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A constructive characterization of 5-connected graphs

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Abstract

Let $\mathscr{H}_5$ denote the set of 5-connected graphs G which have a vertex x such that $G - x$ is a 4-connected 4-regular graph. Two edges of a graph are said to be "independent" if they have no common end vertex. A set of edges of a graph is said to be "semi-independent" if there exists a pair of edges having a common end vertex and all other pairs of edges in the set are independent. Let G be a 5-connected graph. We consider the following operation on G : (I) delete a vertex x from G, (II) add two new vertices x_1 and x_2, and (III) join x_i to $N_i \cup \{x_{3-i}\}$ for $i = 1, 2$, where $N_G(x) = N_1 \cup N_2$, $|N_i| \geq 4$ for $i = 1, 2$. We call this operation "vertex-splitting". If $|N_1 \cap N_2| \leq 1$, the vertex-splitting is said to be "semi-proper". We also consider the following operation on G: (I) choose 3 semi-independent edges of G, (II) subdivide each of the chosen edges by one vertex, and (III) identify the new vertices arising from the subdivision (we replace the resulting pair of multiple edges by a simple edge). We call this operation "pseudo edge-binding".

In this paper we prove that every 5-connected graph can be obtained from a graph in $\mathscr{H}_5$ by repeated applications of edge additions, semi-proper vertex-splittings and pseudo edge-bindings.

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1 Introduction

In this paper, we deal with finite undirected graphs with neither self-loops nor multiple edges, unless otherwise stated.

For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices of G and the set of edges of G, respectively. Let G be a graph. For a vertex $x \in V(G)$, let $N_G(x)$ denote the neighborhood of x in G and let $\deg_G(x)$ denote the degree of x. For a subset $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S. Let K_n and $K_{n,m}$ denote the complete graph on n vertices and the complete bipartite graph with partite sets of cardinalities n and m, respectively. Take a cycle and a vertex, and join the vertex to all vertices of the cycle. We call the resulting graph a *wheel*; in other words, a wheel G is a 3-connected graph which has a vertex x such that $G - x$ is a cycle. Let k be an integer such that $k \geq 3$. An edge of a k-connected graph G is said to be *k-removable* if its removal results in a k-connected graph. A k-connected graph is said to be *minimally* k-connected if it has no k-removable edges. Let $\mathscr{H}_k$ denote the set of k-connected graphs G which have a vertex x such that $G - x$ is a $(k-1)$-connected $(k-1)$-regular graph. Note that $\mathscr{H}_3$ is the set of wheels.

Let G be a k-connected graph. Tutte introduced the following operation on G: (I) delete a vertex x from G; (II) add two new vertices x_1 and x_2; (III) join x_i to $N_i \cup \{x_{3-i}\}$ for $i = 1, 2$, where $N_G(x) = N_1 \cup N_2$ and $|N_i| \geq k - 1$ for $i = 1, 2$. This operation is called *vertex-splitting*. If $N_1 \cap N_2 = \emptyset$ (or $|N_1 \cap N_2| \leq 1$), the vertex-splitting is said to be *proper* (or *semi-proper*).

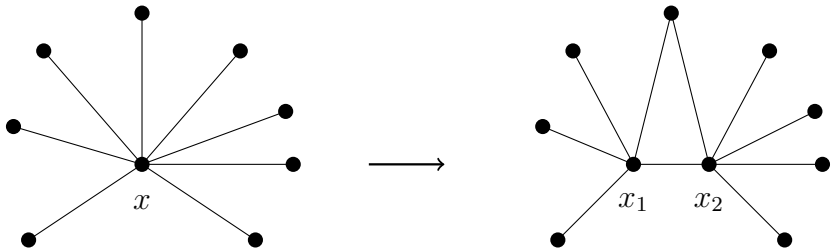

Figure 1: Semi-proper vertex-splitting ($k = 5$).

Tutte proved the following theorem, Tutte's Wheel Theorem, for 3-connected graphs. This is one of the most important generating theorems, which marks the beginning of a continual study of operations in graphs which preserve connectivity.

