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A note on regular factors in vertex-deleted subgraphs of regular bipartite graphs

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Abstract

Let G be a k-regular bipartite graph with bipartition (X, Y) and let ℓ be a positive integer such that $\ell \leq \frac{k}{2}$. If $\lambda(G) \geq 2\ell - 1$ then for every $u \in X$ and $v \in Y$, the graph $G - \{u, v\}$ contains an ℓ-factor.

1 Introduction and terminology

All graphs considered are assumed to be simple and finite. We refer the reader to [2] for standard graph theoretic terms not defined in this paper.

Let G be a graph. The degree $d_G(x)$ of a vertex x in G is the number of edges of G incident with x. If X and Y are subsets of $V(G)$ such that $X \cap Y = \emptyset$, the set and the number of the edges of G joining X to Y are denoted by $E_G(X, Y)$ and $e_G(X, Y)$ respectively. For any set X of vertices in G, the neighbour set of X in G is denoted by $N_G(X)$. If e is an edge of G having u and v as end-vertices, then the edge e is also denoted by uv. A graph G is k-regular if $d_G(x) = k$ for all $x \in V(G)$.

A bipartite graph is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition (X, Y) is called a bipartition of the graph. If $|X| = m$, $|Y| = n$, and each vertex of X is joined to each vertex of Y, such a graph is called a complete bipartite graph and is denoted by $K_{m,n}$.

A vertex cut of G is a subset S of $V(G)$ such that $G - S$ is disconnected. If G is a non-complete graph, we define the vertex connectivity $\kappa(G)$ of G to be the minimum number of elements of a vertex cut. If G is a complete graph, $\kappa(G)$ is defined as $|V(G)| - 1$. A graph G is said to be k-connected if $\kappa(G) \geq k$.

An edge cut of G is a subset of $E(G)$ of the form $E_G(S, \overline{S})$, where S is a nonempty proper subset of $V(G)$. If G is nontrivial and E' is an edge cut, then $G - E'$ is disconnected; we then define the edge connectivity $\lambda(G)$ to be the minimum number of elements of an edge cut. A graph G is said to be k-edge-connected if $\lambda(G) \geq k$.

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A matching or set of independent edges in a graph G is a set of edges without common end-vertices. Let k be a positive integer. A k-factor of a graph G is a spanning subgraph H of G, such that $d_H(x) = k$ for every $x \in V(G)$.

The following theorem, due to Petersen, is chronologically the first result on k-factors in regular graphs.

Petersen's Theorem [6]: *Every 3-regular, 2-edge-connected graph has a 1-factor.*

There are several results which generalize Petersen's theorem. One of them is the following.

Bäbler's Theorem [1]: *Every r-regular, $(r-1)$-edge-connected graph of even order has a 1-factor.*

The next theorem examines the existence of a 1-factor in vertex-deleted subgraphs of a regular graph.

Theorem 1 [3]: *Let G be a $2r$-regular, $2r$-edge-connected graph of odd order and u be any vertex of G. Then the graph $G - \{u\}$ has a 1-factor.*

The following theorem generalizes Theorem 1.

Theorem 2 [4]: *Let G be a $2r$-regular, $2r$-edge-connected graph of odd order and m be an integer such that $1 \leq m \leq r$. Then for every $u \in V(G)$, the graph $G - \{u\}$ has an m-factor.*

The main purpose of this paper is to focus on bipartite graphs and obtain for them, the following result which is similar to Theorem 2.

Theorem 3: *Let G be a k-regular bipartite graph with bipartition (X, Y) and let ℓ be a positive integer such that $\ell \leq \frac{k}{2}$. If $\lambda(G) \geq 2\ell - 1$ then for every $u \in X$ and $v \in Y$, the graph $G - \{u, v\}$ contains an ℓ-factor.*

We note at this point that the two conditions of Theorem 3 are in some sense best possible as we will show later.

For the proof of Theorem 3, we will use the following theorem.

Ore-Ryser ℓ-factor theorem [5]: *Let G be a bipartite graph with bipartition (X, Y) such that $|X| = |Y|$. Then G does not have an ℓ-factor if and only if there exists a subset T of Y such that*

$$\ell|T| > r_1 + 2r_2 + \cdots + (\ell - 1)r_{\ell-1} + \ell(r_\ell + \cdots + r_\Delta)$$

where $R_i = \{x \in X : |E_G(x, T)| = i\}$, $r_i = |R_i|$, and Δ is the maximum degree of G.

