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2-Domination edge subdivision in trees

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Abstract

A set S of vertices in a graph G is a 2-dominating set of G if every vertex not in S has at least two neighbors in S, where two vertices are neighbors if they are adjacent. The 2-domination number of G, denoted by $\gamma_2(G)$, is the minimum cardinality among all 2-dominating sets in G. A γ_2-set of G is a 2-dominating set of G of cardinality $\gamma_2(G)$. The 2-domination subdivision number of G, denoted by $\mathrm{sd}_2(G)$, is the minimum number of

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edges which must be subdivided in order to increase the 2-domination number. If T is a tree of order $n \geq 3$, then $\mathrm{sd}_2(T) \leq 2$. We show that $\mathrm{sd}_2(T) = 1$ if and only if the set of vertices that belong to no γ_2-set of G is nonempty. A graph G is γ_2-q-critical if q is the least number such that for every subset of edges S of cardinality q, the graph produced by subdivision of S has a greater 2-domination number. If T is a γ_2-q-critical tree of order $n \geq 3$, then we prove that $q \leq n-2$. Among other results, we characterize γ_2-q-critical trees when q is large, namely $q \in \{n-4, n-3, n-2\}$. We also characterize γ_2-1-critical trees.

1 Introduction

In this paper, we continue the study of 2-domination critical trees upon edge subdivision. A *dominating set* of a graph G is a set S of vertices of G such that every vertex not in S has a neighbor in S, where two vertices are neighbors if they are adjacent. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A thorough treatise on domination in graphs can be found in the so-called "domination books" [8, 9, 10, 11].

A 2-*dominating set* of a graph G is a set S of vertices of G such that every vertex not in S has at least two neighbors in S. The 2-*domination number* $\gamma_2(G)$ of G is the minimum cardinality of a 2-dominating set of G. A γ_2-*set of* G is a 2-dominating set of G of cardinality $\gamma_2(G)$. We denote by $\mathcal{A}_2(G)$, $\mathcal{S}_2(G)$, and $\mathcal{N}_2(G)$ the set of vertices in G that belong to every, some, or no γ_2-set of G, respectively. Hence a vertex in $\mathcal{A}_2(G)$ belongs to every γ_2-set of G, a vertex in $\mathcal{S}_2(G)$ belongs to some γ_2-set of G, and a vertex in $\mathcal{N}_2(G)$ belongs to no γ_2-set of G. The concept of 2-domination in graphs, and more generally of k-domination in graphs, is very well studied (see, for example, [3, 5]). An excellent survey on 2-domination in graphs can be found in the book chapter by Hansberg and Volkmann [7].

The *subdivision* of an edge $e = uv$ in a graph G consists of deleting the edge e from $E(G)$, adding a new vertex w to $V(G)$, and adding the new edges uw and vw to $E(G)$. In this case, we say that the edge e has been *subdivided*. If S is an arbitrary set of edges of G, we denote the graph obtained from G by subdividing every edge in S by G_S. In particular, if $S = \{e\}$, we simply denote G_S by G_e. Thus, G_e is the graph obtained from G by subdividing the edge e.

The 2-*domination subdivision number* of a graph G, denoted by $\mathrm{sd}_2(G)$, is the minimum number of edges which must be subdivided (where each edge can be subdivided at most once) in order to increase the 2-domination number. Since the 2-domination number of the graph K_2 does not increase when its edge is subdivided, it is only of interest to consider the 2-domination subdivision numbers for connected graphs of order at least 3. The 2-domination subdivision number was defined by Atapour, Sheikholeslami, Hansberg, Volkmann, and Khodkar [1].

A graph G is γ_2-q-critical if q is the least number such that for every subset of edges S of cardinality q, the graph produced by subdivision of S has a greater 2-

domination number. In particular, a graph G is γ_2-critical if subdividing any edge increases $\gamma_2(G)$. Among other results, we characterize γ_2-1-critical trees.

1.1 Notation and terminology

For graph theory notation and terminology, we generally follow [10]. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. Two vertices u and v of G are *adjacent* if $uv \in E(G)$, and are called *neighbors*. The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of neighbors of v, while the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N_G(v)$. In general, for a subset $S \subseteq V(G)$, its *open neighborhood* is the set $N_G(S) = \cup_{v \in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$.

The *degree* of a vertex v in G is the number of neighbors of v in G, and is denoted by $\deg_G(v)$, and so $\deg_G(v) = |N_G(v)|$. The maximum (minimum) degree among the vertices of G is denoted by $\Delta(G)$ ($\delta(G)$, respectively). An *isolated vertex* is a vertex of degree 0, and a graph is *isolate-free* if it contains no isolated vertex. A vertex of degree 1 is called a *leaf*, and its (unique) neighbor is called a *support vertex*. The edge incident with a leaf is called a *pendant edge*. A *strong support vertex* is a vertex with at least two leaf neighbors, and a *weak support vertex* is a vertex with exactly one leaf neighbor.

We denote the set of leaves in a tree T by $L(T)$ and the set of support vertices of T by $S(T)$. The *diameter* of a graph G, denoted $\mathrm{diam}(G)$, is the maximum distance among all pairs of vertices of G. A *diametrical path* in G is a shortest path whose length is equal to the diameter of G. In particular, the diameter of a tree T is the length of a longest path in T, and a diametrical path in T is a longest path in T. A *star* is the complete bipartite graph $K_{1,s}$ where $s \geq 1$. A *double star* $S(r, s)$, for $1 \leq r \leq s$, is a tree with exactly two (adjacent) vertices that are not leaves, with one of the vertices having r leaf neighbors and the other s leaf neighbors.

A *rooted tree* T distinguishes one vertex r called the *root*. Let T be a tree rooted at vertex r. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. The root r does not have a parent in T and all its neighbors are its children. A *descendant* of v is a vertex x such that the unique (r, x)-path contains v. Thus, every child of v is a descendant of v. A *grandchild* of v is a descendant of v at distance 2 from v. Let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$. For $k \geq 1$ an integer, we let $[k]$ denote the set $\{1, \ldots, k\}$ and we let $[k]_0 = [k] \cup \{0\} = \{0, 1, \ldots, k\}$.

2 Main results

In Section 3, we characterize trees with largest possible 2-domination number.

Theorem 2.1 *If T is a tree of order $n \geq 3$, then $\gamma_2(T) = n - 1$ if and only if T is a star $K_{1,n-1}$ or T is a double star $S(1, n-3)$.*

If T is a tree of order $n \geq 3$, then Atapour et al. [1] showed that either $\mathrm{sd}_2(T) = 1$ or $\mathrm{sd}_2(T) = 2$. We present a simple characterization of trees T for which $\mathrm{sd}_2(T) = 1$ in Theorem 2.2, a proof of which is given in Section 4.

Theorem 2.2 *A tree T satisfies $\mathrm{sd}_2(T) = 1$ if and only if $\mathcal{N}_2(T) \neq \emptyset$.*

In Section 5, we study γ_2-q-critical trees for $q \geq 1$. If T is a γ_2-q-critical tree of order $n \geq 3$, then we prove that $q \leq n - 2$. The following results characterize γ_2-q-critical trees for large values of q. Proofs of Theorems 2.3, 2.4, and 2.5 are given in Section 5.

Theorem 2.3 *If T is a γ_2-q-critical tree of order $n \geq 3$, then $q = n - 2$ if and only if T is a double star $S(1, n - 3)$.*

Theorem 2.4 *No tree T is γ_2-q-critical of order $n \geq 5$ where $q = n - 3$.*

A characterization of γ_2-q-critical trees of order $n \geq 6$ is given by the following theorem, where $\mathcal{H}$ is a family of trees defined in Section 5.

Theorem 2.5 *If T is a γ_2-q-critical tree of order $n \geq 6$, then $q = n - 4$ if and only if $T \in \mathcal{H}$.*

A characterization of γ_2-1-critical trees is given by the following theorem, where $\mathcal{F}$ is a family of labeled trees (T, S) defined in Section 6.1. A proof of Theorem 2.6 is given in Section 6.

Theorem 2.6 *A tree T is γ_2-1-critical if and only if $(T, S) \in \mathcal{F}$ for some labeling S.*

We remark that the above results were motivated by research problems generated in the first author's PhD thesis [4].

3 Preliminary observations and results

In this section, we present preliminary results on the 2-domination number of a tree. We first present properties of 2-dominating sets in trees. If T is a tree of order $n \geq 3$, then every 2-dominating set of a tree T contains all leaves of T. In particular, $L(T) \subseteq \mathcal{A}_2(T)$. If v is a vertex of degree at least 2 in T, then the set $V(T) \setminus \{v\}$ is a 2-dominating set of T, implying that $\gamma_2(T) \leq n - 1$. We state these observations formally as follows.

