for a vertex $v_i \in M_0$ that had a leaf probed, or for a vertex v_i that is not in M_0, the next turn for the chaser will be to use the strategy in Lemma 3.1 on that subgraph. The subgraph will have at most $(\Delta - 2) - 1$ leaves that have not been probed, which means that this strategy will locate the runner in at most $\left\lceil \frac{(\Delta-2)-2}{k} \right\rceil \leq \left\lceil \frac{\Delta-q-4}{k} \right\rceil$ additional turns. If the runner is identified on S_i for a vertex $v_i \in M_0$ that did not have a leaf probed, then applying the strategy of Lemma 3.1 will take at most $\left\lceil \frac{(\Delta-2)-1}{k} \right\rceil = \left\lceil \frac{\Delta-q-4}{k} \right\rceil$ additional turns. Thus, $\mathrm{loc}_k(T_m) \leq \left\lceil \frac{\Delta-q-4}{k} \right\rceil + 1$.

If $q \geq 0$, then on the first turn the chaser will probe x, y, and one leaf on each vertex in M_i for $0 \leq i \leq q$. Since $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_q$, a vertex in M_i will have $q - i + 1$ leaves probed. If $\sum_{i=0}^{q} |M_i| < k - 2$, the chaser will probe an additional leaf in M_0, then in M_1, and so forth until $k-2$ leaves on T_m have been probed. Thus, after the chaser's first turn, each S_i has at most $\Delta - 2 - (q+1)$ leaves that have not been probed. Since the star subgraph that the runner is on is identified with the first probe, the chaser will then implement the strategy from Lemma 3.1 to locate the runner on that star. This takes at most $\left\lceil \frac{(\Delta-2)-(q+1)-1}{k} \right\rceil = \left\lceil \frac{\Delta-q-4}{k} \right\rceil$ additional turns. Therefore,

$$\mathrm{loc}_k(T_m) \leq \left\lceil \frac{\Delta - q - 4}{k} \right\rceil + 1.$$

□

4 Cartesian products

Next, we study the runner locating game on the Cartesian products of graphs. The Cartesian product of G and H, written $G \,\square\, H$, is the graph with vertex set $V(G) \times V(H)$, where $(u, v) \sim (u', v')$ if and only if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$. The metric dimension of the Cartesian product of graphs has been studied in [7, 9, 14, 18], and Boshoff and Roux [5] studied the localization number of Cartesian products.

In the following sections, we give upper bounds on the runner-locating number and k-location number for the Cartesian products of even cycles, the product of a path and a cycle, and the product of a path and a complete bipartite graph.

In our proofs, we consider $G \,\square\, H$ embedded in the plane with vertices indexed matrix-style with indices starting at 0. Unless otherwise specified, vertex $v_{i,j} \in V(G \,\square\, H)$ is in the i^{th} row and j^{th} column where i increases from top to bottom and j increases from left to right. We use $V_{i,*}$ and $V_{*,j}$ to indicate the entirety of row i or column j, respectively, of $G \,\square\, H$.

4.1 Cartesian Product of Even Cycles

Cáceres et al. [7] determined the metric dimension of the Cartesian product of two cycles:

Theorem 4.1 (Cáceres et al., [7]). *For all $m, n \geq 3$,*

$$\dim(C_m \,\square\, C_n) = \begin{cases} 3 & \text{if } m \text{ or } n \text{ is odd} \\ 4 & \text{otherwise.} \end{cases}$$

Boshoff and Roux [5] later determined the localization number of these graphs as follows.

Theorem 4.2 (Boshoff and Roux, [5]). *For $m \geq n \geq 3$,*

$$\zeta(C_m \,\square\, C_n) = \begin{cases} 3 & \text{if } m = n = 3, \text{ or } m \text{ is even and } n = 4 \\ 2 & \text{otherwise.} \end{cases}$$

Additionally, they showed that in the localization game, the runner can be located with two probes in three turns on $C_{2n} \,\square\, C_{2m}$ if $m, n \geq 4$ and in four turns on $C_{2m} \,\square\, C_6$ if $m \geq 3$; using 3 probes on $C_{2m} \,\square\, C_4$, the runner can be located in 2 turns.

In contrast, the strategy in Figure 3 shows that $\zeta^-(C_3 \,\square\, C_3) = 2$ and $\mathrm{loc}_k(C_3 \,\square\, C_3) \leq 4$. In the figure, we envision $C_3 \,\square\, C_3$ embedded in the plane in the natural way. On each turn, the chaser probes $\{v_1, v_2\}$ and each other vertex of the graph is labeled with the distance from the probed set to that vertex, $(P(v_1), P(v_2))$. Vertices that the runner could not be occupying due to the no-backtrack condition or because he could not have been occupying an adjacent vertex on the previous turn without being located are identified with B and M respectively.