Theorem A [7] *Every 3-connected graph can be obtained from a graph in $\mathscr{H}_3$ by repeated applications of edge additions and proper vertex-splittings.*

Two edges of a graph are said to be *independent* if they have no common end vertex, and a set of edges of a graph is said to be *independent* if any two edges in the set are independent. A set of edges of a graph is said to be *semi-independent* if there exists a pair of edges having a common end vertex and all other pairs of edges in the set are independent. Let $k \geq 3$ be an integer, and let G be a k-connected graph. We

consider the following operation on G: (I) if k is even, then choose $\frac{k}{2}$ independent edges of G and, if k is odd, then choose $\lceil \frac{k}{2} \rceil$ semi-independent edges of G; (II) subdivide each of the chosen edges by one vertex; (III) identify the new vertices arising from the subdivision (in the case where k is odd, we replace the resulting pair of multiple edges by a simple edge). If k is even, then we call this operation *edge-binding* and if k is odd, then we call this operation *pseudo edge-binding*.

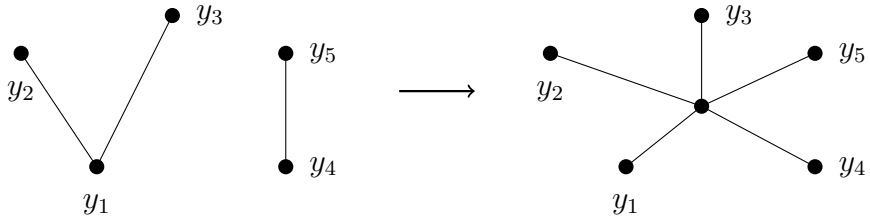

Figure 2: Pseudo edge-binding ($k = 5$).

We add that this operation with k even was first defined for multi-graphs (for multi-graphs, it is not necessary to choose independent edges, and we keep the resulting multiple edges as they are), and was used in a generating theorem for k-edge-connected k-regular multi-graphs which was obtained independently by Kotzig [4] and Simmons [6] (see also Problem 52 in Section 6 of Lovász [5]).

In an earlier paper, we gave the following 4-connected analogue of Tutte's Wheel Theorem.

Theorem B [1] *Every 4-connected graph can be obtained from a graph in $\mathscr{H}_4$ by repeated applications of edge additions, proper vertex-splittings and edge-bindings.*

In this paper, we prove the following 5-connected analogue of Tutte's Wheel Theorem.

Main Theorem *Every 5-connected graph can be obtained from a graph in $\mathscr{H}_5$ by repeated applications of edge additions, semi-proper vertex-splittings and pseudo edge-bindings. In other words, if G is a 5-connected graph, then there exists a sequence $G_1, G_2, \cdots, G_m$ $(m \geq 1)$ of 5-connected graphs with $G_1 \in \mathscr{H}_5$ and $G_m = G$ such that for each i $(2 \leq i \leq m)$, G_i is obtained from G_{i-1} by an edge addition, a semi-proper vertex-splitting or a pseudo edge-binding.*

In words, the Main Theorem shows that we can generate all 5-connected graphs starting with graphs in $\mathscr{H}_5$. The class $\mathscr{H}_5$ of graphs which we start with is rather large. However, in [2], we have recently obtained another theorem which generates all 4-connected 4-regular graphs from two graphs, K_5 and $K_{4,4}$. Thus the Main Theorem and the result in [2] together show that we can generate all 5-connected graphs from K_6 and the join of K_1 and $K_{4,4}$.