Observation 3.1 *If T is a tree of order $n \geq 3$, then the following properties hold.*
(a) *Every 2-dominating set of T contains all leaves of T.*
(b) *$L(T) \subseteq \mathcal{A}_2(T)$.*
(c) *$\gamma_2(T) \leq n - 1$.*

Lemma 3.2 *If T is a tree of order $n \geq 2$ and diameter d, then $\gamma_2(T) \leq n - \left\lfloor \frac{d}{2} \right\rfloor$.*

Proof. The 2-domination number of a path P_k of order $k \geq 2$ is given by $\gamma_2(P_k) = \lceil \frac{1}{2}(k+1) \rceil$. In particular, if P is a diametrical path in a tree T of order n and of length $d = \mathrm{diam}(T)$, then P has order $d+1$ and $\gamma_2(P) = \lceil \frac{1}{2}(d+2) \rceil = d + 1 - \lfloor \frac{d}{2} \rfloor$. Let D_p be a γ_2-set of the path P. The set $(V(T) \setminus V(P)) \cup D_p$ is a 2-dominating set of T, and so $\gamma_2(T) \leq (n - (d+1)) + (d + 1 - \lfloor \frac{d}{2} \rfloor) = n - \lfloor \frac{d}{2} \rfloor$. □

The *boundary* of a set X in a graph G is the set of all vertices not in X that have a neighbor in X.

Lemma 3.3 *Let T be a tree of order $n \geq 3$ and diameter d, and let $P\colon v_0 v_1 \ldots v_d$ be a diametrical path in T. If S is the set of vertices in the boundary of the set $V(P)$ that are not leaves of T, then $\gamma_2(T) \leq n - |S| - 2$.*

Proof. The set $D = V(T) \setminus (S \cup \{v_1, v_{d-1}\})$ is a 2-dominating set of T, and so $\gamma_2(T) \leq |D| = n - |S| - 2$. □

We are now in a position to prove Theorem 2.1. Recall the statement of the theorem.

Theorem 2.1. *If T is a tree of order $n \geq 3$, then $\gamma_2(T) = n - 1$ if and only if T is a star $K_{1,n-1}$ or T is a double star $S(1, n-3)$.*

Proof. Suppose firstly that T is a star $K_{1,n-1}$ or T is a double star $S(1, n-3)$. If T is a star $K_{1,n-1}$, then by Observation 3.1(b), $L(T) \subseteq \mathcal{A}_2(T)$, and so $\gamma_2(T) \geq n - 1$. By Observation 3.1(c) we infer that $\gamma_2(T) = n - 1$. If T is a double star $S(1, n-3)$, then T contains $n - 2$ leaves, all of which belong to every 2-dominating set of T. Further in order to 2-dominate the support vertex of degree 2, at least one additional vertex is required in the 2-dominating set. Hence, $\gamma_2(T) \geq n - 1$, and we infer that $\gamma_2(T) = n - 1$.

Suppose next that T is a tree of order $n \geq 3$ satisfying $\gamma_2(T) = n - 1$. Let D be a γ_2-set of T, and let x be the (unique) vertex belonging to T that does not belong to the set D. By Observation 3.1(a), the vertex x is not a leaf, and so $\deg_T(x) \geq 2$. If at least two neighbors of x, say x_1 and x_2, are not leaves, then the set $(D \setminus \{x_1, x_2\}) \cup \{x\}$ is a 2-dominating set of T, and so $\gamma_2(T) \leq n - 2$, a contradiction. Hence all neighbors of x, except possibly at most one, are leaves. If all neighbors of x are leaves, then T is a star $K_{1,n-1}$ (with central vertex x), as desired. Hence we may assume that x has exactly one neighbor, say y, that is not a leaf. If y has a neighbor w different from x that is not a leaf, then $D \setminus \{w\}$ is a 2-dominating set of T, once again contradicting the minimality of the set D. Therefore, all neighbors of y different from x are leaves. If both $\deg_T(x) \geq 3$ and $\deg_T(y) \geq 3$, then $D \setminus \{y\}$ is a 2-dominating set of T, contradicting the minimality of the γ_2-set D of T. Hence, $\deg_T(x) = 2$ or $\deg_T(y) = 2$, implying that the tree T is a double star $S(1, n-3)$.□

We shall need the following elementary observation about the 2-domination number of a graph obtained by subdividing an edge.

Observation 3.4 *If e is an edge of a graph G, then $\gamma_2(G) \leqslant \gamma_2(G_e) \leqslant \gamma_2(G) + 1$.*

4 2-Domination subdivision in trees

If T is a tree of order $n \geq 3$, then in 2008 Atapour et al. [1] showed that either $\mathrm{sd}_2(T) = 1$ or $\mathrm{sd}_2(T) = 2$. Subsequently, in 2012 Atapour et al. [2] characterized trees T satisfying $\mathrm{sd}_2(T) = 2$ (which therefore also yielded a characterization of trees T satisfying $\mathrm{sd}_2(T) = 1$). In this section we present a proof of Theorem 2.2, which provides a simple characterization of trees T for which $\mathrm{sd}_2(T) = 1$. For this purpose, we prove two lemmas. Adopting our earlier notation, recall that if T is a tree and $e \in E(T)$, then we denote by T_e the tree obtained from T by subdividing the edge e. Further recall that $\mathcal{N}_2(T)$ is the set of vertices in T that belong to no γ_2-set of T.

Lemma 4.1 *If T is a tree and $\mathcal{N}_2(T) \neq \emptyset$, then $\mathrm{sd}_2(T) = 1$.*

Proof. Let T be a tree for which $\mathcal{N}_2(T) \neq \emptyset$. Let u be an arbitrary vertex in the set $\mathcal{N}_2(T)$. Since every 2-dominating set of T contains all its leaves, $L(T) \subseteq \mathcal{A}_2(T)$. Thus, u is not a leaf, and so $\deg_T(u) \geq 2$. Let $v \in N_T(u)$ and let $e = uv$. We now consider the tree T_e, and we let w be the new vertex of degree 2 resulting from subdividing the edge e. We show that $\gamma_2(T) < \gamma_2(T_e)$. Let D_e be a γ_2-set of T_e, and so D_e is a 2-dominating of T_e and $|D_e| = \gamma_2(T_e)$.

Suppose that $w \notin D_e$. Thus, $\{u, v\} \subseteq D_e$ in order to 2-dominate the vertex w. In this case, the set D_e is also a 2-dominating set of T, and so $\gamma_2(T) \leq |D_e|$. However since $u \in D_e$ and $u \in \mathcal{N}_2(T)$, the set D_e is not a γ_2-set of T, implying that $\gamma_2(T) < |D_e| = \gamma_2(T_e)$.

Suppose that $w \in D_e$ and $D_e \cap \{u, v\} = \emptyset$. In this case, the set $D = (D_e \setminus \{w\}) \cup \{u\}$ is a 2-dominating set of T, and so $\gamma_2(T) \leq |D|$. Since $u \in D$ and $u \in \mathcal{N}_2(T)$, the set D is not a γ_2-set of T, implying that $\gamma_2(T) < |D| = |D_e| = \gamma_2(T_e)$, as desired.

Suppose that $w \in D_e$ and $D_e \cap \{u, v\} \neq \emptyset$. By the minimality of the set D_e, exactly one of u and v belongs to D_e. Thus either $D_e \cap \{u, v, w\} = \{u, w\}$ or $D_e \cap \{u, v, w\} = \{v, w\}$. If $u \in D_e$, then we let $D = (D_e \setminus \{w\}) \cup \{v\}$, while if $v \in D_e$, then we let $D = (D_e \setminus \{w\}) \cup \{u\}$. In both cases $u \in D$, and the set D is a 2-dominating set of T, and so $\gamma_2(T) \leq |D|$. Since $u \in D$ and $u \in \mathcal{N}_2(T)$, the set D is not a γ_2-set of T, implying that $\gamma_2(T) < |D| = |D_e| = \gamma_2(T_e)$.