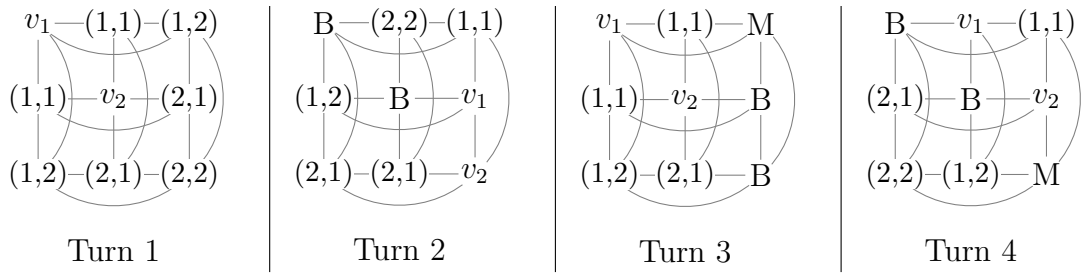


Figure 3: A four-turn strategy to locate a runner on $C_3 \square C_3$ with two probes.

Since $\zeta^-(G) \leq \zeta(G)$, we know that the number of probes needed to locate the runner is at most 2. However, the methods used by Boshoff and Roux to show $\zeta(C_{2m} \square C_{2n}) = 2$ for $n \geq 3$ required 3 or 4 turns; we will show that with one additional probe and the no-backtrack condition, a runner can be located on the Cartesian product of two even cycles of any length in at most 2 turns. This, along with the fact that $\zeta^-(C_3 \square C_3) = 2$, shows that the no-backtrack condition can in fact make a difference in both the number of probes needed and the speed with which the runner is located (even at the cost of additional probes).

Theorem 4.3. *If $n, m \geq 2$, $\mathrm{loc}_3(C_{2m} \square C_{2n}) = 2$.*

Proof. We will partition the vertices of $C_{2m} \square C_{2n}$ into four $m \times n$ sets which we refer to as quadrants:

$$Q_1 = \left\{ \bigcup_{i=0}^{m-1} V_{i,*} \cap \bigcup_{j=n}^{2n-1} V_{*,j} \right\}$$
$$Q_2 = \left\{ \bigcup_{i=0}^{m-1} V_{i,*} \cap \bigcup_{j=0}^{n-1} V_{*,j} \right\}$$
$$Q_3 = \left\{ \bigcup_{i=m}^{2m-1} V_{i,*} \cap \bigcup_{j=0}^{n-1} V_{*,j} \right\}$$
$$Q_4 = \left\{ \bigcup_{i=m}^{2m-1} V_{i,*} \cap \bigcup_{j=n}^{2n-1} V_{*,j} \right\}.$$

Note that these are named in correspondence with quadrants in the usual coordinate plane; i.e., Q_1 is the upper-right corner of the graph, Q_2 is the upper-left, etc. Our strategy leverages the ability to distinguish vertices in diagonal quadrants on a single turn.

For any two vertices $v_{i,j}$ and $v_{s,t}$ in $V(C_{2m} \square C_{2n})$, the distance between them is given by the taxi-cab metric on a torus, that is:

$$\mathrm{dist}(v_{i,j}, v_{s,t}) = \min\{|i-s|, 2m - |i-s|\} + \min\{|j-t|, 2n - |j-t|\}. \tag{1}$$

On the first turn, the chaser probes $L_1 = \{v_{0,0}, v_{0,n}, v_{2m-1,2n-1}\}$. Table 1 gives the distance from an arbitrary vertex in the given quadrant to each of the vertices in L_1.

Location of $v_{i,j}$	$\text{dist}(v_{i,j}, v_{0,0})$	$\text{dist}(v_{i,j}, v_{0,n})$	$\text{dist}(v_{i,j}, v_{2m-1,2n-1})$
Q_1	$i-j+2n$	$i+j-n$	$i-j+2n$
Q_2	$i+j$	$i-j+n$	$i+j+2$
Q_3	$2m-i+j$	$2m+n-i-j$	$2m-i+j$
Q_4	$2m+2n-i-j$	$2m-n-i+j$	$2m+2n-i-j-2$

Table 1: Simplified distances from a vertex $v_{i,j}$ to the vertices in L_1 in Theorem 4.3.

Let $R_1 = R(L_1)$. The next claim is the key observation in our proof.

Claim. *Assuming the runner is not located,* $R_1 \subseteq Q_1 \cup Q_3$. *More specifically,* $R_1 \subseteq (Q_1 - V_{0,*} - V_{*,n}) \cup (Q_3 - V_{m,*} - V_{*,0})$.

