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Spectral measures of balance for signed graphs

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Abstract

A connected signed graph Σ with underlying graph G is balanced if and only if either spectra (of the adjacency matrix) of G and Σ coincide or the largest eigenvalue of G coincides with the largest eigenvalue of Σ. These spectral criteria for balance are known and they are extended to the Laplacian spectrum, as well. In this paper, we generalize these criteria by proposing $\beta_A(\Sigma) = \sum_{i=1}^{n} \big(\lambda_i(\Sigma) - \lambda_i(G)\big)^2$ and $\gamma_A(\Sigma) = \lambda_1(G) - \lambda_1(\Sigma)$ as spectral measures of balance, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of the corresponding (signed) graph. Also, analogous measures based on Laplacian eigenvalues are defined. We discuss and compute them for several classes of signed graphs and prove a sequence of lower or upper bounds for each of them. In particular, relationships with the frustration index (a structural measure of balance) are established.

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1 Introduction

A signed graph has its edges declared either positive or negative. Formally, for a finite unoriented unsigned graph without loops or multiple edges G with the edge set $E(G)$ and a function $\sigma\colon E(G) \longrightarrow \{+1,-1\}$, a *signed graph* Σ is the ordered pair (G,σ). In this context, G is referred to as the *underlying graph* of Σ and σ is a *sign function* or a *signature*. The number of vertices of Σ is denoted by n and called the *order*.

Many notions about unsigned graphs extend directly to signed graphs. For example, a signed graph is connected or regular if the same holds for its underlying graph. The degree of a vertex in Σ is simply its degree in the corresponding underlying graph G. Similarly, the diameter of Σ is the diameter of G.

We proceed with some notions exclusive to signed graphs. The sign of a cycle in a signed graph is the product of its edge signs. If U is a set of vertices of Σ, the switched signed graph is obtained by reversing the signs of edges of Σ having exactly one end in U.

One of the fundamental concepts in the framework of signed graphs is the concept of balance. A signed graph or its subgraph is called *balanced* if every cycle in it, if any, is positive. Equivalently, it is balanced if and only if it switches to its underlying graph (considered as a signed graph with the all-positive signature) [16].

There are many invariants measuring how far a signed graph is from being balanced, and we only mention the *frustration index* $f(\Sigma)$, that is the minimum number of edges whose removal results in a balanced signed graph. Evidently, Σ is balanced if and only if $f(\Sigma)=0$.

We proceed to introduce the standard matrices associated with signed graphs. The *adjacency matrix* $A(\Sigma)=(a_{uv})$ is the $n\times n$ matrix such that $a_{uv}=\sigma(uv)$ if u and v are adjacent, and 0 otherwise. The *Laplacian matrix* $L(\Sigma)$ is $D(\Sigma)-A(\Sigma)$, where $D(\Sigma)$ is the diagonal matrix of vertex degrees. The *eigenvalues* of Σ are the eigenvalues of $A(\Sigma)$, and they are denoted by $\lambda_1,\lambda_2,\dots,\lambda_n$ along with an assumption that they are indexed non-increasingly. Similarly, the *Laplacian eigenvalues* of Σ are the eigenvalues of $L(\Sigma)$, denoted by $\mu_1,\mu_2,\dots,\mu_n$, along with the same assumption on their indexing. In some occasions, we will also use additional notation to distinguish the eigenvalues (or the Laplacian eigenvalues) of distinct signed graphs. The *spectrum* and the *Laplacian spectrum* of Σ are defined along the same lines.