In all the above cases, we have $\gamma_2(T) < \gamma_2(T_e)$, as claimed. Thus we have shown that if there exists a vertex $u \in \mathcal{N}_2(T)$, then subdividing any edge incident with u in the tree T increases the 2-domination number. In particular, there exists an edge in T which, when subdivided, increases the 2-domination number, implying that $\mathrm{sd}_2(T) = 1$. □

Lemma 4.2 *If T is a tree and $\mathrm{sd}_2(T) = 1$, then $\mathcal{N}_2(T) \neq \emptyset$.*

Proof. Let T be a tree satisfying $\mathrm{sd}_2(T) = 1$. Thus there exists an edge $e = xy$ such that $\gamma_2(T_e) > \gamma_2(T)$. We show that $\mathcal{N}_2(T) \neq \emptyset$. Suppose, to the contrary, that $\mathcal{N}_2(T) = \emptyset$. Thus every vertex in T belongs to some γ_2-set of T. In particular, there exists a γ_2-set, D_x, of T containing x and there exists a γ_2-set, D_y, of T containing y.

If there exists a γ_2-set of T that contains both x and y, then such a set is also a 2-dominating set of T_e, implying that $\gamma_2(T_e) \leq \gamma_2(T)$, a contradiction. Hence, no γ_2-set of T contains both x and y. Thus, $x \notin D_y$ and $y \notin D_x$. We now consider the forest $T - xy$. Let T_x and T_y be the components of $T - xy$ containing x and y, respectively. We let

$$D_{xx} = D_x \cap V(T_x) \quad \text{and} \quad D_{xy} = D_x \cap V(T_y)$$

and

$$D_{yx} = D_y \cap V(T_x) \quad \text{and} \quad D_{yy} = D_y \cap V(T_y).$$

Thus, $D_x = D_{xx} \cup D_{xy}$ and $D_y = D_{yx} \cup D_{yy}$. Let $|D_{xx}| = k$ and let $|D_{yx}| = t$. We note that $|D_{xy}| = |D_x| - |D_{xx}| = \gamma_2(T) - k$ and $|D_{yy}| = |D_y| - |D_{yx}| = \gamma_2(T) - t$. We now consider the sets

$$D = D_{xx} \cup D_{yy} \quad \text{and} \quad S = D_{xy} \cup D_{yx}.$$

The set D is a 2-dominating set of T, and so $\gamma_2(T) \leq |D|$. By construction, the set D contains both x and y. By our earlier observations, no γ_2-set of T contains both x and y. In particular, the set D is not a γ_2-set of T, implying that $\gamma_2(T) < |D| = |D_{xx}| + |D_{yy}| = k + (\gamma_2(T) - t)$, and so $k - t > 0$. Since $k, t \in \mathbb{N}$, we infer that $k - t \geq 1$. Since the set $S \cup \{w\}$ is a 2-dominating set of T_e, we therefore have that

$$\begin{aligned} \gamma_2(T_e) \leq |S| + 1 &= |D_{xy}| + |D_{yx}| + 1 \\ &= (\gamma_2(T) - k) + t + 1 \\ &= \gamma_2(T) - (k - t) + 1 \\ &\leq \gamma_2(T), \end{aligned}$$

contradicting the fact that $\gamma_2(T_e) > \gamma_2(T)$. Hence, $\mathcal{N}_2(T) \neq \emptyset$. □

As an immediate consequence of Lemmas 4.1 and 4.2, we have the characterization of trees with 2-domination subdivision number equal to 1 given in Theorem 2.2. Recall the statement of the theorem.

Theorem 2.2. *A tree T satisfies $\mathrm{sd}_2(T) = 1$ if and only if $\mathcal{N}_2(T) \neq \emptyset$.*

5 γ_2-q-Critical trees

In this section, we present general results on γ_2-q-critical trees. Recall that a tree T is γ_2-q-critical if the smallest subset of edges (where each edge in T can be subdivided at most once) whose subdivision necessarily increases $\gamma_2(T)$ has cardinality q. Thus if T is γ_2-q-critical and F is an arbitrary subset of q edges of T, then subdividing the q edges in F increases the 2-domination number of T. However, there exists a subset S of $q - 1$ edges of T such that subdividing these $q - 1$ edges in S does not increase the 2-domination number of T. We shall need the following result due to Fink and Jacobson [6].

Theorem 5.1 ([6]) *If G is a graph of order n and size m, then $\gamma_2(G) \geq n - \frac{1}{2}m$.*

As an immediate consequence of Theorem 5.1, we have the following lower bound on the 2-domination number of a graph with k edges subdivided.

Corollary 5.2 *If G_k is the graph obtained from a graph G of order n and size m by subdividing $k \geq 1$ edges, then $\gamma_2(G_k) \geq n - \frac{1}{2}(m-k)$.*

As a consequence of Corollary 5.2, if T is a γ_2-q-critical tree, then we have the following upper bound on q.

Lemma 5.3 *If T is a γ_2-q-critical tree of order $n \geq 3$, then $q \leq n-2$.*

Proof. Let T be a γ_2-q-critical tree of order $n \geq 3$ and suppose, to the contrary, that $q = n-1$. Thus there exists a set F of $n-2$ edges of T such that subdividing these $n-2$ edges in F does not increase the 2-domination number of T. By Observation 3.1(c), $\gamma_2(T) \leq n-1$. Let T_F be the tree obtained from T by subdividing the $n-2$ edges that belong to F. By Corollary 5.2 and by supposition, $\gamma_2(T) = \gamma_2(T_F) \geq n - \frac{1}{2}((n-1)-(n-2)) = n - \frac{1}{2} > n-1 \geq \gamma_2(T)$, a contradiction. □

We next characterize γ_2-q-critical trees with large values of q. By Lemma 5.3, $q \leq n-2$. We first prove Theorem 2.3. Recall the statement of the theorem.

Theorem 2.3. *If T is a γ_2-q-critical tree of order $n \geq 4$, then $q = n-2$ if and only if T is a double star $S(1, n-3)$.*

Proof. Suppose that T is a double star $S(1, n-3)$ where $n \geq 4$. Let u and v be the two (adjacent) vertices of T that are not leaves, where u has $n-3$ leaf neighbors and v has exactly one leaf neighbor. By Theorem 2.1, $\gamma_2(T) = n-1$. We show that there exists a set F of $n-3$ edges such that subdividing these $n-3$ edges in F does not increase the 2-domination number of T. Let $D_u = L(T) \cup \{u\}$ and let $D_v = L(T) \cup \{v\}$. Let D be a γ_2-set of T. By Observation 3.1(b), $L(T) \subseteq \mathcal{A}_2(T)$, and so the set D contains the $n-2$ leaves of T. Since $\gamma_2(T) = n-1$, we infer that either $D = D_u$ or $D = D_v$. We note that both sets D_u and D_v are γ_2-sets of T. Let F be the set of $n-3$ edges that join u to its leaf neighbors. The set D_u is a 2-dominating set of the tree obtained from T by subdividing the $n-3$ edges in F. Hence there exists a set of $n-3$ edges such that subdividing these $n-3$ edges in F does not increase the 2-domination number of T. By Lemma 5.3, the 2-domination number of the tree obtained from T by subdividing any set of $n-2$ edges increases the 2-domination number of T. We therefore infer that the tree T is a γ_2-q-critical tree where $q = n-2$.

Suppose next that T is a γ_2-q-critical tree of order $n \geq 4$, where $q = n-2$. Hence there exists a set F of $n-3$ edges such that subdividing these $n-3$ edges in F does not increase the 2-domination number of T. Let T_F be the tree obtained from T by subdividing the $n-3$ edges that belong to F. By Corollary 5.2 and by supposition, $\gamma_2(T) = \gamma_2(T_F) \geq n - \frac{1}{2}((n-1)-(n-3)) = n-1 \geq \gamma_2(T)$. Consequently, we must

have equality throughout this inequality chain. In particular, $\gamma_2(T) = n - 1$. By Theorem 2.1, either T is a star $K_{1,n-1}$ or T is a double star $S(1, n-3)$. If T is a star $K_{1,n-1}$, then $\gamma_2(T) = n - 1$ (with the set of leaves in T as the unique γ_2-set of T) and $\gamma_2(T_e) = n$ for every edge $e \in E(T)$, implying that T is γ_2-1-critical. Since $n \geq 4$, this contradicts our supposition that T is a γ_2-$(n-2)$-critical tree. Hence, T is a double star $S(1, n-3)$, as claimed. □

We next prove Theorem 2.4. Recall the statement of the theorem.