This paper consists of nine sections. In Section 2, after giving some notation, we introduce two operations, *semi-proper edge-contraction* and *proper lifting* which

are the inverse operations of *semi-proper vertex-splitting* and *pseudo edge-binding*, respectively. We let $\mathscr{B}(5)$ denote the set of connected non-tree graphs on five vertices with maximum degree 3. In Section 3, we classify the graphs in $\mathscr{B}(5)$, and we prove basic lemmas for investigating 5-connected graphs. In Section 4, we give some sufficient conditions for a degree 5 vertex of a 5-connected graph to be properly 5-liftable. In Section 5, we show that if G is a minimally 5-connected graph with $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$ and x is a vertex of G with $\deg_G(x) = 5$, then $G[N_G(x)]$ is not isomorphic to any of the graphs in $\mathscr{B}(5)$ (the term 5-*liftable* and the symbols $E^{**}(G)$ and $\mathscr{L}^*(G)$ will be defined in Section 2). In Sections 6 through 8, we show that a minimally 5-connected graph with $|V(G)| \geq 7$ and $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$ has a dominating vertex y and, if y dominates z, then y also dominates each neighbor of z other than y (the term *dominating* will be defined in Section 2). In Section 9, we give a proof of the Main Theorem.

We remark that arguments in Sections 3 through 6 are rather technical. Thus we suggest that after reading Section 2, readers not interested in technical details should read the first three paragraphs in Section 3, the statements of Lemmas 3.2, 3.4, 3.5 and 3.6, the first three paragraphs in Section 4 and the statements of Lemmas 5.3, 6.1 and 6.5, and proceed to Section 7 (Lemmas 4.3 and 4.5 are used in the proof of Lemma 8.3 (1) (i) in Section 8, but we believe that readers will understand the idea of the proof of Lemma 8.3 (1) (i) without referring to Lemmas 4.3 and 4.5).

2 Preliminaries

In this section, after giving some more notation, we introduce two operations, *semi-proper edge-contraction* and *proper lifting*, which are the inverse operations of *semi-proper vertex-splitting* and *pseudo edge-binding*, respectively. We also give some preliminary results.

Let G be a graph. For a subgraph A of G, when there is no ambiguity, we simply write A for $V(A)$. Thus $|A|$, $G - A$ and $N_G(A)$ mean $|V(A)|$ and $G - V(A)$ and $N_G(V(A))$, respectively. Also for a subgraph A of G and a subset S of $V(G)$, $A \cap S$ and $A \cup S$ stand for $V(A) \cap S$ and $V(A) \cup S$, respectively.

For a subset $S \subseteq V(G)$, let $N_G(S) = \cup_{x \in S} N_G(x) - S$. For a vertex $x \in V(G)$, let $E_G(x)$ denote the set of edges incident with x. Since G is simple, $\deg_G(x) = |N_G(x)| = |E_G(x)|$. For an edge e of G, let $V(e)$ denote the set of end vertices of e. For $X, Y \subseteq V(G)$ with $X \cap Y = \emptyset$, let $E_G(X, Y)$ denote the set of edges between X and Y. If $X = \{x\}$, we write $E_G(x, Y)$ for $E_G(\{x\}, Y)$. Let $\Delta(G)$ denote the maximum degree of G. Let $V_k(G)$ denote the set of vertices of G having degree k and let $V_{\geq k}(G)$ denote the set of vertices of G having degree at least k. Let $\kappa(G)$ denote the connectivity of G. For two graphs G and H, we let $G \cup H$ and $G + H$ denote the union of G and H, and the join of G and H, respectively. Moreover, for a positive integer m, let mG stand for the union of m copies of G.

Let G be a connected graph. A subset $S \subseteq V(G)$ is said to be a separator of G if $G - S$ is disconnected. If $|S| = s$, then a separator S is said to be an s-separator. A

