AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **93(1)** (2025), Pages 216–223

Intersecting families of 3-sets

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Abstract

An intersecting family of sets is a collection of sets so that each pair of sets has a nonempty intersection. We characterize all intersecting families of 3-sets using six infinite collections of families and eight finite families. Some of these families have interpretations as combinatorial designs, and can be represented using graph embeddings.

1 Introduction

An **intersecting family of sets** is a collection of sets so that each pair of sets has a nonempty intersection. A k**-set** is a set of size k. A **star** is a family of sets that all contain a common element.

Theorem 1. (**Erdős-Ko-Rado (EKR) Theorem** [2]) *If S is an intersecting family of k-subsets of $[n] = \{1, 2, \ldots, n\}$, then S contains at most $\binom{n-1}{k-1}$ sets. When $n > 2k$, equality holds exactly for stars.*

Katona [7] has a short proof of this theorem. Hilton and Milner [6] showed that the largest intersecting family of k-subsets of $[n]$ with no common element has at most $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ sets.

A **transversal** of a family S is a set of elements T such that every set in S has a nonempty intersection with T. The **transversal number** (or **covering number**) $\tau(S)$ is the minimum size of a transversal.

Every set in an intersecting family must be a transversal of the family. The EKR Theorem determines the maximum size of an intersecting family of k-sets with $\tau(S) = 1$. Hilton and Milner [6] solved the analogous problem for $\tau(S) = 2$, and Frankl [3] did likewise for $\tau(S) = 3$.

There are many other generalizations of the EKR Theorem such as requiring that every intersection contain at least t elements, or requiring that each set of $r > 2$ sets have a nonempty intersection. See [5] for a survey of many such generalizations. However, these theorems are generally focused on the maximum number of sets in a

ISSN: 2202-3518

family with some property. In this paper, we consider the question of describing all possible intersecting families of k-sets for some k (mainly $k = 3$).

Large intersecting families of 3-sets are described in [8], and some examples of small intersecting families of 3-sets are given in [4]. This paper is the first to contain a complete characterization of all intersecting families of 3-sets.

2 Intersecting Set Families

A characterization of intersecting families of 2-sets is immediate. We use 12 to mean the set $\{1, 2\}$, and similarly for other sets, following the usual notation for graphs.

Proposition 2. *In an intersecting family S of 2-sets, either all sets in S have a common element or S is isomorphic to $\{12, 13, 23\}$.*

An intersecting family is called maximal (saturated) if no further sets can be added without destroying the intersecting property. Now we characterize the intersecting families of 3-sets.

Theorem 3. *Let S be a maximal intersecting family of 3-sets. Let B be the set of all minimal transversals of S of size at most 2. Then S and B are isomorphic to one of the following.*

(We use 123 to mean the set $\{1, 2, 3\}$, and similarly for other sets. Note that $12x$ means that there is some set S' so that for any $x \in S'$, $\{1, 2, x\}$ is a set in the family.)

1. $B = \{1\}$, *all sets in S have a common element (S is a star).*
2. $B = \{12, 13, 23\}$, $S = \{12x, 13y, 23z\}$
3. $B = \{12, 13, 14\}$, $S = \{12x, 13y, 14z, 234\}$
4. $B = \{12, 13\}$, $S = \{12x, 13y, 234, 235, 145\}$
5. $B = \{12\}$, *S is one of the following.*

a. $S = \{12x, 134, 135, 145, 234, 235, 245\}$

b. $S = \{12x, 134, 235, 246, 156, 136, 145\}$

c. $S = \{12x, 134, 235, 246, 156, 136, 236\}$

6. $B = \emptyset$, *S is one of the following.*

a. $S = \{123, 124, 125, 134, 135, 145, 234, 235, 245, 345\}$ *(3-sets of [5])*

b. $S = \{124, 135, 236, 123, 146, 156, 256, 245, 345, 346\}$ *(K_6 on projective plane)*

c. $S = \{124, 135, 236, 123, 146, 156, 256, 245, 345, 125\}$

d. $S = \{124, 135, 236, 123, 146, 156, 256, 136, 345, 125\}$

e. $S = \{124, 135, 236, 123, 235, 156, 256, 136, 345, 125\}$

f. $S = \{124, 135, 236, 456, 134, 345, 125, 156, 245, 146\}$

g. $S = \{123, 124, 134, 234, 127, 347, 135, 245, 146, 236\}$ *(the 4-plus family)*

h. $S = \{124, 135, 236, 456, 167, 257, 347\}$ *(the Fano plane)*

Proof. If every set contains a common element (say 1), then $B = \{1\}$, and S is a star.