Theorem 2.4. *There is no tree T that is γ_2-q-critical of order $n \geq 5$ where $q = n-3$.*

Proof. Let T be a γ_2-q-critical of order $n \geq 5$, and suppose, to the contrary, that $q = n - 3$. Thus there exists a set S of $n - 4$ edges such that subdividing these $n - 4$ edges in S does not increase the 2-domination number of T, but the 2-domination number of the tree obtained from T by subdividing any set of $n - 3$ edges increases the 2-domination number of T. By Corollary 5.2 and by supposition, $\gamma_2(T) = \gamma_2(T_S) \geq n - \frac{1}{2}((n-1) - (n-4)) = n - \frac{3}{2}$, implying that $\gamma_2(T) \geq n - 1$. By Observation 3.1(c), $\gamma_2(T) \leq n - 1$. Consequently, $\gamma_2(T) = n - 1$. By Theorem 2.1, either T is a star $K_{1,n-1}$ or T is a double star $S(1, n-3)$. If T is a star $K_{1,n-1}$, then T is γ_2-1-critical, while if T is a double star $S(1, n-3)$, then by Theorem 2.3 the tree T is γ_2-$(n-2)$-critical. In both cases we contradict our supposition that T is a γ_2-q-critical tree of order $n \geq 5$ where $q = n - 3$. □

We frequently use the following property of γ_2-1-critical trees.

Proposition 5.4 *A tree T is γ_2-1-critical if and only if every γ_2-set of T is independent.*

Proof. Suppose first that there exists a γ_2-set D of T that is not independent. Let u and v be two adjacent vertices in D and let $e = uv$. The set D is a 2-dominating set of T_e, and so $\gamma_2(T_e) \leq |D| = \gamma_2(T)$. Thus, T is not γ_2-1-critical. Conversely, suppose that T is not γ_2-1-critical. Let $e = v_1v_2$ be an edge such that $\gamma_2(T_e) = \gamma_2(T)$ and let v be the new vertex obtained by subdividing an edge e. Let D be a γ_2-set of T_e.

Suppose that $v \in D$. If $v_i \in D$ for some $i \in [2]$, then $(D \setminus \{v\}) \cup \{v_{3-i}\}$ is a γ_2-set of T containing two adjacent vertices, namely v_1 and v_2. If neither v_1 nor v_2 belongs to the set D, then in order to 2-dominate the vertex v_i, at least one neighbor of v_i different to v belongs to D for $i \in [2]$. Let u_1 be a neighbor of v_1 different from v that belongs to D. In this case, we infer that $(D \setminus \{v\}) \cup \{v_1\}$ is a γ_2-set of T containing two adjacent vertices, namely u_1 and v_1. Suppose next that $v \notin D$. In this case, $\{v_1, v_2\} \subseteq D$ and the set D is a γ_2-set of T containing two adjacent vertices, namely v_1 and v_2. In both cases, we produce a γ_2-set of T that is not independent. Thus, not every γ_2-set of T is independent. □

If removing all leaves from a tree T produces a path P, then the tree T is called a *caterpillar* and the path P is called the *spine* of the caterpillar. If P is the path $v_1v_2 \ldots v_p$, and the vertex v_i has k_i leaf neighbors in T for $i \in [p]$, then we refer to

the sequence $(k_1, k_2, \ldots, k_p)$ as the *code* of the caterpillar. We note that $k_1, k_p \geq 1$. Further if $p \geq 3$, then $k_i \geq 0$ for $i \in \{2, \ldots, p-1\}$. In what follows, if F is a set of edges in a tree T, then we denote the tree obtained from T by subdividing the edges that belong to F by T_F.

Proposition 5.5 *If T is a caterpillar of order n with spine $P\colon v_1v_2v_3$ and code (k_1, k_2, k_3), then the following properties hold.*

(a) *If $k_1, k_3 \geq 2$ and $k_2 \neq 1$, then T is γ_2-1-critical.*
(b) *If $k_1, k_3 \geq 2$ and $k_2 = 1$, then $\gamma_2(T) = n - 2$ and T is γ_2-q-critical where $q = \max\{k_1, k_3\} + 1$.*
(c) *If exactly one of k_1 or k_3 equals one, then $\gamma_2(T) = n - 2$ and T is γ_2-q-critical where $q = k_2 + 1$ and $k_2 \leq n - 6$.*

Proof. Let T be a caterpillar with spine $P\colon v_1v_2v_3$ and code (k_1, k_2, k_3). We consider three possibilities.

(a) Suppose that $k_1, k_3 \geq 2$ and $k_2 \neq 1$. If $k_i \geq 2$ for all $i \in [3]$, then $\gamma_2(T) = n-3$ and the set $L(T)$ of leaves of T is the unique γ_2-set of T. If $k_2 = 0$, then $\gamma_2(T) = n-2$ and the set $L(T)\cup\{v_2\}$ is the unique γ_2-set of T. In both cases, we obtain a unique γ_2-set of T that is independent. Therefore, by Proposition 5.4, the tree T is γ_2-1-critical.

(b) Suppose that $k_1, k_3 \geq 2$ and $k_2 = 1$. Without loss of generality, we may assume that $k_1 \geq k_3$. Let D be an arbitrary γ_2-set of T. By Observation 3.1(b), the set D contains the $n-3$ leaves of T. Since $k_1, k_3 \geq 2$, both v_1 and v_3 are 2-dominated by their leaf neighbors. In order to 2-dominate the vertex v_2, the set D contains exactly one vertex from the spine of the path P, and so $\gamma_2(T) = n - 2$. Moreover the set $L(T) \cup \{v_i\}$ is a γ_2-set of T for $i \in [3]$. In particular, $D^* = L(T) \cup \{v_1\}$ is a γ_2-set of T. The set D^* remains a 2-dominating set if we subdivide the k_1 edges joining v_1 and its k_1 leaf neighbors. Hence there exists a set of k_1 edges which, when subdivided, does not increase the 2-domination number. Moreover if F is an arbitrary set of $k_1 + 1$ edges of T, then it necessarily includes at least one of the edges incident with v_2 or v_3. By our earlier observations, we infer that $\gamma_2(T_F) \geq |D^*| + 1 = \gamma_2(T) + 1$. Hence, T is a γ_2-q-critical tree where $q = k_1 + 1 = \max\{k_1, k_3\} + 1$.

(c) Suppose that exactly one of k_1 or k_3 equals 1. Without loss of generality, we may assume that $k_1 = 1$ (and so, $k_3 \geq 2$). In this case, $n = k_2+k_3+4$. Moreover, $\gamma_2(T) = n - 2$ and the set $D = L(T) \cup \{v_2\}$ is a γ_2-set of T. If $k_2 = 0$, then $n = k_3 + 4$ and subdividing any edge of T increases the 2-domination number, implying that T is γ_2-q-critical where $q = 1 = k_2 + 1$. If $k_2 \geq 1$, then the set D remains a 2-dominating set if we subdivide all k_2 edges joining v_2 and its leaf neighbors. Hence there is a set of k_2 edges which when subdivided does not increase the 2-domination number. Moreover if F is an arbitrary set of $k_2 + 1$ edges of T, then $\gamma_2(T_F) \geq |L(T)| + 2 = \gamma_2(T) + 1$. Hence, T is γ_2-q-critical where $q = k_2 + 1$. □

As an application of Proposition 5.5, we show next that, for every $n \geq 6$, there exists a γ_2-q-critical tree for any given positive integer $q \in [n-5]$.

Corollary 5.6 *For $n \geq 6$ and $q \in [n-5]$ there exists a γ_2-q-critical tree of order n.*

Proof. Let T be a caterpillar of order n with spine $P\colon v_1v_2v_3$ and code (k_1, k_2, k_3). If $q \in [n-5]$, then letting $k_1 = 1$, $k_2 = q-1$ and $k_3 = n-3-q$, we note that $k_3 \in \{2, \ldots, n-4\}$, and so by Proposition 5.5(b) the caterpillar T is γ_2-q-critical. □

We next define a family $\mathcal{H}$ of γ_2-q-critical trees where $q = n-4$ as follows.

- Let $\mathcal{H}_1 = \{K_{1,4}\}$.
- Let $\mathcal{H}_2$ be the family of all caterpillars with spine $P\colon v_1v_2v_3$ and code (k_1, k_2, k_3) where $k_1 = k_3 = 1$ and $k_2 \geq 0$.
- Let $\mathcal{H}_3$ be the family of all caterpillars with spine $P\colon v_1v_2v_3v_4$ and code (k_1, k_2, k_3, k_4) where $k_1 = k_4 = 1$ and $k_2, k_3 \geq 0$.
- Let $\mathcal{H}_4$ be the family of all caterpillars with spine $P\colon v_1v_2v_3v_4$ and code (k_1, k_2, k_3, k_4) where $k_1 = 1$, $k_2 \geq 0$, $k_3 = 0$, and $k_4 \geq 2$.
- Let $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3 \cup \mathcal{H}_4$.