separator S is said to be *trivial* if $G-S$ has a component of order one; S is said to be *nontrivial* if it is not trivial, i.e., if each component of $G-S$ has at least two vertices. Let S be a separator of G. An induced subgraph A of G is called an *S-fragment* of G if A is a nonempty union of components of $G-S$ such that $V(G)-A-S\neq\emptyset$. If A is an S-fragment of G, then $G-A-S$ is also an S-fragment of G. Let H be a subgraph of G and let S be a separator of G. We say that S *divides* H (or S is an *H-dividing separator*) if every component of $G-S$ intersects with H, i.e., $A\cap H\neq\emptyset$ for every S-fragment A of G. An induced subgraph A of G is called a *fragment* of G if A is an *S-fragment* for some $\kappa(G)$-separator S of G. In this case, having in mind the fact that $S=N_G(A)$, we let $\bar{A}$ stand for $G-A-S$. Note also that from $S=N_G(A)$, we see that neither $N_G(x)\cap A$ nor $N_G(x)\cap\bar{A}$ is empty for every $x\in S$. For a positive integer t, we let $\mathscr{F}_t(G)$ denote the set of fragments A of G such that $|A|,|\bar{A}|\geq t$. Thus $\mathscr{F}_1(G)$ is the set of fragments of G. A fragment $A\in\mathscr{F}_t(G)$ is said to be *$\mathscr{F}_t(G)$-minimum* if $|A|\leq|B|$ for any $B\in\mathscr{F}_t(G)$. For $x,y\in V(G)$ with $x\neq y$, if $N_G(x)\cup\{x\}\subseteq N_G(y)\cup\{y\}$, then we say that *$y$ dominates x* and, *x is dominated by y*. Note that if y dominates x, then $yx\in E(G)$. A vertex is said to be *dominating* if it dominates some other vertex.

Let $k\geq 3$ be a fixed integer and let G be a k-connected graph. An edge e of G is said to be *k-contractible* if G/e, the graph obtained from G by contracting e, is k-connected. Note that, in the contraction, we replace each resulting pair of multiple edges by a simple edge. If e is not k-contractible, then e is said to be *k-noncontractible*. Note that if $|V(G)|\geq k+2$, then e is k-noncontractible if and only if there exists a k-separator S of G which contains $V(e)$. Let $E_c(G)$ and $E_n(G)$ denote the set of k-contractible edges of G and the set of k-noncontractible edges of G, respectively. If at most one multiple edge arises in the contraction, then we say that the contraction is *semi-proper*. A k-contractible edge e is said to be *semi-properly k-contractible* if the contraction is semi-proper; in other words, if there is at most one triangle which contains e, then a k-contractible edge e is semi-properly k-contractible. Let $E^{**}(G)$ denote the set of semi-properly k-contractible edges of G. By definition, $xy\in E^{**}(G)$ if and only if $xy\in E_c(G)$ and $|N_G(x)\cap N_G(y)|\leq 1$. A semi-proper edge-contraction is the inverse operation of a semi-proper vertex-splitting.

Let again G be a k-connected graph. For $S\subseteq V(G)$, a set M of independent (resp. semi-independent) edges on S, not necessarily belonging to $E(G)$, is called a *perfect matching* (resp. *perfect pseudo-matching*) on S if $\cup_{e\in M}V(e)=S$. We consider the following operation on G: (I) delete a vertex x of degree k from G, (II) if k is even, then add a perfect matching M on $N_G(x)$ and, if k is odd, then add a perfect pseudo-matching M on $N_G(x)$, and (III) replace each resulting pair of multiple edges by a simple edge. We call this operation *lifting* on x with M and let $G^{(x;M)}$ denote the resulting graph. If no multiple edge arises in the lifting, i.e., $M\cap E(G)=\emptyset$, we call the lifting *proper*.

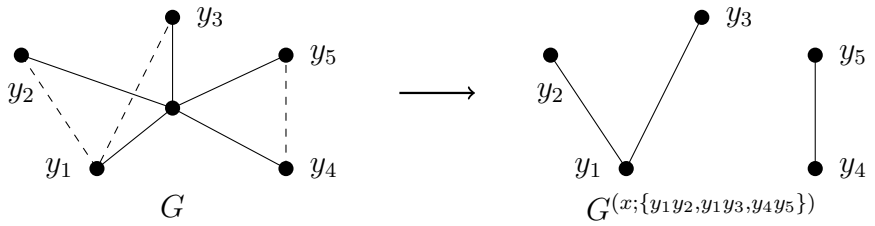

Figure 3: Proper lifting (dotted lines indicate that $\{y_1y_2, y_1y_3, y_4y_5\} \cap E(G) = \emptyset$).