2 Proof of the main result

Proof of Theorem 3: Suppose that the theorem does not hold. Then there exist a positive integer ℓ, where $\ell \leq \frac{k}{2}$, and vertices $u \in X$, $v \in Y$ such that $G - \{u, v\}$ does not have an ℓ-factor. If we define $G_1 = G - \{u, v\}$ then by Ore-Ryser Theorem there exists $T \subseteq Y - \{v\}$ such that

$$\ell|T| > r_1 + 2r_2 + \cdots + \ell(r_\ell + \cdots + r_k) \tag{1}$$

where $R_i = \{x \in X - \{u\} : |E_{G_1}(x,T)| = i\}$, $|R_i| = r_i$ for $i = 1, 2, \ldots, k$.

Clearly $r_\ell + \cdots + r_k \leq |T| - 1$ since otherwise (1) does not hold. Define $r_\ell + \cdots + r_k = |T| - t$, where t is a positive integer. Then (1) yields

$$r_1 + 2r_2 + \cdots + (\ell - 1)r_{\ell-1} \leq \ell t - 1. \tag{2}$$

At this point, we consider the following two cases.

Case 1: $t = 1$

We have

$$\begin{aligned} k|T| &= e_G(T, R_1 \cup \cdots \cup R_k \cup \{u\}) \\ &= e_G(T, R_1 \cup \cdots \cup R_{\ell-1}) + e_G(T, R_\ell \cup \cdots \cup R_k \cup \{u\}) \\ &= e_G(T, R_1 \cup \cdots \cup R_{\ell-1}) + k(|T| - 1) - e_G(R_\ell \cup \cdots \cup R_k, Y - T) \\ &\quad + k - e_G(\{u\}, Y - T) \end{aligned} \tag{3}$$

since $r_\ell + \cdots + r_k = |T| - 1$ and $d_G(u) = k$. Furthermore, $E_G(T, R_1 \cup \cdots \cup R_{\ell-1}) \cup E_G(R_\ell \cup \cdots \cup R_k \cup \{u\}, Y - T)$ is an edge cut of G and so we have

$$e_G(T, R_1 \cup \cdots \cup R_{\ell-1}) + e_G(R_\ell \cup \cdots \cup R_k \cup \{u\}, Y - T) \geq \lambda(G) \geq 2\ell - 1.$$

Thus by using (2),

$$e_G(R_\ell \cup \cdots \cup R_k \cup \{u\}, Y - T) \geq \ell. \tag{4}$$

Hence by considering (2) and (4), (3) yields

$$k|T| \leq (\ell - 1) + k(|T| - 1) + k - \ell$$

which is a contradiction. Therefore this case cannot occur.

Case 2: $t \geq 2$

We have

$$\begin{aligned} e_{G_1}(T, N_{G_1}(T)) &= k|T| - e_G(\{u\}, T) \\ &= \sum_{i=1}^{\ell-1} i r_i + \sum_{i=\ell}^{k} i r_i \\ &\leq t\ell - 1 + k(|T| - t) \quad \text{by using (2).} \end{aligned}$$

Hence $k|T| - k \leq t\ell - 1 + k|T| - tk$, which yields

$$\ell \geq \frac{tk + 1 - k}{t} \geq k + \frac{1}{t} - \frac{k}{t} > \frac{k}{2} \quad \text{since } t \geq 2$$

contradicting the hypothesis of the theorem. Therefore this case also cannot occur.

3 Sharpness

We will show in this section that the conditions of Theorem 3 are in some sense best possible. We will first notice that the upper bound on ℓ is in some sense best possible. For this purpose we will describe a family of graphs G which constitutes counterexamples to an opposite claim. Let H_0 be a simple k-regular, k-connected bipartite graph with bipartition (X_0, Y_0) and let $w \in Y_0$. Define $H_1 = H_0 - \{w\}$ and let $\{u_{0,1}, u_{0,2}, \ldots, u_{0,k}\}$ be the set of vertices having degree $k-1$ in H_1. We also consider a copy of $K_{k-1,k}$ with bipartition (X_1, Y_1) where $Y_1 = \{u_{1,1}, u_{1,2}, \ldots, u_{1,k}\}$ denotes the set of vertices having degree $k-1$ in $K_{k-1,k}$. We form the family of graphs mentioned above, by adding to graphs H_1 and $K_{k-1,k}$ the independent edges $u_{0,1}u_{1,1}, u_{0,2}u_{1,2}, \ldots, u_{0,k}u_{1,k}$. Clearly G is a k-regular, k-edge-connected bipartite graph with bipartition $(X_0 \cup X_1, (Y_0 - \{w\}) \cup Y_1)$. Let $u \in X_1$, $v \in Y_0 - \{w\}$ and define $G_1 = G - \{u, v\}$. If we assume that $k \geq 3$ is odd and let $\ell = \frac{k+1}{2}$, then G is k-edge-connected with $k = 2\ell - 1$. Furthermore G_1 does not have an ℓ-factor for $\ell = \frac{k+1}{2}$ because if we let $T = Y_1$, then