Proposition 5.7 *If $T \in \mathcal{H}$ has order n, then T is γ_2-q-critical where $q = n-4$.*

Proof. Let $T \in \mathcal{H}$ have order n. We consider four possibilities.

(a) Suppose that $T \in \mathcal{H}_1$. Thus, $T = K_{1,4}$. In this case, $\gamma_2(T) = 4$ (with the set of leaves in T as the unique γ_2-set of T) and $\gamma_2(T_e) = n$ for every edge $e \in E(T)$, implying that T is γ_2-1-critical. Equivalently, T is γ_2-q-critical where $q = n-4$.

(b) Suppose that $T \in \mathcal{H}_2$ has order n. Thus, T is a caterpillar with spine $P\colon v_1v_2v_3$ and code (k_1, k_2, k_3) where $k_1 = k_3 = 1$ and $k_2 \geq 0$. In this case, $\gamma_2(T) = n-2$ and the set $L(T) \cup \{v_2\}$ is the unique γ_2-set of T. If $k_2 = 0$, then T is a path P_5, and $n = 5$ and T is γ_2-q-critical where $q = 1 = n-4$. If $k_2 \geq 1$, then subdividing the $n-5$ edges joining v_2 and its leaf neighbors does not increase the 2-domination number. Moreover if F is an arbitrary set of $n-4$ edges of T, then F contains at least one of the edges on the diametrical path in T. We infer that $\gamma_2(T_F) \geq |L(T)| + 2 = \gamma_2(T) + 1$. Hence, T is γ_2-q-critical where $q = n-4$.

(c) Suppose that $T \in \mathcal{H}_3$ has order n. Thus, T is a caterpillar with spine $P\colon v_1v_2v_3v_4$ and code (k_1, k_2, k_3, k_4) where $k_1 = k_4 = 1$, $k_2, k_3 \geq 0$, and $n = k_2 + k_3 + 6$. If D is an arbitrary γ_2-set of T, then $L(T) \subseteq D$. Further, the set D contains at least one of v_1 and v_2 in order to 2-dominate the vertex v_1, and the set D contains at least one of v_3 and v_4 in order to 2-dominate the vertex v_4. Thus, $\gamma_2(T) \geq |L(T)| + 2 = n-2$. The set $D_{2,3} = L(T) \cup \{v_2, v_3\}$, for example,

is a 2-dominating set of T, and so $\gamma_2(T) \le n-2$. Consequently, $\gamma_2(T) = n-2$. The set $D_{2,3}$ remains a 2-dominating set if we subdivide k_2+k_3+1 edges, namely the k_2 edges joining v_2 and its leaf neighbors, the k_3 edges joining v_3 and its leaf neighbors, and the edge v_2v_3. Hence there exists a set of $n-5$ edges which, when subdivided, does not increase the 2-domination number. Moreover, if F is an arbitrary set of $n-4$ edges of T, then it necessarily includes at least one of the edges incident with v_1 or v_4. By our earlier observations, we infer that $\gamma_2(T_F) \ge |D|+1 = \gamma_2(T)+1$. Hence, T is a γ_2-q-critical tree where $q = n-4$.

(d) Suppose that $T \in \mathcal{H}_4$ has order n. Thus, T is a caterpillar with spine P: $v_1v_2v_3v_4$ and code (k_1, k_2, k_3, k_4) where $k_1 = 1$, $k_2 \ge 0$, $k_3 = 0$, $k_4 \ge 2$, and $n = k_2 + k_4 + 5$. If D is an arbitrary γ_2-set of T, then $L(T) \subseteq D$. Further the set D contains at least one of v_1 and v_2 in order to 2-dominate the vertex v_1, and the set D contains at least one of v_3 and v_4 in order to 2-dominate the vertex v_4. Thus, $\gamma_2(T) \ge |L(T)| + 2 = n-2$. The set $D_{2,4} = L(T) \cup \{v_2, v_4\}$, for example, is a 2-dominating set of T, and so $\gamma_2(T) \le n-2$. Consequently, $\gamma_2(T) = n-2$. The set $D_{2,4}$ remains a 2-dominating set if we subdivide k_2+k_4 edges, namely the k_2 edges joining v_2 and its leaf neighbors and the k_4 edges joining v_4 and its leaf neighbors. Hence there exists a set of $n-5$ edges which, when subdivided, does not increase the 2-domination number. Moreover if F is an arbitrary set of $n-4$ edges of T, then it necessarily includes at least one of the edges incident with v_1 or v_3. By our earlier observations, we infer that $\gamma_2(T_F) \ge |D|+1 = \gamma_2(T)+1$. Hence, T is a γ_2-q-critical tree where $q = n-4$.

In all four cases, the tree T is γ_2-q-critical where $q = n-4$. This completes the proof of Proposition 5.7. □

We are now in a position to characterize γ_2-q-critical trees where $q = n-4$. Recall the statement of Theorem 2.5.

Theorem 2.5. *If T is a γ_2-q-critical tree of order $n \ge 6$, then $q = n-4$ if and only if $T \in \mathcal{H}$.*

Proof. If $T \in \mathcal{H}$ has order n, then by Proposition 5.7, the tree T is γ_2-q-critical where $q = n-4$. Conversely, suppose that T is a γ_2-q-critical tree of order $n \ge 6$ satisfying $q = n-4$. Thus, there exists a set of $n-5$ edges which, when subdivided, does not increase the 2-domination number. Moreover if F is an arbitrary set of $n-4$ edges of T, then subdividing the edges in F increases the 2-domination number, that is, $\gamma_2(T_F) \ge \gamma_2(T)+1$. By Corollary 5.2, $\gamma_2(T) = \gamma_2(T_F) \ge n - \frac{1}{2}((n-1)-(n-4)) = n - \frac{5}{2}$, implying that $\gamma_2(T) \ge n-2$. By Observation 3.1(c), $\gamma_2(T) \le n-1$. Hence either $\gamma_2(T) = n-1$ or $\gamma_2(T) = n-2$.

Suppose that $\gamma_2(T) = n-1$. In this case, by Theorem 2.1, either T is a star $K_{1,n-1}$ or T is a double star $S(1, n-3)$. If T is a double star $S(1, n-3)$, then by Theorem 2.3 T is γ_2-q-critical with $q = n-2$, contradicting our supposition that T is γ_2-q-critical with $q = n-4$. Hence, T is a star $K_{1,n-1}$. The tree T is therefore γ_2-1-critical. By supposition, T is γ_2-q-critical with $q = n-4$, implying that $n = 5$; that is, $T = K_{1,4} \in \mathcal{H}_1$.

Suppose next that $\gamma_2(T) = n - 2$. If T is not a caterpillar, then the set S in the statement of Lemma 3.3 is not empty, and so $\gamma_2(T) \le n - |S| - 2 \le n - 3$, a contradiction. Hence, T is a caterpillar. Let T have diameter d, and so $\mathrm{diam}(T) = d$. By Lemma 3.2 we have $\gamma_2(T) \le n - \left\lfloor \frac{d}{2} \right\rfloor$, and so $d \le 5$. If $d = 2$, then T is a star, and so $\gamma_2(T) = n - 1$ by Theorem 2.1, a contradiction. Thus, $3 \le d \le 5$.

Suppose that $d = 3$. Thus, T is a double star $S(r, s)$ where $1 \le r \le s$. If $r = 1$, then T is a double star $S(1, n - 3)$, and so $\gamma_2(T) = n - 1$ by Theorem 2.1, a contradiction. Hence, $r \ge 2$, implying that $n = r+s+2 \ge s+4$, and so $s \le n-4$. We note that $\gamma_2(T) = r+s$ (and the set $L(T)$ is the unique γ_2-set of T). Subdividing any edge of T increases the 2-domination number, implying that T is γ_2-1-critical. By supposition, T is γ_2-q-critical with $q = n - 4$, implying that $n = 5$, a contradiction. Hence, $d \ne 3$.

Suppose that $d = 4$. Let T be a caterpillar of order n with spine $P\colon v_1v_2v_3$ and code (k_1, k_2, k_3). We may assume that $1 \le k_1 \le k_3$. If $k_3 \ge 2$, then we infer from Proposition 5.5 that T is γ_2-q-critical. Since all parts of Proposition 5.5 imply $q \le n - 5$, we contradict our supposition that $q = 4$. Hence, $k_1 = k_3 = 1$, implying that $T \in \mathcal{H}_2$. Hence if $d = 4$, then $T \in \mathcal{H}$.