Proof. Let $v = v_{i,j}$ and $v' = v_{s,t}$ be vertices in R_1. Suppose $v, v' \in Q_1$; then, consulting Table 1, we see that $\text{dist}(v, v_{0,0}) = \text{dist}(v', v_{0,0})$ and $\text{dist}(v, v_{0,n}) = \text{dist}(v', v_{0,n})$. This means $(i-j+2n) = (s-t+2n)$ and $(i+j-n) = (s+t-n)$, yielding the system of equations

$$\begin{aligned} 0 &= i-j-s+t \\ 0 &= i+j-s-t \end{aligned}$$

This system is consistent only when $i = s$ and $j = t$, implying $v = v'$, a contradiction. Similar arguments for each quadrant imply that v and v', if indistinguishable, cannot be in the same quadrant.

Now we consider the possible options for v and v' to be in distinct quadrants. The table below shows the systems of equations created by considering the distances from the pair of vertices in the given quadrants to the probed vertices $v_{0,0}$ and $v_{2m-1,2n-1}$, for all possible pairs except for $v \in Q_1$ and $v' \in Q_3$.

v	v'	System of equations
Q_1	Q_2	$0 = (i-j+2n) - s - t$ $0 = (i-j+2n) - s - t - 2$
Q_1	Q_4	$0 = (i-j+2n) - 2m - 2n + s + t$ $0 = (i-j+2n) - 2m - 2n + s + t + 2$
Q_2	Q_3	$0 = (i+j) - 2m + s - t$ $0 = (i+j+2) - 2m + s - t$
Q_2	Q_4	$0 = (i+j) - 2m - 2n + s + t$ $0 = (i+j+2) - 2m - 2n + s + t + 2$
Q_3	Q_4	$0 = (2m-i+j) - 2m - 2n + s + t$ $0 = (2m-i+j) - 2m - 2n + s + t + 2$

Each of these systems is inconsistent, meaning that v and v' cannot be in that pair of quadrants.

This leaves us only to consider the case where $v \in Q_1$ and $v' \in Q_3$. We have the following system established by considering distances to $v_{0,0}$ and $v_{0,n}$:

$$\begin{aligned} 0 =& (i - j + 2n) - 2m + s - t \\ 0 =& (i + j - n) - 2m - n + s + t, \end{aligned}$$

which implies

$$i + s = 2m \quad \text{and} \quad j + t = 2n. \tag{2}$$

Note that this system is consistent. The possible indices for $v \in Q_1$ and $v' \in Q_3$, together with Equation (2), imply

$$1 \leq i < m < s \leq 2m - 1 \quad \text{and} \quad 1 \leq t < n < j \leq 2n - 1,$$

which gives the desired result. □

Since the runner can only move a distance of one each turn, we know

$$R_1' \subseteq Q_1 \cup Q_3 \cup V_{0,*} \cup V_{m,*} \cup V_{*,0} \cup V_{*,n}.$$

For the second turn, the chaser probes $L_2 = \{v_{m,0}, v_{m,n}, v_{m-1,2n-1}\}$, effectively shifting L_1 half way around the torus along C_{2m}. With this probe, we effectively swap the relative distances to Q_1 with Q_4 and Q_2 with Q_3. As such, the claim implies

$$R_2 \subseteq (Q_2 - V_{0,*} - V_{*,0}) \cup (Q_4 - V_{m,*} - V_{*,n}).$$

However, $R_1' \cap ((Q_2 - V_{0,*} - V_{*,0}) \cup (Q_4 - V_{m,*} - V_{*,n})) = \emptyset$, which implies $R_2 = \emptyset$ and the runner must be located after two turns. □

4.2 Cycles and Paths

We now turn our attention to Cartesian products of graphs from different families. First, we consider the product of a path and a cycle. Cáceres et al. [7] showed:

Theorem 4.4 (Cáceres et al., [7])**.** *For all $\ell \geq 2$ and $n \geq 3$,*

$$\dim(P_\ell \,\square\, C_n) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 3 & \text{if } n \text{ is even.} \end{cases}$$

If n is odd, determining $\zeta^-(P_\ell \,\square\, C_n)$ amounts to deciding if $P_\ell \,\square\, C_n$ is locatable. However, in the case that n is even and at least 10, we find that $\zeta^-(P_\ell \,\square\, C_n) < \dim(P_\ell \,\square\, C_n)$, and that the runner can be located in two turns.

Theorem 4.5. *If $n \geq 5$ and $\ell \geq 2$, then $\zeta^-(P_\ell \,\square\, C_{2n}) \leq 2$ and $\mathrm{loc}_2(P_\ell \,\square\, C_{2n}) = 2$.*

Proof. For any two vertices in $V(P_\ell \,\square\, C_{2n})$, say $v_{i,j}$ and $v_{s,t}$ with $j \le t$, the distance between them is given by

$$\operatorname{dist}(v_{i,j}, v_{s,t}) = |s-i| + \begin{cases} t-j & \text{if } 0 \le t-j \le n \\ j-t \pmod{2n} & \text{if } n < t-j \le 2n-1. \end{cases} \tag{3}$$

For the remainder of this proof, arithmetic on column indices will be computed modulo $2n$.