If B contains an element of size 2, then all elements of B have size 2. Then B is an intersecting family of 2-sets, so by Proposition 2, either B is a star or $B \cong \{12, 13, 23\}$. If the latter, then any other 3-set in S contains one element of B and one other number (Family 2).

If B is a star, $B = \{12, 13, \ldots, 1k\}$, then S contains a 3-set that does not contain 1 (else $B = \{1\}$). It intersects each element of B, so $|B| = k - 1 \leq 3$.

If $|B| = 3$, $B = \{12, 13, 14\}$, and $S = \{12x, 13y, 14z, 234\}$ (Family 3).

If $|B| = 2$, $B = \{12, 13\}$. There must be at least one set of the form $\{2, 3, a_i\}$, $a_i \in A$. To avoid $23 \in B$, there must be a set that contains 1 and all elements of A, so $|A| \leq 2$. Thus $S = \{12x, 13y, 234, 235, 145\}$ (Family 4).

If $|B| = 1$, $B = \{12\}$. Let R be the set of elements of sets not of the form $12x$. To avoid a transversal other than 12, $|R| \geq 5$. Thus there must be a 3-set that contains 1 but not 2, and vice versa. Call them 134 and 235. If $|R| = 5$, every 3-set of R is possible except 345, producing Family 5a.

If $|R| = 6$, then to avoid small transversals, 6 must be contained in two sets whose only intersection is 6. To intersect each existing set, these must be (up to symmetry) 156 and 246 or 136 and 246. If we have sets 156 and 246, there may be a set containing $\{3, 6\}$, say 136. Then 145 may exist, but not 245. If there is no set containing $\{4, 5\}$, then 236 may exist. No other sets are possible, so we have families 5b and 5c.

If we have sets 136 and 246, 5 must be contained in another set avoiding $\{2, 3\}$. It must be 145 or 156, and to avoid the previous case, assume it is 145. The only other set that can be added is 234. However, this family is equivalent to Family 5c, as can be seen by swapping 4 and 6.

Now we cannot have $|R| > 6$, as for example $\{1, 7, z\}$ cannot intersect 235 and 246 for any choice of z.

Suppose $B = \emptyset$. We claim that for every 3-set abc in S, there is another 3-set cde so that $\{a, b\} \cap \{d, e\} = \emptyset$. Otherwise, if every 3-set containing c also contained a or b, then $\{a, b\}$ would intersect all sets.

Let $n = |\cup_{t \in S} t|$. Thus $n \geq 5$. If $n = 5$, then all 3-sets intersect. Thus the family of 3-sets of $[5]$ is maximal. Thus we assume $n \geq 6$.

Let 124 and 135 be 3-sets in S. Now 6 must be contained in some set. If there is a set containing 1 and 6, there must another that does not contain 1, say 236.

Now assume $n = 6$ and $123 \in S$. There must be a set in S not intersecting $\{2, 3\}$. It must contain 1, 6, and either 4 or 5. Thus S must contain at least one of 146 or 156, and similarly at least one of 256 or 246 and at least one of 345 or 346. Of these 6 sets, we can omit 0, 1, 2, or 3 of them, instead including their complements. Omitting 2 or 3 can be done in two nonequivalent ways (see the table).

To determine whether these families are actually nonisomorphic, we count the number of appearances of each element (r^s means element r appears s times). Options 1 and 3A are equivalent under the permutation swapping 3 and 5. Applying the permutation (1)(32465) to 2A produces 2B, showing they are equivalent. The other four families are distinct since they have distinct element counts. We designate them families 6b-6e.