A vertex x of G is said to be *k-liftable* if there is a lifting on x which preserves k-connectedness; in other words, if there is a perfect (semi-perfect) matching M on $N_G(x)$ such that $G^{(x;M)}$ is k-connected, then x is k-liftable. A vertex x of G is said to be *properly k-liftable* if there is a proper lifting on x which preserves k-connectedness. Let $\mathscr{L}^*(G)$ denote the set of properly k-liftable vertices of G. A proper lifting is the inverse operation of an edge-binding or a pseudo edge-binding according as k is even or odd. We now state lemmas, which we use in later sections. The following result is well known.

Lemma 2.1(Halin [3]) *Every minimally k-connected graph contains a vertex of degree k.* □

The following lemma can be found in [1].

Lemma 2.2 *Let G be a graph with $\kappa(G) \geq 3$. Let A be a fragment of G and let $U \subset N_G(A)$. If $|N_G(U) \cap A| < |U|$, then $A = N_G(U) \cap A$.* □

The following lemma can be found in [2].

Lemma 2.3 *Let G be a graph with $\kappa(G) \geq 3$. Let A be a $\mathscr{F}_2(G)$-minimum fragment of G. Then the following hold.*

(1) *If $|A| \geq 3$, then $|N_G(x) \cap A| \geq 2$ for any $x \in N_G(A)$.*
(2) *If $|A| \geq 4$, then $|N_G(\{x, y\}) \cap A| \geq 3$ for any $\{x, y\} \subseteq N_G(A)$.*

The following lemma summarizes elementary properties of dominating vertices in minimally k-connected graphs.

Lemma 2.4 *Let $k \geq 3$, and let G be a minimally k-connected graph with $|G| \geq k+2$. Suppose that G has two vertices y, z such that y dominates z.*

(1) *If G has a k-separator S which contains z, then $y \in S$.*
(2) *$N_G(z) \cup \{z\} - \{y\} \subseteq V_k(G)$.*
(3) *$y \in V_{\geq k+1}(G)$.*
(4) *There is no vertex other than y which dominates a vertex of $N_G(z) - \{y\}$.*

Proof. (1) Let A be an S-fragment of G. Since $z \in S$ and S is a k-separator of G, neither $N_G(z) \cap A$ nor $N_G(z) \cap \bar{A}$ is empty. Suppose that $y \notin S$, say $y \in A$. Since S

separates A and $\bar{A}$, we have $N_G(y) \cap \bar{A} = \emptyset$. Since $N_G(z) \cap \bar{A} \neq \emptyset$, this contradicts the assumption that y dominates z. This contradiction shows $y \in S$ and (1) is shown.

(2) Suppose that $N_G(z) \cup \{z\} - \{y\} \not\subseteq V_k(G)$, say $x \in (N_G(z) \cup \{z\} - \{y\}) \cap V_{\geq k+1}(G)$. Since y dominates z, $x \in N_G(y)$. Let $G' = G - xy$. Since G is minimally k-connected, G' has a $(k-1)$-separator S' which separates x and y. Let A' be the S'-fragment of G' which contains x. Since $x \in V_{\geq k+1}(G)$, we see that $x \in V_{\geq k}(G')$, which implies that $|A'| \geq 2$. Hence $S = S' \cup \{x\}$ is a k-separator of G such that $y \notin S$. We show that $z \in S$. If $x = z$, then $z \in \{x\} \subseteq S$. Hence we may assume $x \neq z$. Then $x \in N_G(z)$, which implies that $xz \in E(G')$. Since $xz, yz \in E(G')$ and S' separates x and y, we see that $z \in S' \subseteq S$. Thus it is shown that $z \in S$. Since $y \notin S$, this contradicts (1), and this contradiction shows (2).

(3) Suppose that $y \in V_k(G)$. Since $N_G(y) \cup \{y\} \supseteq N_G(z) \cup \{z\}$, we observe that $N_G(\{z, y\}) = N_G(z) \cap N_G(y)$. Since $|G| \geq k+2$ and $|N_G(z) \cap N_G(y)| = k-1$, we see that $V(G) - (N_G(\{z, y\}) \cup \{z, y\}) \neq \emptyset$. Hence $N_G(z) \cap N_G(y)$ separates $\{z, y\}$ and $V(G) - (N_G(\{z, y\}) \cup \{z, y\})$, which contradicts the k-connectedness of G. This contradiction shows $y \in V_{\geq k+1}(G)$ and (3) is shown.