Our strategy for catching the runner consists of probing the first and last vertex in a single column on each turn; these vertices are the endpoints of a copy of P_ℓ. The first turn will either immediately locate the runner or identify the row and two possible columns where he may be hiding. This will allow us to locate the runner in the second turn by probing the first and last vertices of one of the columns he may have moved to.

To see this, suppose the chaser probes the set $L = \{v_{0,j}, v_{\ell-1,j}\}$ (the first and last vertices in column j). Let $d_0 = P(v_{0,j})$, $d_{\ell-1} = P(v_{\ell-1,j})$, and $s = d_0 + d_{\ell-1}$. By Equation (3), $s = \ell - 1 + 2k$, where k is the distance between column j and the column in which the runner is hiding. Note that $0 \le k \le n$. This identifies the runner to be in row $d_0 - k$ and column $j+k$ or $j-k$. If $k = 0$, then the runner is in column $V_{*,j}$ and is immediately located at vertex $v_{d_0,j}$. If $k = n$, then the runner is in the antipodal column, $V_{*,j+n}$, and is immediately located at $v_{d_0-n,j+n}$. If $0 < k < n$, then $R(L) = \{v_{d_0-k,j+k}, v_{d_0-k,j-k}\}$.

We now describe the specific strategy for the chaser. On the first turn, probe $L_1 = \{v_{0,0}, v_{\ell-1,0}\}$. The argument in the previous paragraph shows that if the runner is not located, then $R_1 = \{v_{d_0-k,k}, v_{d_0-k,-k}\}$, where $d_0 = P(v_{0,0}), d_{\ell-1} = P(v_{\ell-1,0})$, and $k = \frac{1}{2}(d_0 + d_{\ell-1} - \ell + 1)$, and

$$R_1' \subseteq \{V_{*,-k-1}, V_{*,-k}, V_{*,-k+1}, V_{*,k-1}, V_{*,k}, V_{*,k+1}\} \cap \{V_{d_0-k-1,*}, V_{d_0-k,*}, V_{d_0-k+1,*}\}.$$

For the second turn, the chaser again probes the endpoints of a column; the exact column depends on the value of k. If $k < \lfloor \frac{n}{2} \rfloor$ or $k = \lceil \frac{n}{2} \rceil$, they probe the endpoints of column $V_{*,-k-1}$, so $L_2 = \{v_{0,-k-1}, v_{\ell-1,-k-1}\}$. If $k > \lceil \frac{n}{2} \rceil$ or $k = \lfloor \frac{n}{2} \rfloor$, they probe the endpoints of column $V_{*,k-1}$, so $L_2 = \{v_{0,k-1}, v_{\ell-1,k-1}\}$. In each case, since $n \ge 5$, every column that intersects R_1' is a unique distance from the column that is probed, so the runner is located. (Note that if n is even, either column $V_{*,-k-1}$ or $V_{*,k-1}$ can be probed and yield the same result.) □

4.3 Paths and Complete Bipartite Graphs

We now turn our attention to $P_\ell \,\square\, K_{n,m}$. Saputro et al. [15] determined the metric dimension of a complete multipartite graph.

Theorem 4.6 (Saputro et al., [15])**.** *If $G = K_{n_1,n_2,\ldots,n_k}$, with $2 \le n_1 \le \cdots \le n_k$, then* $\dim(G) = |V(G)| - k$.

They also determined that the metric dimension of the Cartesian product of a complete multipartite graph and a path is almost always equal to the metric dimension of the complete multipartite graph itself. Specifically, they showed:

Theorem 4.7 (Saputro et al., [15]). *Let $G = K_{n_1,n_2,\ldots,n_k}$. If $\ell \geq 2$, then*

$$\dim(P_\ell \,\square\, G) = \begin{cases} \dim(G) + 1 & \text{if } G = K_{n,1}, K_{n,2}, \text{ or } K_{n,2,1} \text{ with } n = 1,2,3,4 \\ \dim(G) & \text{otherwise.} \end{cases}$$

Combining these results and restricting to the bipartite case, we see that

Theorem 4.8 (Saputro et al., [15]). *If $\ell \geq 2$,*

- $\dim(P_\ell \,\square\, K_{2,2}) = 3$;
- $\dim(P_\ell \,\square\, K_{2,3}) = 4$; *and*
- $\dim(P_\ell \,\square\, K_{n,m}) = n + m - 2$ *if* $3 \leq n \leq m$.

We will show that for ℓ and n at least 5, the k-location number is significantly less than the metric dimension; at most n chasers are needed to locate a runner. In the general case, this can be done in at most $m - 1$ turns, and for balanced or nearly-balanced complete bipartite graphs, only three turns are needed.