We know from Acharya [2] that a signed graph is balanced if and only if its spectrum coincides with the spectrum of its underlying graph. A refinement has recently been obtained by the third author of this paper and states that a connected signed graph $\Sigma=(G,\sigma)$ is balanced if and only if $\lambda_1(\Sigma)=\lambda_1(G)$ [12]. Both criteria are significant since, in contrast to the frustration index and other structural criteria (see [15]), they are polynomially computed. Accordingly, we define

$$\beta_A(\Sigma)=\sum_{i=1}^{n}(\lambda_i(\Sigma)-\lambda_i(G))^2, \tag{1}$$

and, for Σ connected,

$$\gamma_A(\Sigma) = \lambda_1(G) - \lambda_1(\Sigma), \tag{2}$$

and propose these numerical quantities as spectral measures of balance. Clearly, $\beta_A(\Sigma) = 0$ (and $\gamma_A(\Sigma) = 0$) holds if and only if Σ is balanced. It is worth mentioning that $\beta_A(\Sigma)$ can be seen as a particular case of the spectral distance between n-vertex signed graphs defined in the same way, i.e., when G is replaced by any signed graph. A generalization considers the ℓ_p norm instead of ℓ_2 [14]. We know from [12] that $\gamma_A(\Sigma) \geq 0$ holds for every signed graphs Σ.

In the spirit of the previous measures, we introduce $\beta_L(\Sigma) = \sum_{i=1}^{n}(\mu_i(\Sigma) - \mu_i(G))^2$ and, for Σ connected, $\gamma_L(\Sigma) = \mu_n(\Sigma)$. For the latter measure, it is known that the least Laplacian eigenvalue of a connected signed graph is zero if and only if it is balanced [5, 15].

In this paper we discuss and compute the previously mentioned measures for signed cycles and signed complete graphs. We also establish some lower and some upper bounds on these invariants. In particular, they are related to the frustration index.

In the beginning of Section 2 we compute $\beta_A(\Sigma)$ and $\gamma_A(\Sigma)$, when Σ is any signed graph whose underlying graph is the Petersen graph. This part can be seen as an illustrative example. We also show that $\beta_A(\Sigma) = \beta_L(\Sigma)$ and $\gamma_A(\Sigma) = \gamma_L(\Sigma)$ hold for every regular signed graph Σ. We also discuss the spectral measures in the case of signed cycles and signed complete graphs. Section 3 is devoted to bounds expressed in terms of certain structural invariants including the frustration index.

2 The Petersen graph, a cycle or a complete graph in the role of an underlying graph

We warm-up with the Petersen graph. According to Zaslavsky [17] there are exactly six equivalence classes of signatures under the combination of switching and isomorphism of the Petersen graph. The corresponding representatives are illustrated in Fig. 1; they include the Petersen graph itself (i.e., the all-positive signature) as well as the negation of the Petersen graph (i.e., the all-negative signature). It is a matter of algebraic computation to obtain the results listed in Table 1.

Since a signed graph built on an r-regular graph G has $r - \lambda_i(\Sigma)$ as the Laplacian eigenvalues, the following theorem needs no further explanation.

Theorem 2.1. *If $\Sigma = (G, \sigma)$ is a signed graph such that the underlying graph G is regular, then $\beta_A(\Sigma) = \beta_L(\Sigma)$ and $\gamma_L(\Sigma) = \gamma_A(\Sigma)$.*

Hereafter, whenever we deal with a regular signed graph Σ, we suppress the results on $\beta_L(\Sigma)$ and $\gamma_L(\Sigma)$.

We proceed with n-vertex cycles C_n. Evidently, here we have only two switching classes: a positive signed cycle C_n and a negative signed cycle C_n^-. The spectrum of