(4) Assume that $w \in V(G)$ dominates $x \in N_G(z) - \{y\}$. We show $w = y$. Suppose that $w \neq y$. Since y dominates z and $x \in N_G(z) - \{y\}$, we observe that $xy \in E(G)$. Since w dominates x, (2) assures us that $N_G(x) - \{w\} \subseteq V_k(G)$. Since $y \in N_G(x) - \{w\}$, this contradicts (3). This contradiction shows $w = y$ and (4) is shown. □

We conclude this section by introducing a notion which we will use in later sections. Let now H be a graph. A triplet $(A; B; C)$ is said to be a *separating triplet of* H if (i) A, B and C are mutually disjoint subsets of $V(H)$, (ii) $V(H) = A \cup B \cup C$, (iii) neither A nor C is empty, and (iv) $E_H(A, C) = \emptyset$. Thus $(A; B; C)$ is a separating triplet if and only if B is a separator and A and C are (the vertex sets of) disjoint B-fragments such that $A \cup C = V(H) - B$. We regard two separating triplets $(A; B; C)$ and $(C; B; A)$ as the same. Let $\mathscr{T}(H)$ denote the set of separating triplets of H.

3 5-connected graphs

In the first half of this section, we classify the connected non-tree graphs on 5 vertices whose maximum degree is three (see Lemma 5). In the latter half of this section, we give some basic tools for investigating 5-connected graphs.

Let G be a 5-connected graph. For a given vertex $x \in V(G)$ and $W \subseteq V(G)$, we let $W_x = N_G(x) \cap W$. For a given vertex $x \in V(G)$ and an induced subgraph A of G, we let A_x denote $V(A)_x$ or $G[V(A)_x]$; in particular, G_x denotes $G[N_G(x)]$.

Let $\mathscr{B}(5)$ be the set of connected non-tree graphs on 5 vertices whose maximum degree is three. To describe $\mathscr{B}(5)$, we need to introduce some specific graphs on 5 vertices. For a graph G, we denote the complementary graph of G by $\tilde{G}$, and let $\mathrm{diam}(G)$ denote the diameter of G. A tree G with $\mathrm{diam}(G) = 2$ is called a *star*. In

other words, a star is a complete bipartite graph $K_{1,n}$. A tree G with $\mathrm{diam}(G) = 3$ is called a *double star*. Each double star is obtained from two stars $K_{1,n}$ and $K_{1,m}$ by joining their centers. We denote the resulting double star by $D_{n,m}$. By the definition, $D_{1,1} = P_4$. Let H be the graph on 4 vertices obtained from K_3 by adding one pendant vertex, i.e., $H = K_1 + (K_1 \cup K_2)$. We observe that H has one pendant edge and three non-pendant edges. Let Ch (Chair) stand for the graph obtained from H by subdividing one of its non-pendant edges by one vertex. We observe that the complementary graph of Ch is isomorphic to the graph obtained from H by subdividing its pendant edge by one vertex. Let Bu (Bull) stand for the graph obtained from H by adding another pendant vertex so that $\Delta(Bu) = 3$. We observe that Bu is a self-complementary graph.

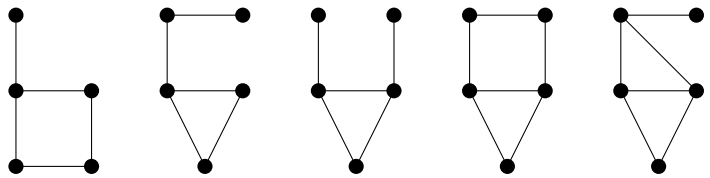

Figure 4: Some graphs in $\mathscr{B}(5)$.