Theorem 4.9. *For $\ell \geq 2$ and $5 \leq n \leq m$, $\zeta^-(P_\ell \,\square\, K_{n,m}) \leq n$.*

Furthermore,

$$\mathrm{loc}_n(P_\ell \,\square\, K_{n,m}) \leq \begin{cases} m - 1 & \text{if } m \geq n + 2 \\ 3 & \text{if } m = n \text{ or } m = n + 1. \end{cases}$$

We will prove this theorem through three lemmas. We first establish some notation to help with the proofs. Let $V(K_{n,m}) = \{u_0, u_1, \ldots, u_{n-1}\} \cup \{v_0, v_1, \ldots, v_{m-1}\}$, and $V(P_\ell) = \{w_0, w_1, \ldots, w_{\ell-1}\}$. Then, we let $u_{i,j}$ and $v_{i,j}$ denote the vertices (w_i, u_j) and (w_i, v_j) in $P_\ell \,\square\, K_{n,m}$, respectively. Let $V(P_\ell \,\square\, K_{n,m}) = U \cup V$, where, for $0 \leq i \leq \ell - 1$, $U = \{u_{i,j}\}$ with $0 \leq j \leq n - 1$ and $V = \{v_{i,j}\}$ with $0 \leq j \leq m - 1$. Laying out the vertices in a grid as in Figure 4, we assume vertices are indexed from top to bottom and columns from left to right, with U to the left of V; thus, $u_{0,0}$ is the top left corner of the grid. We denote column j of U or V as $U_{*,j}$ or $V_{*,j}$, respectively, and similarly for rows.

Note that $u_{i,j} \sim u_{i',j'}$ if and only if $j = j'$ and $|i - i'| = 1$; $v_{i,j} \sim v_{i',j'}$ if and only if $j = j'$ and $|i - i'| = 1$; and $u_{i,j} \sim v_{i',j'}$ if and only if $i = i'$. Thus,

$$\mathrm{dist}(u_{i,j}, u_{s,t}) = \mathrm{dist}(v_{i,j}, v_{s,t}) = \begin{cases} |i - s| & \text{if } j = t \\ |i - s| + 2 & \text{if } j \neq t, \end{cases}$$

and

$$\mathrm{dist}(u_{i,j}, v_{s,t}) = |i - s| + 1.$$

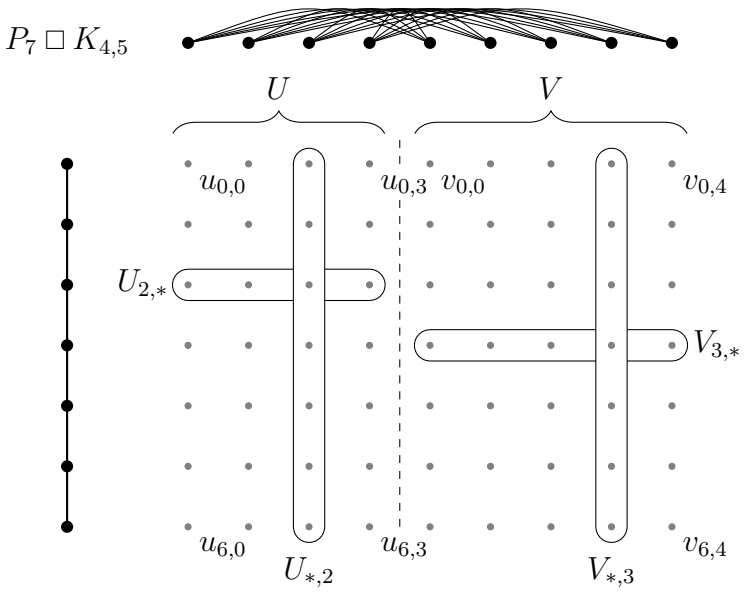


Figure 4: An example of the indexing and notation used in the proof of Theorem 4.9, on $P_7 \square K_{4,5}$.

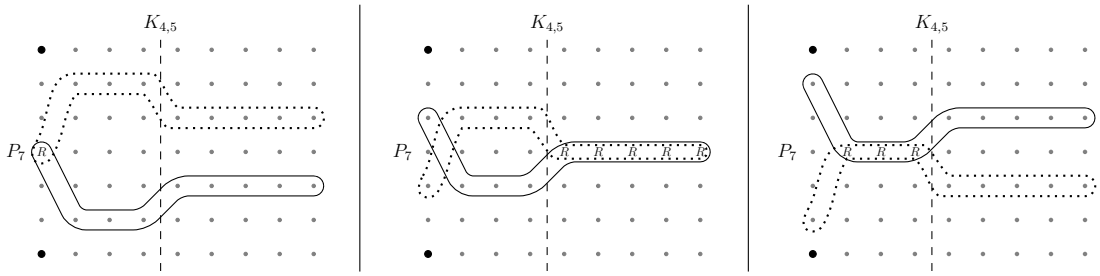


Figure 5: Visualization of R in $P_7 \square K_{4,5}$, when $P(u_{0,0}, u_{6,0})$ is $(3,3)$, $(4,4)$, and $(5,5)$, respectively. The solid and dotted curves enclose vertices of same distance from bottom and top probes, respectively.