Suppose that $d = 5$. Let T be a caterpillar with spine $P\colon v_1v_2v_3v_4$ and code (k_1, k_2, k_3, k_4). If $k_i \ge 2$ for at least three values of $i \in [4]$, then $\gamma_2(T) \le n - 3$, a contradiction. Hence, $k_i \ge 2$ for at most two values of $i \in [4]$. We may assume that $1 \le k_1 \le k_4$. If $k_1 \ge 2$, then $L(T) \cup \{v_2\}$ is a 2-dominating set of T, and so $\gamma_2(T) \le n - 3$, a contradiction. Hence, $k_1 = 1$. If $k_3, k_4 \ge 2$, then $L(T) \cup \{v_2\}$ is a 2-dominating set of T, and once again $\gamma_2(T) \le n-3$, a contradiction. Hence, $k_4 = 1$ or $k_3 \le 1$. If $k_4 = 1$, then $T \in \mathcal{H}_3$. Hence we may assume that $k_4 \ge 2$, implying that $k_3 \le 1$. If $k_3 = 1$, then $L(T) \cup \{v_2\}$ is a 2-dominating set of T, and once again $\gamma_2(T) \le n - 3$, a contradiction. Hence, $k_3 = 0$. Thus, $k_1 = 1$, $k_2 \ge 0$, $k_3 = 0$ and $k_4 \ge 2$, implying that $T \in \mathcal{H}_4$. Hence if $d = 5$, then $T \in \mathcal{H}$. □

6 γ_2-1-Critical trees

In this section, we characterize γ_2-1-critical trees. For this purpose, we first define a family $\mathcal{F}$ of labeled trees in Section 6.1 adopting the terminology in [10]. Thereafter in Section 6.2 we present a proof of Theorem 6.2.

6.1 The family $\mathcal{F}$

In this section, we define a family $\mathcal{F}$ of labeled trees (T, S) where T is a tree and S is a labeling that assigns to every vertex v of T a label, called the *status* of v and denoted by $\mathrm{sta}(v)$, where $\mathrm{sta}(v) \in \{A, B\}$. We define (T_1, S_1) as the *labeled base tree* of the family $\mathcal{F}$, where T_1 is a path of order 3 given by $v_1v_2v_3$ and the labeling S_1 assigns to the two leaves status A and to the central vertex status B. The labeled base tree (T_1, S_1) is illustrated in Figure 1.

A labeled tree (T, S) belongs to the family $\mathcal{F}$, if there is a sequence $(T_1, S_1), \ldots,$

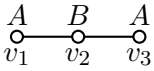


Figure 1: The labeled base tree (T_1, S_1)

(T_k, S_k) of labeled trees where (T_1, S_1) is the labeled base tree defined earlier, $(T, S) = (T_k, S_k)$, and for $k \geq 2$ and $i \in [k-1]$, the labeled tree (T_{i+1}, S_{i+1}) is obtained from the labeled tree (T_i, S_i) by applying one of the operations $\mathcal{O}_j$, $j \in [3]$, given below to a vertex $v \in V(T_i)$.

Operation $\mathcal{O}_1$. Assume $\mathrm{sta}(v) = B$ in (T_i, S_i). The labeled tree (T_{i+1}, S_{i+1}) is obtained from the labeled tree (T_i, S_i) by adding to it a new vertex x and the edge vx, and letting $\mathrm{sta}(x) = A$. Operation $\mathcal{O}_1$ is illustrated in Figure 2.

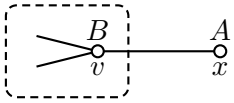


Figure 2: Operation $\mathcal{O}_1$

Operation $\mathcal{O}_2$. Assume $\mathrm{sta}(v) = A$ in (T_i, S_i). The labeled tree (T_{i+1}, S_{i+1}) is obtained from the labeled tree (T_i, S_i) by adding to it a path xy and the edge vx, and letting $\mathrm{sta}(x) = B$ and $\mathrm{sta}(y) = A$. Operation $\mathcal{O}_2$ is illustrated in Figure 3.

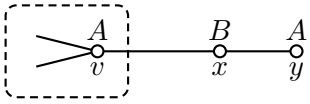


Figure 3: Operation $\mathcal{O}_2$

Operation $\mathcal{O}_3$. Assume $\mathrm{sta}(v) = B$ in (T_i, S_i). The labeled tree (T_{i+1}, S_{i+1}) is obtained from the labeled tree (T_i, S_i) by adding to it a path xyz and the edge vy, and letting $\mathrm{sta}(x) = \mathrm{sta}(z) = A$ and $\mathrm{sta}(y) = B$. Operation $\mathcal{O}_3$ is illustrated in Figure 4.

To illustrate operations $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$, let $(T_1, S_1), (T_2, S_2), \ldots, (T_6, S_6)$ be the labelled trees illustrated in Figure 5. The labeled tree (T_1, S_1) is the base tree illustrated in Figure 1. The labeled tree (T_2, S_2) is obtained from (T_1, S_1) by applying operation $\mathcal{O}_1$, the labeled tree (T_3, S_3) is obtained from (T_2, S_2) by applying operation $\mathcal{O}_2$, the labeled tree (T_4, S_4) is obtained from (T_3, S_3) by applying operation $\mathcal{O}_3$, the labeled tree (T_5, S_5) is obtained from (T_4, S_4) by applying operation $\mathcal{O}_1$, and the labeled tree (T_6, S_6) is obtained from (T_5, S_5) by applying operation $\mathcal{O}_2$. All labeled trees (T_i, S_i) belong to the family $\mathcal{F}$. In particular, the labeled tree $(T, S) = (T_6, S_6)$ belongs to the family $\mathcal{F}$.

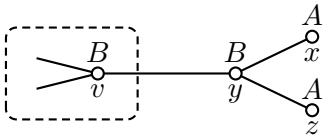
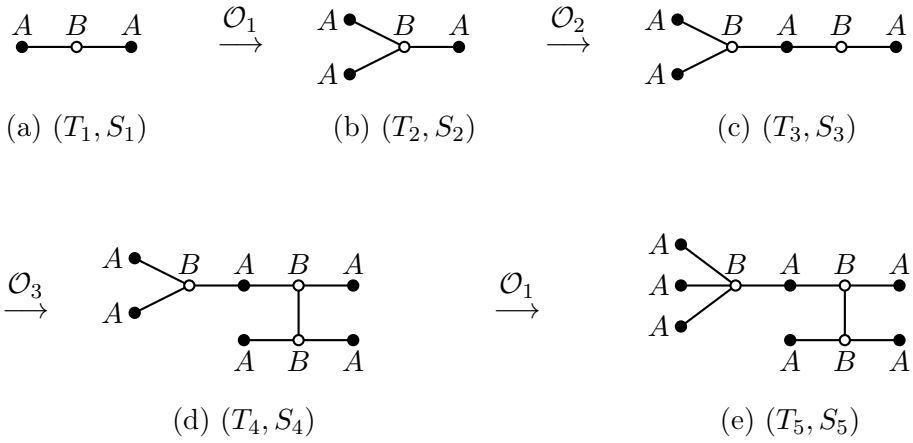


Figure 4: Operation $\mathcal{O}_3$



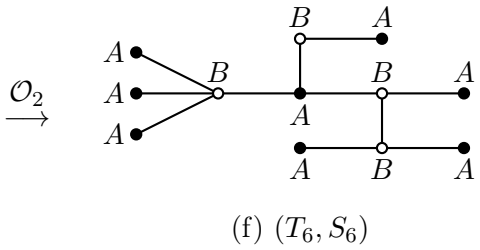


Figure 5: A labeled tree $(T, S) = (T_6, S_6)$ in the family $\mathcal{F}$

If $(T, S) \in \mathcal{F}$, then we let $S_A(T)$ and $S_B(T)$ be the sets of vertices of status A and B, respectively, in the labeled tree (T, S). The following observation is immediate from the way in which each tree in the family $\mathcal{F}$ is constructed. Recall that $L(T)$ and $S(T)$ denote the set of leaves and support vertices in a tree T, respectively.

Observation 6.1 *If $(T, S) \in \mathcal{F}$, then the following properties hold.*
(a) *$L(T) \subseteq S_A(T)$ and $S(T) \subseteq S_B(T)$.*
(b) *If $v \in S_A(T)$, then $N_T(v) \subseteq S_B(T)$.*
(c) *If $v \in S_B(T)$, then $|N_T(v) \cap S_A(T)| \geq 2$.*
(d) *The set $S_A(T)$ is an independent set.*

Recall that if T is a tree and $e \in E(T)$, then T_e is the tree obtained from T by subdividing the edge e. We proceed further with the following lemma.