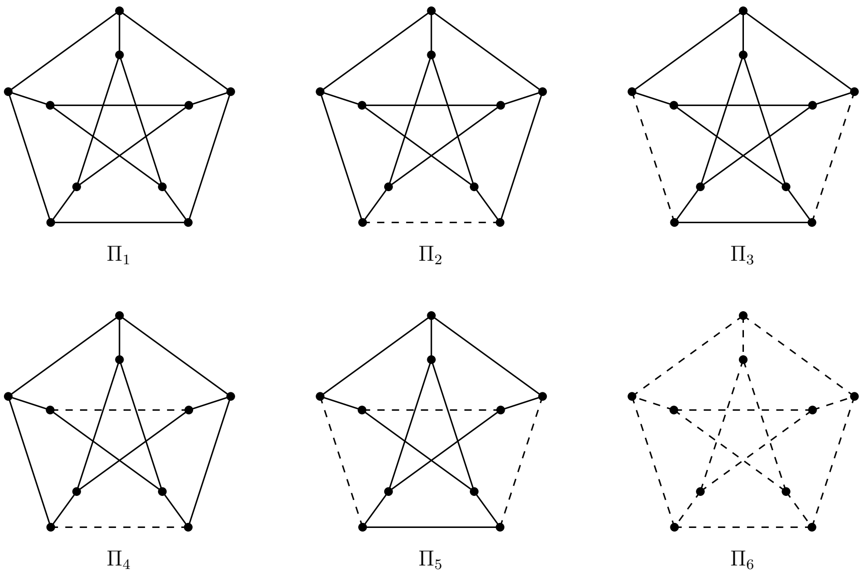

Figure 1: Representatives of six switching classes of the Petersen graph.

Table 1: Values of β_A and γ_A for signed graphs of Figure 1

Σ	$\beta_A(\Sigma)$	$\gamma_A(\Sigma)$
Π_1	0	0
Π_2	3.95044	0.221543
Π_3	5.59874712	0.438477
Π_4	9.372103	0.510711
Π_5	10.80650450	0.763932
Π_6	16	1

the latter one is $\lambda_j(C_n^-) = 2\cos[(2j+1)\pi/n],\ 0 \leq j \leq n-1$ [10], and for the positive cycle C_n, it is comprised of $\lambda_j(C_n) = 2\cos(2j\pi/n),\ 0 \leq j \leq n-1$ [4, Example 1.1.4].

Theorem 2.2. *We have*

$$\beta_A(C_n^-) = 8n\sin^2\Big(\frac{\pi}{2n}\Big)$$

and

$$\gamma_A(C_n^-) = 4\sin^2\Big(\frac{\pi}{2n}\Big).$$

Proof. Taking into account properties of the cosine function, we deduce that the eigenvalues λ_j (a common notation for both $\lambda_j(C_n^-)$ and $\lambda_j(C_n)$) are ordered as $\lambda_0 \geq \lambda_{n-1} \geq \lambda_1 \geq \lambda_{n-2} \geq \lambda_2 \geq \lambda_{n-3} \geq \cdots \geq \lambda_{k-1} \geq \lambda_{k+1} \geq \lambda_k$, for $n = 2k+1$. Similarly, for $n = 2k$ we have $\lambda_0 \geq \lambda_{n-1} \geq \lambda_1 \geq \lambda_{n-2} \geq \lambda_2 \geq \lambda_{n-3} \geq \cdots \geq \lambda_{k+1} \geq \lambda_{k-1} \geq \lambda_k$. Therefore,

$$\beta_A(C_n^-) = \sum_{j=0}^{n-1}\Big(2\cos\frac{(2j+1)\pi}{n} - 2\cos\frac{2j\pi}{n}\Big)^2$$

$$= 8\sin^2\big(\frac{\pi}{2n}\big)\Big(n - \sum_{j=0}^{n-1}\cos\frac{(4j+1)\pi}{n}\Big)$$
$$= 8n\sin^2\big(\frac{\pi}{2n}\big),$$

since $\sum_{j=0}^{n-1}\cos\frac{(4j+1)\pi}{n} = 0$. As the sine function increases in $[0, \frac{\pi}{2}]$, we have $\gamma_A(C_n^-) = 4\sin^2\big(\frac{\pi}{2n}\big)$. □

Here is a corollary concerning the limit case. The proof is elementary.

Corollary 2.3. $\lim_{n\to\infty}\beta_A(C_n^-) = 0 = \lim_{n\to\infty}\gamma_A(C_n^-)$.

We continue with complete signed graphs $K_n^\sigma = (K_n, \sigma)$. In the forthcoming Theorem 2.5 we offer an upper bound for $\beta_A(K_n^\sigma)$. The all-negative complete signed graph is denoted by $-K_n$. The following lemma is needed.