Lemma 3.1 $\mathscr{B}(5) = \{Ch, \widetilde{Ch}, Bu, \tilde{D}_{1,2}, \tilde{P}_5, K_{2,3}, 2K_1 + (K_1 \cup K_2)\}$.

Proof. Let G be a connected graph on 5 vertices with $\Delta(G) = 3$. We take into consideration the condition that G is not a tree only in the last stage of the proof. Let $V(G) = \{x_1, x_2, \cdots, x_5\}$. Since $\Delta(G) = 3$, we observe that $V_3(G) \neq \emptyset$.

First we consider the case where G has an edge whose end vertices both have degree 3, say $x_1, x_2 \in V_3(G)$ and $x_1x_2 \in E(G)$. Since $|G| = 5$, we see that $N_G(x_1) \cap N_G(x_2) \neq \emptyset$. Since $x_1x_2 \in E(G)$, $|N_G(x_1) \cap N_G(x_2)| \leq 2$. We consider the subcase where $|N_G(x_1) \cap N_G(x_2)| = 2$, say $N_G(x_1) \cap N_G(x_2) = \{x_3, x_4\}$. Since G is connected, we see that $N_G(x_5) \cap \{x_3, x_4\} \neq \emptyset$. Hence we have either $G \cong 2K_1 + (K_1 \cup K_2)$ ($|N_G(x_5) \cap \{x_3, x_4\}| = 2$) or $G \cong \tilde{D}_{1,2}$ ($|N_G(x_5) \cap \{x_3, x_4\}| = 1$). We consider the subcase where $|N_G(x_1) \cap N_G(x_2)| = 1$, say $N_G(x_1) \cap N_G(x_2) = \{x_3\}$. We may assume that $N_G(x_1) = \{x_2, x_3, x_4\}$ and $N_G(x_2) = \{x_1, x_3, x_5\}$. If $x_3 \in V_3(G)$, then we have either $|N_G(x_1) \cap N_G(x_3)| = 2$ or $|N_G(x_2) \cap N_G(x_3)| = 2$, and are reduced to the previous subcase. Thus we may assume that $x_3 \in V_2(G)$ and $N_G(x_3) = \{x_1, x_2\}$. Thus we have either $G \cong \tilde{P}_5$ ($x_4x_5 \in E(G)$) or $G \cong Bu$ ($x_4x_5 \notin E(G)$).

Next we consider the case where $V(e) \not\subseteq V_3(G)$ for any $e \in E(G)$. Let $x_1 \in V_3(G)$ and let $N_G(x_1) = \{x_2, x_3, x_4\}$. Then $\deg_G(x_i) \in \{1, 2\}$ for $i = 2, 3, 4$, which implies that $|E(G[\{x_2, x_3, x_4\}])| \leq 1$. We consider the subcase where $|E(G[\{x_2, x_3, x_4\}])| = 1$, say $E(G[\{x_2, x_3, x_4\}]) = \{x_2x_3\}$. Since $\deg_G(x_2) = \deg_G(x_3) = 2$ and G is connected, we observe that $x_4x_5 \in E(G)$. Hence, in this subcase, we have $G \cong \widetilde{Ch}$. Finally we consider the subcase where $E(G[\{x_2, x_3, x_4\}]) = \emptyset$. Since G is connected, we see that $N_G(x_5) \cap \{x_2, x_3, x_4\} \neq \emptyset$. Hence, in this subcase, we have

$G \cong K_{2,3}$ ($\{x_2, x_3, x_4\} \subseteq N_G(x_5)$), $G \cong Ch$ ($|N_G(x_5) \cap \{x_2, x_3, x_4\}| = 2$) or $G \cong D_{1,2}$ ($|N_G(x_5) \cap \{x_2, x_3, x_4\}| = 1$).