Our overall strategy is to probe vertices in the first and last rows of U and one vertex in V, to successively eliminate columns of U where the runner could be hiding. Figure 5 shows the possible sets R that can result from probing the first and last rows of U. Combining this with probing a vertex in V reduces the number of columns that must be considered, as shown in the following lemma.

Lemma 4.1. *For $\ell, m, n \geq 3$, suppose the chaser probes the two vertices in the set $L = \{u_{i,j}, u_{i,j'}\}$. If the runner is in column j or j' of U, then $|R| \leq 2$ and $R \subseteq U_{*,j}$ or $R \subseteq U_{*,j'}$, respectively. Additionally, if $i = 0$ or $i = \ell - 1$, then the runner is located.*

Proof. Suppose the chaser probes $L = \{u_{i,j}, u_{i,j'}\}$. If the runner is at $u_{s,j}$, then $P(L) = (c, c+2)$ where $c = |i - s|$; if he is at $u_{s,j'}$, then $P(L) = (c+2, c)$ for the same value of c. Thus, if $P(L) = (c, c+2)$, then $R = \{u_{i-c,j}, u_{i+c,j}\}$, and if

$P(L) = (c+2, c)$, then $R = \{u_{i-c,j'}, u_{i+c,j'}\}$. This means that if $i = 0$ or $i = \ell - 1$, the runner is located. □

Note that the result also holds if the chaser probes two vertices in the same row and distinct columns of V.

Now we can show how to locate the runner on $P_\ell \,\square\, K_{n,m}$. First, we consider the case where $m \geq n + 2$.

Lemma 4.2. *For $\ell \geq 2$ and $5 \leq n \leq m-2$, $\zeta^-(P_\ell \square K_{n,m}) \leq n$ and $\mathrm{loc}_n(P_\ell \square K_{n,m}) \leq m - 1$.*

Proof. On turn $t = 1, 2, 3, \ldots, m$, the chaser probes

$$L_t = S \cup \{v_{0,t-1}\}$$

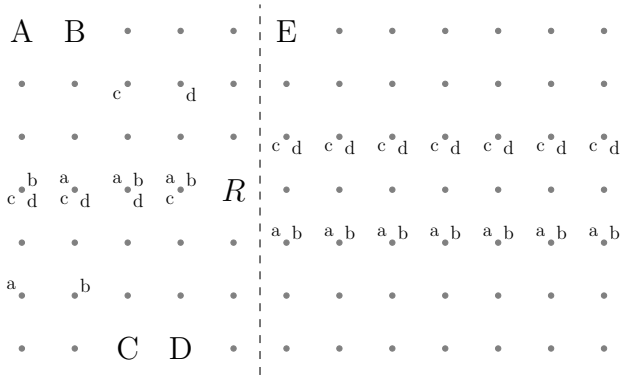
where

$$S = \left\{u_{0,i} \colon 0 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor \right\} \cup \left\{u_{\ell-1,i} \colon \left\lfloor \frac{n-2}{2} \right\rfloor + 1 \leq i \leq n-2 \right\}.$$

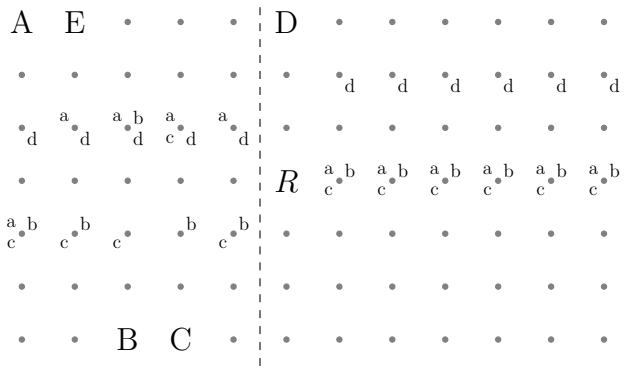
Notice that the $n-1$ vertices of S consist of the first vertex of columns 0 through $\lfloor \frac{n-2}{2} \rfloor$ in U, and the last vertex of columns $\lfloor \frac{n-2}{2} \rfloor + 1$ through $n-2$ in U. Since $n \geq 5$, this means at least two vertices of S are in each of the first and last rows of U. Thus, Lemma 4.1 implies that the runner is located if he is occupying a vertex in any but the last column of U.