Lemma 2.4 ([3]). *The largest eigenvalue of a signed graph Σ of order n lies in $[1,\ n-1]$. It is equal to 1 when Σ switches to $-K_n$.*

Now, we formulate the result.

Theorem 2.5. *For a complete signed graph K_n^σ, $\beta_A(K_n^\sigma) \leq 2n(n-2) = \beta_A(-K_n)$.*

Proof. Note that the eigenvalues of K_n are $\lambda_1 = n-1$ and $\lambda_i = -1$, for $2 \leq i \leq n$. Let $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$ be the eigenvalues of $A(K_n^\sigma)$. Then

$$\begin{aligned}\beta_A(K_n^\sigma) &= \big(\nu_1 - (n-1)\big)^2 + \sum_{i=2}^{n}\big(\nu_i + 1\big)^2\\ &= \sum_{i=1}^{n}\nu_i^2 + 2\Big(\sum_{i=2}^{n}\nu_i - (n-1)\nu_1\Big) + (n-1)^2 + (n-1)\\ &= \sum_{i=1}^{n}\nu_i^2 - 2n\nu_1 + n(n-1).\end{aligned}$$

In the last step, we have used the fact that the trace is zero which makes $\sum_{i=2}^{n}\nu_i = -\nu_1$. Since $\sum_{i=1}^{n}\nu_i^2 = n(n-1)$ (as follows by considering the trace of $A^2(K_n^\sigma)$), by employing Lemma 2.4 we find

$$\beta_A(K_n^\sigma) \leq n(n-1) - 2n + n(n-1) = 2n(n-2) = \beta_A(-K_n),$$

as desired. □

We prove the following comparison of complete signed graphs.

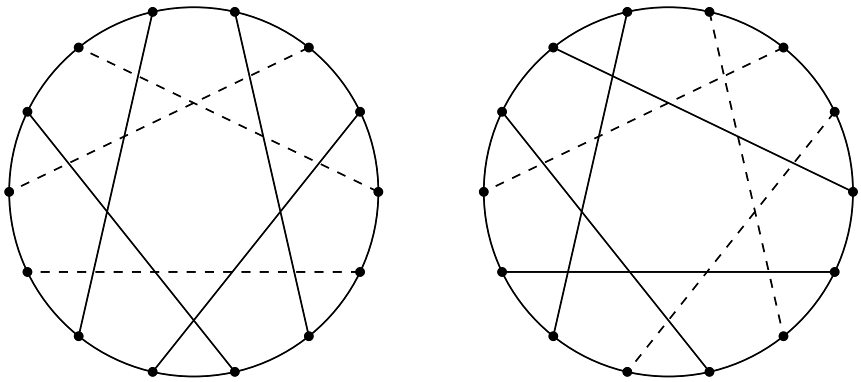

Figure 2: Two signatures on the Heawood graph for Example 2.7.

Theorem 2.6. *For complete signed graphs* $K_n^{\sigma_1}$ *and* $K_n^{\sigma_2}$, $\beta_A(K_n^{\sigma_1}) \leq \beta_A(K_n^{\sigma_2})$ *if and only if* $\gamma_A(K_n^{\sigma_1}) \leq \gamma_A(K_n^{\sigma_2})$.

Proof. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$ be the eigenvalues of $K_n^{\sigma_1}$ and $K_n^{\sigma_2}$, respectively. Now, $\beta_A(K_n^{\sigma_1}) \leq \beta_A(K_n^{\sigma_2})$ implies $(\lambda_1 - n+1)^2 + \sum_{i=2}^{n}(\lambda_i+1)^2 \leq (\nu_1 - n+1)^2 + \sum_{i=2}^{n}(\nu_i+1)^2$. From $\sum_{i=2}^{n}(\lambda_i+1)^2 = n(n-1) - \lambda_1^2 - 2\lambda_1 + n - 1$ and the analogous equality for $\sum_{i=2}^{n}(\nu_i+1)^2$, we deduce $-\lambda_1 \leq -\nu_1$. Hence, $0 \leq (n-1) - \lambda_1 \leq (n-1) - \nu_1$, proving that $\gamma_A(K_n^{\sigma_1}) \leq \gamma_A(K_n^{\sigma_2})$.