Lemma 6.2 *If $(T, S) \in \mathcal{F}$, then the following properties hold.*
(a) *$S_A(T)$ is the unique γ_2-set of T.*
(b) *The tree T is γ_2-1-critical.*

Proof. (a) Let $(T, S) \in \mathcal{F}$. We show firstly that every 2-dominating et of T has cardinality at least $|S_A(T)|$. Let D be an arbitrary 2-dominating set of T. If $S_A(T) \subseteq D$, then $|D| \geq |S_A(T)|$, as desired. Hence we may assume that at least one vertex of status A does not belong to D. Let D_A and D_B be the set of vertices in D that belong to A and B, respectively, and so $D_A = D \cap S_A(T)$ and $D_B = D \cap S_B(T)$. Further, let $\overline{D}_A = S_A(T) \setminus D_A$, and so $\overline{D}_A$ is the set of vertices of status A that do not belong to D. By assumption, $\overline{D}_A \neq \emptyset$.

By Observation 6.1(b), if $v \in \overline{D}_A$, then $N_T(v) \subseteq S_B(T)$. Since D is a 2-dominating set of T, each vertex in $\overline{D}_A$ therefore has at least two neighbors in D_B. We now consider the forest F induced by the set $\overline{D}_A \cup D_B$. Every edge in F is incident with a vertex of $\overline{D}_A$ and a vertex of D_B, and so F has size at least $2|\overline{D}_A|$, that is, $m(F) \geq 2|\overline{D}_A|$. Thus since F is a forest, we have $2|\overline{D}_A| \leq m(F) \leq n(F) - 1 = |\overline{D}_A| + |D_B| - 1$, and so $|D_B| \geq |\overline{D}_A| + 1$. Thus, $|D| = |D_A| + |D_B| \geq |D_A| + |\overline{D}_A| + 1 = |S_A(T)| + 1$. Hence, if $S_A(T) \not\subseteq D$, then $|D| \geq |S_A(T)| + 1$. As shown earlier, if $S_A(T) \subseteq D$, then $|D| \geq |S_A(T)|$. Since D is an arbitrary 2-dominating set of T, we infer that $\gamma_2(T) \geq |S_A(T)|$.

By Observation 6.1, the set $S_A(T)$ is a 2-dominating set of T, and so $\gamma_2(T) \leq |S_A(T)|$. By our earlier observations, $\gamma_2(T) \geq |S_A(T)|$. Consequently, $\gamma_2(T) = |S_A(T)|$. Let D^* be an arbitrary γ_2-set of T. Thus, $|D^*| = \gamma_2(T) = |S_A(T)|$. If $S_A(T) \not\subseteq D^*$, then by our earlier observations $|D^*| \geq |S_A(T)| + 1$, a contradiction. Hence, $S_A(T) \subseteq D^*$. Thus, $|S_A(T)| \leq |D^*| = |S_A(T)|$, and so we must have equality through this inequality chain, implying that $D^* = S_A(T)$. Thus, $S_A(T)$ is the unique γ_2-set of T. This proves part (a).

(b) To prove part (b), let $e = uv$ be an arbitrary edge of T, and consider the tree T_e. Let w be the new vertex of degree 2 resulting from subdividing the edge e. By Observation 6.1, at least one of u and v has status B. Renaming u and v if necessary, we may assume that $v \in S_B(T)$. Let D be a γ_2-set of T_e. We show that $|D| > \gamma_2(T)$.

Suppose firstly that e is a pendant edge in T. Thus, u is a leaf in T and $u \in S_A(T)$, and so uw is a pendant edge in T_e. Since u is a leaf in T_e, we note that $u \in D$. If $w \notin D$, then $v \in D$ in order for the set D to 2-dominate the vertex w. If $w \in D$, then we can simply replace w in D with the vertex v to obtain a new γ_2-set of T_e. Hence we may choose the set D to contain the vertex v. However, such a set D is a 2-dominating of T, implying that $\gamma_2(T) \leq |D| = \gamma_2(T_e)$. Since the set D contains a vertex of status B, namely the vertex v, by part (a) the set D is not a γ_2-set of T, implying that $\gamma_2(T) < |D|$.

Suppose next that e is not a pendant edge in T, and so both u and v have degree at least 2 in T. If $w \notin D$, then $\{u, v\} \subseteq D$ in order to 2-dominate the vertex w. If $w \in D$, then by the minimality of the set D, at most one of u and v belongs to D, and so in this case $|D \cap \{u, v, w\}| \leq 2$. Hence in both cases we have $|D \cap \{u, v, w\}| \in \{1, 2\}$.

Suppose that $|D \cap \{u, v, w\}| = 1$. From our earlier observations, we infer that $D \cap \{u, v, w\} = \{w\}$. In this case, the set $D^* = (D \setminus \{w\}) \cup \{v\}$ is a 2-dominating set of T, and so $\gamma_2(T) \le |D| = \gamma_2(T_e)$. Since the set D^* contains a vertex of status B, namely the vertex v, by part (a) the set D^* is not a γ_2-set of T, implying that $\gamma_2(T) < |D^*| = |D|$.

Suppose that $|D \cap \{u, v, w\}| = 2$. If $D \cap \{u, v, w\} = \{u, v\}$, then the set D is a 2-dominating set of T, and so $\gamma_2(T) \le |D| = \gamma_2(T_e)$. However, since $v \in D$, the set D is not a γ_2-set of T, implying that $\gamma_2(T) < |D|$. Hence we may assume that $D \cap \{u, v, w\} = \{u, w\}$ or $D \cap \{u, v, w\} = \{v, w\}$. If $u \in D$, then we let $D^* = (D \setminus \{w\}) \cup \{v\}$, while if $v \in D$, then we let $D^* = (D \setminus \{w\}) \cup \{u\}$. In both cases, the set D is a 2-dominating set of T, and so $\gamma_2(T) \le |D|$. Since the set D^* contains a vertex of status B, namely the vertex v, by part (a) the set D^* is not a γ_2-set of T, implying that $\gamma_2(T) < |D^*| = |D|$. In all the above cases, we have shown that $\gamma_2(T) < |D| = \gamma_2(T_e)$, implying that the tree T is γ_2-1-critical. This proves part (b). □

6.2 A characterization of γ_2-1-critical trees

Every γ_2-set of a tree T contains all its leaves, and so $L(T) \subseteq \mathcal{A}_2(T)$. Moreover if there is a γ_2-set, D, of a tree T that contains two adjacent vertices u and v, then the set D is a 2-dominating set of the tree T_e where $e = uv$. Thus, in this case subdividing the edge $e = uv$ does not increase the 2-domination number of the tree T. Hence if T is a tree and $\gamma_2(T) < \gamma_2(T_e)$ for every edge e of T, then the tree T has the following elementary properties.

Observation 6.3 *If a tree T is γ_2-1-critical, then the following properties hold.*
(a) $L(T) \subseteq \mathcal{A}_2(T)$.
(b) $S(T) \subseteq \mathcal{N}_2(T)$.
(c) *Each γ_2-set of T is an independent set.*

We are now in a position to characterize γ_2-1-critical trees. Recall the statement of Theorem 2.6.

Theorem 2.6. *A tree T is γ_2-1-critical if and only if $(T, S) \in \mathcal{F}$ for some labeling S.*

Proof. If $(T, S) \in \mathcal{F}$, then by Theorem 6.2(b) the tree T is γ_2-1-critical. Hence it suffices for us to show that if T is γ_2-1-critical tree, then $(T, S) \in \mathcal{F}$ for some labeling S. We proceed by induction on the order n of a γ_2-1-critical tree T.

If $n \in \{1, 2\}$, then T is not γ_2-1-critical. Hence, $n \ge 3$. If $n = 3$, then $T = P_3$ and the labeled tree $(T, S) \in \mathcal{F}$, where S is the labeling associated with the labeled base tree shown in Figure 1. This proves the base cases when $n \le 3$. Let $n \ge 4$ and assume that if T' is a γ_2-1-critical tree of order n' where $n' < n$, then $(T', S') \in \mathcal{F}$ for some labeling S'.

Suppose that T is a star, and so $T = K_{1,n-1}$. In this case, $\gamma_2(T) = n-1$ (with the set of leaves in T as the unique γ_2-set of T) and $\gamma_2(T_e) = n$ for every edge $e \in E(T)$,

implying that T is γ_2-1-critical. Moreover, if S is the labeling that labels the central vertex of the star with status B and labels all leaves with status A, then the labeled tree (T, S) can be obtained from the labeled base tree by repeated applications of Operation $\mathcal{O}_1$, implying that $(T, S) \in \mathcal{F}$. Hence, we may assume that T is not a star, for otherwise the desired result follows. Thus, $\operatorname{diam}(T) \geq 3$.