Case	Family of 3-sets S	Element counts
0	$\{124, 135, 236, 123, 146, 156, 256, 245, 345, 346\}$	$1^5 2^5 3^5 4^5 5^5 6^5$
1	$\{124, 135, 236, 123, 146, 156, 256, 245, 345, 125\}$	$1^6 2^6 3^4 4^4 5^6 6^4$
2A	$\{124, 135, 236, 123, 146, 156, 256, 136, 345, 125\}$	$1^7 2^5 3^5 4^3 5^5 6^5$
2B	$\{124, 135, 236, 123, 146, 156, 134, 245, 345, 125\}$	$1^7 2^5 3^5 4^5 5^5 6^3$
3A	$\{124, 135, 236, 123, 146, 234, 256, 136, 345, 125\}$	$1^6 2^6 3^6 4^4 5^4 6^4$
3B	$\{124, 135, 236, 123, 235, 156, 256, 136, 345, 125\}$	$1^6 2^6 3^6 4^2 5^6 6^4$

Now assume $n = 6$ and $123 \notin S$, so $456 \in S$. Call four 3-sets isomorphic to $\{123, 124, 135, 236\}$ a **4-triangle**. If S contains a 4-triangle, we can rename the elements and return to the previous case. Thus we assume S has no 4-triangle.

At this point, we think of the six elements as vertices of an octahedron, with 1 and 6 on the x-axis, 2 and 5 on the y-axis, and 3 and 4 on the z-axis. The 8 faces of the octahedron are 3-sets, 4 of which are in S. Thus we can have at most 6 of 12 non-face 3-sets in S. Let the reflection of a non-face 3-set be the 3-set in the same plane with 2 common vertices on an axis. (For example, the reflection of 134 is 346.) No 3-set and its reflection can both be in S, as then there would be a 4-triangle (e.g. $\{134, 346, 124, 135\}$ is a 4-triangle).

By symmetry, we arbitrarily choose 134 and 345 to add to S. Now 346 and 234 (reflections) and 256 and 126 (complements) are not in S. Thus we add 125 and 156 to S. Now 235 cannot be in S, since $\{135, 134, 156, 235\}$ is a 4-triangle. We add 245 to S, so 136 cannot be in S, and finally we add 146 to S. It is easily verified that S has no 4-triangles, and so is not isomorphic to any of the previous families.

Now assume $n = 7$. We start with sets 124, 135, 236 and add element 7. Now 7 must be contained in two sets that have no other common elements. To intersect the first three sets, we must have (up to symmetry) either 347 and 127, or 347 and 257.

Assume we add 347 and 127 to S, and 167 and 257 are not in S. Then 6 must be in a set not intersecting $\{2, 3\}$. To intersect all other sets, this must be 146. Similarly, 5 must be in a set not intersecting $\{1, 3\}$, which must be 245. Then 5 and 6 cannot be in any additional 3-sets. However, sets 123, 134, and 234 can be added. Call this the **4-plus family**.

Suppose we start with sets 124, 135, 236 and add 347 and 257. Now 6 must be in a set not intersecting $\{2, 3\}$. If it contains 7, it must be 167. If it does not contain 7, it must be 456.

If 456 is in S, then checking cases shows that no other set containing 1, 6, or 7 except 167 may be added. If 167 is added, we have the Fano plane, and no more sets can be added. If 167 is not added, then only 234 and 235 can be added, producing the 4-plus family.

If 167 is in S and 456 is not in S, then 457, 467, and 567 are not in S, so 123 must be in S. Checking cases shows that no other set containing 4 can be in S, and by symmetry, no other set containing 5 or 6 can be in S. The sets 127, 137, and 237 can all be in S, and when we add them, we find the 4-plus family.

Now assume $n \geq 8$. We start with sets 124, 135, 236 and add elements 7 and 8. Each of 7 and 8 must be contained in two sets that have no other common elements. Now at least one of 167, 257, or 347 (say 167) is in S. If 168 is in S, then so is 258, 348, or 238, none of which intersect 167. If 168 is not in S, then 258 or 348 is in S, neither of which intersect 167. We have a contradiction, so $n \leq 7$. □

3 Analysis and Interpretation

It would be interesting to know if there is a shorter proof of Theorem 3, particularly the $B = \emptyset$ case.