The converse is proved by retracing the steps. □

Example 2.7. An exhaustive search on randomly chosen small underlying graphs and the corresponding signed graphs has revealed many particular examples for which the statement of Theorem 2.6 remains valid. However, this statement does not hold in general, and a counterexample consists of the Heawood graph and two signatures illustrated in Fig. 2.

Indeed, since the signed graphs we are dealing with are bipartite, their spectrum is symmetric with respect to the origin, and the non-negative part of the spectrum of the underlying graph is $[3, \sqrt{2}^6]$, whereas the same parts for two illustrated signed graphs (see Fig. 2), say Σ_1 and Σ_2 respectively, are $[2.548^2, 1.763^2, 1, 0.629^2]$ and $[2.681, 2.323, 2, \sqrt{2}^2, 0.642, 0^2]$, respectively. Accordingly, $\beta_A(\Sigma_1) = 6.276 < \beta_A(\Sigma_2) = 7.734$, but $\gamma_A(\Sigma_1) = 0.452 > \gamma_A(\Sigma_2) = 0.319$.

3 Bounds for β_A, β_L and γ_A

This section presents a collection of upper and lower bounds for the invariants indicated in its title.

3.1 Bounds for β_A

We first extend the upper bound of Theorem 2.5 to any signed graph.

Theorem 3.1. *For a signed graph Σ of order n, $\beta_A(\Sigma) \in [0,\, 2n(n-2)]$.*

This result can be proved by examining the eigenvalues of a signed graph Σ and its underlying graph G. An elegant way to perform this is based on the Schatten 2-norm applied to the adjacency matrices A_Σ and A_G; the details are given in the proof of Theorem 3.3 below.

3.2 Bounds for β_L

It is interesting that Theorem 3.1 remains unchanged when $\beta_A(\Sigma)$ is replaced with $\beta_L(\Sigma)$. In what follows we will prove this and relate $\beta_L(\Sigma)$ to the frustration index of Σ.

We recall that, for a vector $(x_1, x_2, \dots, x_n)^\intercal \in \mathbb{R}^n$, its ℓ_p-norm ($1 \leq p < \infty$) is $\|(x_1, x_2, \dots, x_n)^\intercal\|_{\ell_p} = \big(\sum_{i=1}^{n} |x_i|^p\big)^{1/p}$. For an $n \times n$ matrix B with real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, the Schatten p-norm is $\|B\|_{S_p} = \|(\lambda_1, \lambda_2, \dots, \lambda_n)^\intercal\|_{\ell_p}$. In particular, for the adjacency matrix $A(\Sigma)$ of a signed graph and $p = 2$, $\|A(\Sigma)\|_{S_2}^2 = (\|(\lambda_1, \lambda_2, \dots, \lambda_n)^\intercal\|_{\ell_2})^2 = 2|E(\Sigma)|$.

For two symmetric $n \times n$ real matrices A and B, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n$, respectively, we have

$$\|(\lambda_1, \lambda_2, \dots, \lambda_n)^\intercal - (\lambda'_1, \lambda'_2, \dots, \lambda'_n)^\intercal\|_{\ell_p} \leq \|A - B\|_{S_p}\,. \tag{3}$$

This inequality is a consequence of the Wielandt-Hoffman inequality; for an explicit proof see [1, Theorem 2.1].

In what follows, we will use the following upper bound for the frustration index $f(\Sigma)$ of an arbitrary signed graph Σ expressed in terms of the number of edges m, cf. [6, 7]:

$$f(\Sigma) \leq \frac{m - \sqrt{m}}{2}. \tag{4}$$

The following lemma is a known result, but for the sake of completeness, we include a short proof.

Lemma 3.2. *Every signed graph Σ switches to a signed graph Σ' such that the number of negative edges of Σ' is the frustration index of Σ.*

Proof. As the removal of $f := f(\Sigma)$ edges results in a balanced signed graph, Σ switches to a signed graph in which at most f edges are negative. Moreover, this number is equal to f, since otherwise the frustration index of Σ would be less than f. □

We are ready to prove the following result.