Now, we consider the remaining possible locations for the runner in $V \cup U_{*,n-1}$. Figure 6 gives an example of the different cases that may result when probing L_1. Suppose that at turn t, the chaser is given $P(u_{0,0}) = p$ and $P(u_{\ell-1,n-2}) = q$. If $p+q = \ell+3$, then the runner is at $u_{p-2,n-1}$, as demonstrated in Figure 6a. Otherwise, we know that $R_t \subseteq V_{p-1,*} = V_{\ell-q,*}$. This shows that if the runner ever goes to U, he will be immediately located.

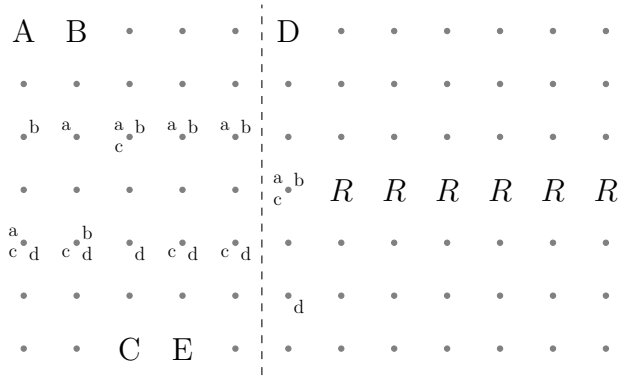
We now show that sequentially probing the first vertex of the columns in V will locate the runner. To that end, let $P(v_{0,t-1}) = d_t$. Then $R_t = (V_{d_t-2,*} - \{v_{d_t-2,t-1}\}) \cup \{v_{d_t,t-1}\}$. Recall that by probing the vertices of U in S, we can determine which row of V the runner is in. This, combined with our knowledge of R_t, means the runner is either located at $v_{d_t,t-1}$ (when $d_t = \ell - q = p - 1$, see Figure 6b) or known to be in $V_{d_t-2,*}$ (Figure 6c). Since the runner cannot move to U without being immediately located, he must remain in one column of V and can only move up or down. Hence, if the runner has not been located after turn t (when vertex $v_{0,t-1}$ was probed), he must be in row $V_{d_t-2,*}$ and column $V_{*,j}$ where $j \geq t$; that is $R_t \subseteq V_{d_t-2,*} \cap \left(\bigcup_{j=t}^{m-1} V_{*,j}\right)$. As the set of columns where the runner may be decreases with each turn, we see that when $t = m-1$, the runner is at $v_{d_{m-1},m-1}$ if $d_{m-1} = p-1$, or at $v_{d_t-2,m}$ otherwise. □



(a) Runner in U but not column of a probe.



(b) Runner in both V and column of a probe.



(c) Runner in V but not column of a probe.

Figure 6: Options for R when probing L_1 on $P_7 \,\square\, K_{5,7}$. The probed vertices are indicated by the upper-case letters A, B, C, D, and E. Vertices at distance $P(A)$ from A are indicated by a, and similarly for B, C, and D. The distance information from vertex E does not give additional information and is not indicated.

When the bipartite graph is roughly balanced, we can leverage the balanced structure to greatly improve the time needed to locate the runner while using the same number of probes. This is accomplished by using the chaser's first turn to narrow the runner's location down to half of a row in either U or V.

Lemma 4.3. *If $\ell \geq 2$, and $5 \leq n \leq m \leq n+1$, then $\zeta^-(P_\ell \,\square\, K_{n,m}) \leq n$ and $\mathrm{loc}_n(P_\ell \,\square\, K_{n,m}) \leq 3$.*

Proof. Let $N = \left\lfloor \frac{n}{2} \right\rfloor$. For the chaser's first turn, probe the top vertex in the first N columns of U and the first $n-N$ columns of V. That is, let

$$L_1 = \{u_{0,i} \colon 0 \leq i \leq N-1\} \cup \{v_{0,i} \colon 0 \leq i \leq n-N-1\}.$$

Since $n \geq 5$ and $N \geq 2$, Lemma 4.1 implies that the runner is located if he is occupying a vertex in any of the columns of U and V that were probed. If the runner is not in one of those columns, then the runner is the same distance, say p, from every vertex in $U \cap L_1$, and the same distance, q from every vertex in $V \cap L_1$. Note that the assumption that the runner is not located implies that $0 < p, q < \ell - 1$. The distance p from vertices in U implies that the runner is either in row $U_{p-2,*}$ or $V_{p-1,*}$, and the distance q from vertices in V implies that the runner is either in row $U_{q-1,*}$ or $V_{q-2,*}$. Putting these together, we see that either $q = p-1$, which means the runner is in $U_{p-2,*}$, or $q = p+1$, which means the runner is in $V_{p-1,*}$.