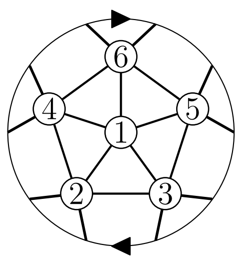
Corollary 4. *(folklore) If $B = \emptyset$, there are at most 10 sets in an intersecting family of 3-sets.*

This is described as 'folklore' in [3, 8]. It follows immediately from Theorem 3. It appears that the first published proof of this result appeared in [9] in 2024. It is about 1.5 pages long.

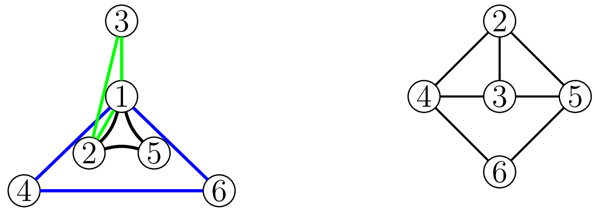
This folklore result is obvious for $n = 5$ and easy for $n = 6$, as there are $\binom{6}{3} = 20$ 3-sets that split into 10 complementary pairs. However, the result is more difficult for $n = 7$.

Note that [4] claims (without proof) to describe all intersecting families of 3-sets with $B = \emptyset$ with (maximum) size 10. Their families 1, 2, 3, 5, 6, and 7 correspond to our families 6a, 6f, 6d, 6e, 6c, and 6g. However, their family 4 is erroneous (146 and 235 have no intersection) and they omit our Family 6b.

Some of the intersecting families of 3-sets can be better understood visually. Note that K_6 embeds on the real projective plane as a triangulation [10]. This embedding has 10 triangular regions, which are the 10 sets in Family 6b. Each region has one or two points in common with each other region. Note that the dual graph of K_6 on the projective plane is the Petersen graph.



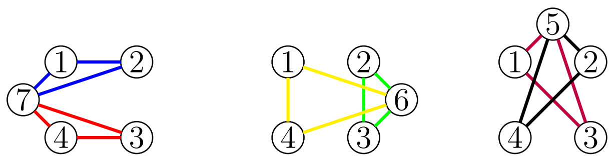
Family 6c has three distinct types of 3-sets. The three 3-sets shown below left can be rotated and reflected in all ways.



Family 6d has sets with element 1 and each pair shown as an edge above right. The other three sets are 245, 345, 236.

To understand Family 6f, consider the two distinct sets of elements $C = \{1, 4, 5\}$ and $D = \{2, 3, 6\}$. Aside from 236, every set in 6f has two elements from C and one from D, and this occurs in all possible ways.

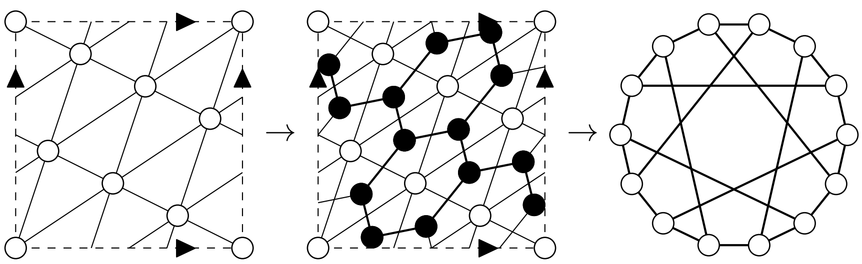
The 4-plus family is so named because it contains the 3-sets of [4], plus three more elements. Each element of $\{5, 6, 7\}$ is contained in two sets, whose intersection with [4] is two disjoint pairs of elements. All three possible pairs occur (12 and 34; 13 and 24; 14 and 23), see below.



There are relationships between some of the 3-set families we have presented. If we start with the 4-plus family and identify 6 and 7, we obtain Family 6e. If we identify 5, 6, and 7 in the 4-plus family (or 4 and 6 in Family 6e) we obtain the 3-sets of [5].