We now consider the γ_2-1-critical tree T of order $n \geq 4$ and diameter $\operatorname{diam}(T) \geq 3$. Among all longest paths in T, let $P\colon v_0v_1 \ldots v_d$ be chosen so that $\deg_T(v_1)$ is a maximum. We note that $d = \operatorname{diam}(T) \geq 3$. We now root the tree T at the vertex $r = v_d$. Necessarily, v_1 is a support vertex of T and all children of v_1 are leaves. In particular, v_0 is a leaf in T. We proceed further with a series of claims.

Claim 1 *If $\deg_T(v_1) = 2$, then $(T, S) \in \mathcal{F}$ for some labeling S.*

Proof. Suppose that $\deg_T(v_1) = 2$. Let D be an arbitrary γ_2-set of T. Since v_1 is a support vertex, by Observation 6.3(b), $v_1 \notin D$. In order to 2-dominate v_1, the vertex v_2 belongs to D. By Observation 6.3(b), we infer that the vertex v_2 is therefore not a support vertex of T. Hence, by our choice of the vertex v_1, every child of v_2 in the rooted tree T is a support vertex of degree 2. Let $\deg_T(v_2) = k + 1$, and so v_2 has $k \geq 1$ children. Further let $C(v_2) = \{p_1, \ldots, p_k\}$ be the set of children of v_2 in T and let $Q(v_2) = \{q_1, \ldots, q_k\}$ be the set of grandchildren of v_2 where q_i is the leaf neighbor of p_i for $i \in [k]$.

We now consider the tree T' obtained from T by deleting all children and grandchildren of v_2, that is, $T' = T - (C(v_2) \cup Q(v_2))$. Let T' have order n', and so $n' = n - 2k$. The vertex v_2 is a leaf in T', and therefore belongs to every γ_2-set of T'. Every γ_2-set of T' can be extended to a 2-dominating set of T by adding to it the k grandchildren of v_2, namely the leaves $q_1, \ldots, q_k$, and so $\gamma_2(T) \leq \gamma_2(T') + k$. On the other hand, by our earlier observations, the γ_2-set D of T contains the vertex v_2 and its k grandchildren, but contains no child of v_2. Hence the set $D \setminus Q(v_2)$ is a 2-dominating set of T', and so $\gamma_2(T') \leq \gamma_2(T) - |Q(v_2)| = \gamma_2(T) - k$. Consequently, $\gamma_2(T) = \gamma_2(T') + k$.

We show next that T' is a γ_2-1-critical tree. Let f be an arbitrary edge of T'. If subdividing the edge f in T' does not increase the 2-domination number of T', that is, if $\gamma_2(T'_f) \leq \gamma_2(T')$, then every γ_2-set of T'_f can be extended to a 2-dominating set of T_f (noting that the leaf v_2 in T' belongs to every γ_2-set of T'_f) by adding to it the k grandchildren of v_2 in T, and so $\gamma_2(T_f) \leq \gamma_2(T'_f) + k \leq \gamma_2(T') + k = \gamma_2(T)$, contradicting our supposition that T is γ_2-1-critical. Hence, T' is γ_2-1-critical. Applying the inductive hypothesis to the γ_2-1-critical T', the labeled tree $(T', S') \in \mathcal{F}$ for some labeling S'. Since v_2 is a leaf in T', by Observation 6.1(a), the vertex v_2 has status A in (T', S'). Applying k applications of Operation $\mathcal{O}_2$ to the labeled tree (T', S'), we can add back the k paths $v_2p_iq_i$ in turn for $i \in [k]$, and assign to vertex q_i the status A and to vertex p_i the status B, thereby producing a labeled tree $(T, S) \in \mathcal{F}$. (□)

By Claim 1, we may assume that $\deg_T(v_1) \geq 3$, for otherwise the desired result follows.

Claim 2 *If* $\deg_T(v_1) \geq 4$*, then* $(T, S) \in \mathcal{F}$ *for some labeling* S*.*

Proof. Suppose that $\deg_T(v_1) \geq 4$. Let $T' = T - v_0$ and let T' have order n', and so $n' = n - 1$. Every γ_2-set of T' can be extended to a 2-dominating set of T by adding to it the vertex v_0, and so $\gamma_2(T) \leq \gamma_2(T') + 1$. On the other hand, if D is a γ_2-set of T, then by our earlier observations the set $D \setminus \{v_0\}$ is a 2-dominating set of T', and so $\gamma_2(T') \leq \gamma_2(T) - 1$. Consequently, $\gamma_2(T) = \gamma_2(T') + 1$.

We show next that T' is a γ_2-1-critical tree. Let f be an arbitrary edge of T'. If subdividing the edge f in T' does not increase the 2-domination number of T', that is, if $\gamma_2(T'_f) \leq \gamma_2(T')$, then every γ_2-set of T'_f can be extended to a 2-dominating set of T_f by adding to it the leaf v_0, and so $\gamma_2(T_f) \leq \gamma_2(T'_f) + 1 \leq \gamma_2(T') + 1 = \gamma_2(T)$, contradicting our supposition that T is γ_2-1-critical. Hence, T' is γ_2-1-critical.

Applying the inductive hypothesis to the γ_2-1-critical T', the labeled tree (T', S') $\in \mathcal{F}$ for some labeling S'. Since T' is γ_2-1-critical, the support vertex v_1 in T' does not belong to any γ_2-set of T'. By Lemma 6.2(b), the set of vertices of status A in $(T', S') \in \mathcal{F}$ is the unique γ_2-set of T'. Therefore, the support vertex v_1 has status B in the labeled tree (T', S'). Applying Operation $\mathcal{O}_1$ to the labeled tree (T', S'), we can add back the deleted vertex v_0 and the edge v_0v_1, and assign to vertex v_0 the status A, thereby producing a labeled tree $(T, S) \in \mathcal{F}$. (□)

By Claim 2, we may assume that $\deg_T(v_1) = 3$, for otherwise the desired result follows. Let u_0 be the child of v_1, different from v_0. Let $T' = T - \{u_0, v_0, v_1\}$ and let T' have order n', and so $n' = n - 3$. Every γ_2-set of T' can be extended to a 2-dominating set of T by adding to it the vertices u_0 and v_0, and so $\gamma_2(T) \leq \gamma_2(T') + 2$. On the other hand, if D is a γ_2-set of T, then by Observation 6.3, $\{u_0, v_0\} \subset D$ and $v_1 \notin D$, implying that the set $D \setminus \{u_0, v_0\}$ is a 2-dominating set of T', and so $\gamma_2(T') \leq \gamma_2(T) - 2$. Consequently, $\gamma_2(T) = \gamma_2(T') + 2$.

We show next that T' is a γ_2-1-critical tree. Let f be an arbitrary edge of T'. If subdividing the edge f in T' does not increase the 2-domination number of T', that is, if $\gamma_2(T'_f) \leq \gamma_2(T')$, then every γ_2-set of T'_f can be extended to a 2-dominating set of T_f by adding to it the vertices u_0 and v_0, and so $\gamma_2(T_f) \leq \gamma_2(T'_f) + 2 \leq \gamma_2(T') + 2 = \gamma_2(T)$, contradicting our supposition that T is γ_2-1-critical. Hence, T' is γ_2-1-critical.

Applying the inductive hypothesis to the γ_2-1-critical T', the labeled tree (T', S') $\in \mathcal{F}$ for some labeling S'. If v_2 has status B in the labeled tree (T', S'), then applying Operation $\mathcal{O}_3$ to the labeled tree (T', S'), we can add back the deleted vertices u_0, v_0, v_1 and the edges $u_0v_1v_0$ and v_1v_2, and assign to vertex v_1 the status B and to the leaves u_0 and v_0 the status A, thereby producing a labeled tree $(T, S) \in \mathcal{F}$. If v_2 has status A in the labeled tree (T', S'), then applying Operation $\mathcal{O}_2$ to the labeled tree (T', S'), we can add back the deleted vertices v_0, v_1 and the edges v_0v_1 and v_1v_2, and assign to vertex v_1 the status B and to the leaf v_0 the status A, thereby producing a labeled tree $(T'', S'') \in \mathcal{F}$. Applying Operation $\mathcal{O}_1$ to the labeled tree (T'', S''), we can add back the deleted vertex u_0 and the edge u_0v_1, and assign to the leaf u_0 the status A, thereby producing a labeled tree $(T, S) \in \mathcal{F}$. Hence we have shown that if T is γ_2-1-critical tree, then $(T, S) \in \mathcal{F}$ for some labeling S. This completes the proof of Theorem 2.6. □

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