Theorem 3.3. *For a signed graph Σ with n vertices and frustration index f,*

(i) $\beta_L(\Sigma) \leq 8f$,

(ii) $\beta_L(\Sigma) \in [0, 2n(n-2)]$.

Proof. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$ denote the Laplacian eigenvalues of the underlying graph G and the signed graph Σ, respectively. By Lemma 3.2, Σ switches to a signed graph Σ' having exactly f negative edges. In addition, Σ' shares the same Laplacian eigenvalues and the same underlying graph. In the light of (3), we compute

$$\begin{aligned}\beta_L(\Sigma) &= \sum_{i=1}^{n}(\nu_i(\Sigma) - \mu_i(G))^2 = (\|(\nu_1, \nu_2, \ldots, \nu_n)^\intercal - (\mu_1, \mu_2, \ldots, \mu_n)^\intercal\|_{\ell_2})^2 \\ &\leq (\|L(\Sigma') - L(G)\|_{S_2})^2 \leq (\|2A(H)\|_{S_2})^2,\end{aligned}$$

where $A(H)$ is the adjacency matrix of the graph H that shares the same vertices and has a positive edge exactly where Σ' has a negative edge. Thus, for $m(H) = |E(H)|$, we have

$$\beta_L(\Sigma) \leq 4(\|A(H)\|_{S_2})^2 = 4 \cdot 2m(H) = 8f,$$

which proves (i).

By taking into account the upper bound (4) and observing that the function $\frac{m-\sqrt{m}}{2}$ is increasing, we obtain

$$\begin{aligned}\beta_L(\Sigma) &\leq 8 \cdot \frac{m(\Sigma) - \sqrt{m(\Sigma)}}{2} \leq 4\Big(\frac{n(n-1)}{2} - \sqrt{\frac{n(n-1)}{2}}\Big) \\ &= 2n(n-1) - \sqrt{8n(n-1)}.\end{aligned}$$

For $n \geq 3$, the latter inequality gives $\beta_L(\Sigma) \leq 2n(n-1) - \sqrt{4n^2} = 2n(n-2)$. For $n = 2$ we have $\beta_L(\Sigma) = 0$, which completes the proof. □

3.3 Bounds for γ_A

Henceforth, $\Sigma = (G, \sigma)$ is a signed graph with n vertices and frustration index f, Σ' is a switching equivalent signed graph with exactly f negative edges (see Lemma 3.2) and H is the subgraph of Σ' obtained by removing all negative edges. We will use the following results:

$$\lambda_1(G) - \lambda_1(H) > \frac{1}{n\lambda_1(G)^{2D}}, \tag{5}$$

where G is connected of diameter D, see [8];

$$\lambda_1(H) \geq \lambda_1(\Sigma), \tag{6}$$

see Theorem 3.1 in [11];

$$\frac{y_{\min}}{y_{\max}} > \frac{1}{\lambda_1(H)^{n-1}}, \tag{7}$$

where H is connected and $y_{\min}$ and $y_{\max}$ are a minimum and a maximum entry of its principal eigenvector (a unit all-positive eigenvector associated with $\lambda_1(H)$), see [9];

$$x_{max}^2 \leq \frac{\Delta}{\Delta + \lambda_1(\Sigma)^2}, \tag{8}$$

where Σ is connected, Δ is the maximum vertex degree in Σ and $x_{\max}$ is a maximum absolute value of entries of the principal eigenvector of Σ (a unit eigenvector associated with $\lambda_1(\Sigma)$), compare [13] where a more general result is established.

Theorem 3.4. *For a connected signed graph Σ with maximum vertex degree Δ, average vertex degree $\overline{d}$ and frustration index f,*

$$\gamma_A(\Sigma) \leq \frac{4f\Delta}{\Delta + \overline{d}^2}.$$

Moreover, if Σ is regular, then

$$\gamma_A(\Sigma) \leq \frac{4f}{n},$$

where n is the order of Σ.