$$\ell|T| > r_1 + 2r_2 + \cdots + \ell(r_\ell + \cdots + r_k)$$

since $|T| = k$, $r_1 = k$, $r_2 + \cdots + r_{k-1} = 0$ and $r_k = |X_1 - \{u\}| = k - 2$.

We next show that the edge-connectivity condition is best possible by describing a family of graphs G having slightly lower edge-connectivity and not having the properties implied by Theorem 3. Let $H(H^*)$ be a bipartite graph obtained from $K_{k,k}$ after the deletion of a matching containing $\ell - 1$ $(2\ell - 2)$ edges, where $2 \leq \ell \leq \frac{k}{2}$. We consider m copies H_i of H^* having bipartitions (X_i, Y_i) for $i = 1, 2, \ldots, m$ and two copies H_0 and H_{m+1} of H having bipartitions (X_0, Y_0), (X_{m+1}, Y_{m+1}) respectively. Define $X'_i = \{x \in X_i \mid d_{H_i}(x) = k - 1\}$, $Y'_i = \{x \in Y_i \mid d_{H_i}(x) = k - 1\}$ for $i = 0, 1, \ldots, m+1$ and let $X'_0 = \{u_{0,1}, u_{0,2}, \ldots, u_{0,\ell-1}\}$, $Y'_0 = \{v_{0,1}, v_{0,2}, \ldots, v_{0,\ell-1}\}$, $X'_{m+1} = \{u_{m+1,1}, u_{m+1,2}, \ldots, u_{m+1,\ell-1}\}$, $Y'_{m+1} = \{v_{m+1,1}, v_{m+1,2}, \ldots, v_{m+1,\ell-1}\}$ and $X'_i = \{u_{i,1}, u_{i,2}, \ldots, u_{i,2\ell-2}\}$, $Y'_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,2\ell-2}\}$ for $i = 1, 2, \ldots, m$. We form the family of graphs mentioned above by adding to graphs $H_0, H_1, \ldots, H_m, H_{m+1}$ the following four sets of independent edges:

$$E_1 = \bigcup_{i=1}^{m} \{u_{i,\ell}v_{i+1,1}, u_{i,\ell+1}v_{i+1,2}, \ldots, u_{i,2\ell-2}v_{i+1,\ell-1}\}$$

$$E_2 = \bigcup_{i=2}^{m+1} \{u_{i,1}v_{i-1,\ell}, u_{i,2}v_{i-1,\ell+1}, \ldots, u_{i,\ell-1}v_{i-1,2\ell-2}\}$$

$$E_3 = \{u_{0,1}v_{1,1}, u_{0,2}v_{1,2}, \ldots, u_{0,\ell-1}v_{1,\ell-1}\}$$

$$E_4 = \{u_{1,1}v_{0,1}, u_{1,2}v_{0,2}, \ldots, u_{1,\ell-1}v_{0,\ell-1}\}.$$

Clearly G is a k-regular, $(2\ell - 2)$-edge-connected graph with bipartition $(X_0 \cup X_1 \cup \cdots \cup X_{m+1}, Y_0 \cup Y_1 \cup \cdots \cup Y_{m+1})$. Let $u \in X_0$, $v \in Y_1 \cup Y_2 \cup Y_3 \cup \cdots \cup Y_{m+1}$ and define $G_1 = G - \{u, v\}$. If we let $T = Y_0$, then

$$\ell|T| > r_1 + 2r_2 + \cdots + \ell(r_\ell + \cdots + r_k)$$

since $|T| = k$, $r_1 = \ell - 1$, $r_2 + \cdots + r_{k-2} = 0$ and $r_{k-1} + r_k = k - 1$. Thus G_1 does not have an ℓ-factor.

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