We will first consider the case where $q = p-1$. This implies that $2 \leq p \leq \ell - 2$, so

$$R_1 = \{u_{p-2,j} : N \leq j \leq n-1\},$$

and

$$R_1' = \{u_{i,j} : p-3 \leq i \leq p-1, N \leq j \leq n-1\} \cup V_{p-2,*}.$$

For the chaser's second turn, she probes the set

$$L_2 = \{u_{p-2,j} : N+1 \leq j \leq n-1\} \cup \{v_{p-3,0}, v_{p-1,1}\} \cup \{v_{p-2,j} : 2 \leq j \leq N\}.$$

We claim that $R_2 = \{v_{p-2,j} : N+1 \leq j \leq m-1\}$.

Suppose $w \in R_1' \cap U$. Then

$$\mathrm{dist}(v_{p-3,0}, w) = \begin{cases} 1 & \text{if } w \in U_{p-3,*} \\ 2 & \text{if } w \in U_{p-2,*} \\ 3 & \text{if } w \in U_{p-1,*} \end{cases}$$

and

$$\mathrm{dist}(v_{p-1,1}, w) = \begin{cases} 3 & \text{if } w \in U_{p-3,*} \\ 2 & \text{if } w \in U_{p-2,*} \\ 1 & \text{if } w \in U_{p-1,*}. \end{cases}$$

Thus, $P(v_{p-3,0}, v_{p-1,1})$ will determine which row of U the runner is in. If $\mathrm{dist}(u, w) = 2$ for each vertex $u \in L_2 \cap U$, then the runner is in column $U_{*,N}$, the only column in

$R_1' \cap U$ that was not probed. Obviously, if $\text{dist}(u, w) = 0$ for any $u \in L_2 \cap U$, the runner is located, which leaves the case that $\text{dist}(u, w) = 1$ for exactly one $u \in L_2 \cap U$ and $\text{dist}(u, w) = 3$ for all others. This identifies the column that the runner is in, which, together with the information gained from probing $\{v_{p-3,0}, v_{p-1,1}\}$, uniquely identifies the runner's location if he is in U.

Thus, $R_2 \subseteq V$. If the runner is in $V_{p-2,j}$ for $2 \leq j \leq N$, then he is distance 0 from a probed vertex, and if he is in $V_{p-2,j}$ for $0 \leq j \leq 1$, he is distance 1 from exactly one probed vertex. This verifies our claim that $R_2 = \{v_{p-2,j} : N + 1 \leq j \leq m - 1\}$. The no-backtrack condition then implies that

$$R_2' = \{u_{p-2,j} : 0 \leq j \leq N\} \cup \{v_{i,j} : p - 3 \leq i \leq p - 1, N + 1 \leq j \leq m - 1\}.$$

For the chaser's third turn, let $S = \{v_{p-2,j} : m - N \leq j \leq m - 2\}$ if n is even, and $S = \{v_{p-2,j} : m - N - 1 \leq j \leq m - 2\}$ if n is odd. Then let

$$L_3 = \{u_{p-2,j} : 1 \leq j \leq N\} \cup S \cup \{v_{p-1,m-1}\}.$$

This probes N columns of U and either N or $N+1$ columns of V. Since $m-N \geq N$ for even n, and $m-N-1 \geq N$ for odd n, there is a probe in every column of $R_2' \cap V$.

If the runner is in $R_2' \cap U$, he will be located since all but one vertex in that set is probed, and the remaining vertex, $u_{p-2,0}$, is distinguishable. If the runner is in V, the probes in S will locate him if he is in that set, or identify if he is in one of those columns. Then the probe at $v_{p-1,m-1}$ can determine whether the runner is in row $V_{p-1,*}$ (a distance of 2 from $v_{p-1,m-1}$) or row $V_{p-3,*}$ (a distance of 4). The remaining vertices in $R_2' \cap V$ are $v_{p-1,m-1}$, which is probed; $v_{p-2,m-1}$, which is the only vertex in $R_2' \cap V$ that is distance 2 from every vertex in S; and $v_{p-3,m-1}$, which is the only vertex in $R_2' \cap V$ that is distance 3 from every vertex in S. Thus, $R_3 = \emptyset$, and the runner is located in at most 3 turns. □

5 Future Work

Since many of our results are upper bounds on the runner-locating number or the k-location number, it would be interesting to explore whether or not these bounds could be improved, or to quantify the change in k-location number as k increases. Additionally, we have seen that Theorem 2.2 is not optimal for $K_{4,4}$, and that when $K_{n,m}$ is nearly balanced, Theorem 4.9 shows that $\text{loc}_n(P_\ell \,\square\, K_{n,m})$ is significantly smaller than when $m \geq n + 2$, using the same number of probes. Exploring the relationship between the number of probes used, the difference between the orders of the partite sets, and the number of turns it takes to locate the runner, may yield interesting results.

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