The Fano plane is a well-known combinatorial object. It is the finite projective plane with 7 points and 7 lines. It is a type of block design, in particular a Steiner triple system with 7 points and 7 triangles. These are all essentially the same idea expressed in different terminology.

The Fano plane can also be represented using a surface embedding. If we embed the complete graph K_7 on the torus, the embedding is a triangulation with 14 regions. The dual graph (the Heawood graph) of this embedding on the torus is bipartite (see below, where the Heawood graph is at right). Picking the triangles corresponding to one partite set gives the Fano plane (see [1]).



Next we analyze the structure of families 6a-6h, which all have nontrivial symmetries. In Table 1, these 8 families are listed along with their element counts. One way to understand the structure of a family is with a graph (see [1] for basic notation). We define a graph on the elements of the family of sets, with an edge whenever any pair is contained in any of the sets in the family. The resulting graphs are listed in Table 1.

We can also describe the automorphism group for each family. Most of these are easily determined, and the automorphism group of the Fano plane is well-known to be the projective special linear group $PSL_2(7)$. For Family 6b (K_6 on projective plane), the automorphism group must be a subgroup of S_6. Any region can map to any of 10 regions, and the vertices of a region can be permuted in any way. Thus the automorphism group has order $10 \cdot 6 = 60$, so it must be the alternating group A_5.

Case	Family of 3-sets S	Element counts	Graph	$Aut(S)$
6a	$S = \{123, 124, 125, 134, 135, 145, 234, 235, 245, 345\}$	$1^6 2^6 3^6 4^6 5^6$	K_5	S_5
6b	$S = \{124, 135, 236, 123, 146, 156, 256, 245, 345, 346\}$	$1^5 2^5 3^5 4^5 5^5 6^5$	K_6	A_5
6c	$S = \{124, 135, 236, 123, 146, 156, 256, 245, 345, 125\}$	$1^6 2^6 3^4 4^4 5^6 6^4$	K_6	S_3
6d	$S = \{124, 135, 236, 123, 146, 156, 256, 136, 345, 125\}$	$1^7 2^5 3^5 4^3 5^5 6^5$	K_6	$Z_2 \times Z_2$
6e	$S = \{124, 135, 236, 123, 235, 156, 256, 136, 345, 125\}$	$1^6 2^6 3^6 4^2 5^6 6^4$	$K_6 - e$	D_4
6f	$S = \{124, 135, 236, 456, 134, 345, 125, 156, 245, 146\}$	$1^6 2^4 3^4 4^6 5^6 6^4$	K_6	$S_3 \times S_3$
6g	$S = \{123, 124, 134, 234, 127, 347, 135, 245, 146, 236\}$	$1^6 2^6 3^6 4^6 5^2 6^2 7^2$	$K_7 - K_3$	S_4
6h	$S = \{124, 135, 236, 456, 167, 257, 347\}$	$1^3 2^3 3^3 4^3 5^3 6^3 7^3$	K_7	$PSL_2(7)$

Table 1: Families of 3-sets with no small transversal.

The **Kneser graph** $KG_{r,k}$ has vertices representing the k-sets of $[r]$ and edges between disjoint subsets. Any independent set in a Kneser graph must have all pairs of k-sets having nonempty intersections. Thus Theorem 3 can be interpreted as describing all maximal independent sets in $KG_{r,3}$.

An **r-uniform hypergraph** (r-graph) H, is a family of r-element subsets of a finite set. The r-sets are called **edges**. Theorem 3 can be interpreted as describing all 3-uniform hypergraphs so that any pair of edges has a nonempty intersection.

It might be possible to find all intersecting families of 4-sets, but this would require a computer search, as there are far more than for 3-sets. Frankl, Ota, and Tokushi [4] found an intersecting family of 4-sets with 42 sets and no smaller transversal. It is unknown if this is the maximum possible size of such a family.

Acknowledgements

Thanks to Peter Frankl and Ben McReynolds for answering questions and providing insight on this topic.

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(Received 25 Mar 2025; revised 4 Sep 2025)