Each equality holds if and only if Σ is balanced.

Proof. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)^\intercal$ and $\mathbf{y}$ be unit eigenvectors corresponding to $\lambda_1(G)$ and $\lambda_1(\Sigma)$, respectively. Then

$$\begin{aligned}
\gamma_A(\Sigma) &= \mathbf{x}^\intercal A(G)\mathbf{x} - \mathbf{y}^\intercal A(\Sigma)\mathbf{y} \\
&\leq \mathbf{x}^\intercal A(G)\mathbf{x} - \mathbf{x}^\intercal A(\Sigma)\mathbf{x} \quad \text{(by the Rayleigh principle)} \\
&= \mathbf{x}^\intercal (A(G) - A(\Sigma))\mathbf{x} \leq 4 \sum_{ij \in E^-(\Sigma)} x_i x_j \qquad (9) \\
&\leq \frac{4f\Delta}{\Delta + \lambda_1^2(G)} \quad \text{(using (8) with } G \text{ in the role of } \Sigma\text{)} \\
&\leq \frac{4f(\Sigma)\Delta}{\Delta + \overline{d}^2},
\end{aligned}$$

where the last inequality follows from $\lambda_1(G) \geq \overline{d}$ which is a well-known result, see [4, Theorem 3.2.1].

If Σ is regular, so is G and then $\mathbf{x} = \frac{1}{\sqrt{n}}(1, 1, \dots, 1)^\intercal$, giving

$$\sum_{ij \in E^-(\Sigma)} x_i x_j = \frac{f}{n}.$$

The desired inequality follows by replacing this into (9).

It remains to consider equality cases. If Σ is balanced, then each side in each inequality is zero, which proves one implication. Conversely, if the first or the second equality give the statement formulation holds, then each inequality in the proof reduce to equality. In particular, this means that $\mathbf{x}$ features as the principal eigenvector of Σ. Using the eigenvalue equation, we deduce that this is possible if and only if Σ has no negative edges, i.e., it is balanced. □

At this point we need the following lemmas; the notation from the beginning of this subsection is used without noting.

Lemma 3.5. *If Σ is connected, so is the induced subgraph H.*

Proof. If H is disconnected, then there exists a negative edge of Σ' lying between two components of H. Let Ξ denote the signed graph obtained by adding this edge to H. By making a switch (in Ξ) with respect to all the vertices in one of the mentioned two components, we arrive at a switching equivalent signed graph having the all-positive signature. Since this signed graph is a subgraph of Σ and has an extra edge relative to H, we deduce that the frustration index of Σ is less than f. This contradiction concludes the proof. □

Lemma 3.6. *If $\mathbf{y}$ is a principal eigenvector of H, then $\mathbf{y}^\intercal A(G)\mathbf{y} \geq \mathbf{y}^\intercal A(H)\mathbf{y}$.*

Proof. By Lemma 3.5, H is connected, and so $\mathbf{y}$ is all-positive. Let F be the subgraph of G induced by the edges that are removed from G to obtain H; in other words, G is decomposed into F and H. Now,

$$\mathbf{y}^\intercal A(G)\mathbf{y} = \mathbf{y}^\intercal \big(A(H) + A(F)\big)\mathbf{y} = \mathbf{y}^\intercal A(H)\mathbf{y} + \mathbf{y}^\intercal A(F)\mathbf{y} \geq \mathbf{y}^\intercal A(H)\mathbf{y},$$

where the inequality follows since $\mathbf{y}$ is all-positive. □

We establish another lower bound.

Theorem 3.7. *Let Σ be a connected signed graph with n vertices and frustration index f. We have*

$$\gamma_A(\Sigma) \geq \frac{2f}{n\Delta_H^{2(n-1)}},$$

where Δ_H is the maximum vertex degree in the all-positive graph H obtained by removing f edges from Σ'.

The equality holds if and only if Σ is balanced.

Proof. Let $\mathbf{y} = (y_1, y_2, \ldots, y_n)^\intercal$ denote the principal eigenvector of H, and $y_{\max}$ a largest entry of $\mathbf{y}$. Observe that $y_{\max} \geq \frac{1}{\sqrt{n}}$. We have

$$\begin{aligned}\gamma_A(\Sigma) &= \lambda_1(G) - \lambda_1(\Sigma') \\ &\geq \lambda_1(G) - \lambda_1(H) \quad \text{(by employing (6))}\end{aligned}$$

$$
\begin{aligned}
&\geq \mathbf{y}^{\intercal}\big(A(G)-A(H)\big)\mathbf{y} \\
&\qquad\qquad \text{(by the Rayleigh principle for } \lambda_1(G)\text{, and by Lemma 3.6)} \\
&= 2\sum_{ij\in E^-(\Sigma)} y_i y_j \\
&\geq 2\sum_{ij\in E^-(\Sigma)} \Big(\frac{y_{max}}{\lambda_1(H)^{n-1}}\Big)^2 \quad \text{(using (7))} \\
&= 2f\Big(\frac{y_{max}}{\lambda_1(H)^{n-1}}\Big)^2 \\
&\geq \frac{2f}{(\sqrt{n}\lambda_1(H)^{n-1})^2} \quad \Big(\text{since } y_{\max}\geq \frac{1}{\sqrt{n}}\Big) \\
&\geq \frac{2f}{n\Delta_H^{2(n-1)}},
\end{aligned}
$$

where for the last inequality we have used $\lambda_1(H) \leq \Delta_H$, see [4, Theorem 3.2.1].

We consider the equality case. If Σ is balanced, then the inequality of this statement reduces to $0 = 0$, i.e., we have the equality.

Conversely, the equality in it implies that G and H share the same principal eigenvector. However, this is possible only if $G \cong H$ by the following argument. Supposing that $G \ncong H$ we deduce the existence of an edge, say $uv \in E(G)$, such that $uv \notin E(H)$ (as H is a subgraph of G). For H we have $y_u = \sum_{uw\in E(H)} y_w$, whereas for G we have

$$y_u = \sum_{uw\in E(G)} y_w = \sum_{uw\in E(H)} y_w + \sum_{uw\in E(G)\setminus E(H)} y_w = y_u + \sum_{uw\in E(G)\setminus E(H)} y_w > y_u,$$

as $\mathbf{y}$ is all-positive and $uv \in E(G) \setminus E(H)$. Now, $G \cong H$ means $f = 0$, i.e., Σ is balanced. □

The last result is similar yet based on (5) instead of (7).

Theorem 3.8. *Let Σ be a connected signed graph with n vertices, maximum vertex degree Δ, diameter D and frustration index f. We have*

$$\gamma_A(\Sigma) > \frac{1}{n\Delta^{2D}}.$$

Proof. We compute

$$
\begin{aligned}
\gamma_A(\Sigma) &= \lambda_1(G) - \lambda_1(\Sigma) \\
&\geq \lambda_1(G) - \lambda_1(H) \quad \text{(using (6))} \\
&> \frac{1}{n\lambda_1(G)^{2D}} \quad \text{(using (5))} \\
&\geq \frac{1}{n\Delta^{2D}},
\end{aligned}
$$

as $\lambda_1(G) \leq \Delta$. □

We conclude this section with some comments. Evidently, lower bounds of Theorems 3.7 and 3.8 are incomparable. The former bound would give a finer estimate whenever $2f\Delta^{2D} \geq \Delta_H^{2(n-1)}$. Any of these bounds in conjunction with the upper bound of Theorem 3.4 gives a range for $\gamma_A(\Sigma)$.

Each bound may be tested on signed graphs of Figs. 1 and 2. For example, since we deal with regular signed graphs, for the Petersen graph the upper bound reads $\gamma_A(\Sigma) \leq \frac{4f}{n} = \frac{2f}{5}$. Say, for $\Sigma \cong \Pi_2$ we have $f = 1$, which implies $0.221543 \approx \gamma_A(\Pi_2) \leq 0.4$.

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