Now we have checked all cases, and we have shown that, except for $D_{1,2}$, which is a tree, no graph other than the desired 7 graphs occurs. Hence we obtain the desired conclusion that $\mathscr{B}(5) = \{Ch, \widetilde{Ch}, Bu, \tilde{D}_{1,2}, \tilde{P}_5, K_{2,3}, 2K_1 + (K_1 \cup K_2)\}$. □

By inspection, we see that there are two connected graphs G on 5 vertices with $\Delta(G) \leq 2$, P_5 and C_5. Thus we see from the proof of Lemma 3.1 that the set of connected graphs G on 5 vertices with $\Delta(G) \leq 3$ is $\mathscr{B}(5) \cup \{D_{1,2}, P_5, C_5\}$.

We proceed to prove lemmas concerning 5-separators in 5-connected graphs. The main result is Lemma 3.6.

Lemma 3.2 *Let G be a 5-connected graph and let S and T be 5-separators of G. Let A and B be an S-fragment and a T-fragment, respectively.*

(1) *If $A \cap B \neq \emptyset$, then $(S \cap B) \cup (S \cap T) \cup (A \cap T)$ is a separator separating $A \cap B$ and $\bar{A} \cup \bar{B}$.*

(2) $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| - 5 = |A \cap T| - |S \cap \bar{B}| = |S \cap B| - |\bar{A} \cap T|$.

(3) *If $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| = 5$ and $A \cap B \neq \emptyset$, then $(S \cap B) \cup (S \cap T) \cup (A \cap T)$ is a 5-separator separating $A \cap B$ and $\bar{A} \cup \bar{B}$; that is to say, $A \cap B$ is a fragment of G and we have $N_G(A \cap B) = (S \cap B) \cup (S \cap T) \cup (A \cap T)$ and $\overline{A \cap B} = \bar{A} \cup \bar{B}$.*

(4) *If $|A \cap T| > |S \cap \bar{B}|$ or $|S \cap B| > |\bar{A} \cap T|$, then $\bar{A} \cap \bar{B} = \emptyset$; that is to say, if $\bar{A} \cap \bar{B} \neq \emptyset$, then $|S \cap \bar{B}| \geq |A \cap T|$ and $|\bar{A} \cap T| \geq |S \cap B|$.*

(5) *If $A \cap B \neq \emptyset$ and $\bar{A} \cap \bar{B} \neq \emptyset$, then $(S \cap B) \cup (S \cap T) \cup (A \cap T)$ is a 5-separator separating $A \cap B$ and $\bar{A} \cup \bar{B}$, and $(S \cap \bar{B}) \cup (S \cap T) \cup (\bar{A} \cap T)$ is a 5-separator separating $\bar{A} \cap \bar{B}$ and $A \cup B$.*

(6) *Suppose $S \cap T \neq \emptyset$, and let $x \in S \cap T$. If $(A \cap B)_x = \emptyset$ and $A \cap B \neq \emptyset$, then we have $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \geq 6$, $|A \cap T| > |S \cap \bar{B}|$, $|S \cap B| > |\bar{A} \cap T|$ and $\bar{A} \cap \bar{B} = \emptyset$.*

(7) *Suppose $S \cap T \neq \emptyset$, and let $x \in S \cap T$. If $\bar{A} \cap \bar{B} \neq \emptyset$ and $(A \cap B)_x = \emptyset$, then $A \cap B = \emptyset$.*

(8) *For each $y \in A$, $|A| \geq |(S \cup A) - N_G(y)|$; in particular, $|A| \geq |S - N_G(y)| + 1$.*

Proof. Statements (1) through (3) are easy observations. Set $S' = (S \cap B) \cup (S \cap T) \cup (A \cap T)$ and $S'' = (S \cap \bar{B}) \cup (S \cap T) \cup (\bar{A} \cap T)$. If we have $|A \cap T| > |S \cap \bar{B}|$ or $|S \cap B| > |\bar{A} \cap T|$, then applying (2) with A and B replaced by $\bar{A}$ and $\bar{B}$, respectively, we get $|S''| < 5$ and, applying (1) with A and B replaced by $\bar{A}$ and $\bar{B}$, respectively, we get $\bar{A} \cap \bar{B} = \emptyset$. This prove (4). We now prove (5). Suppose that $A \cap B \neq \emptyset$ and $\bar{A} \cap \bar{B} \neq \emptyset$. By (4), $|S \cap \bar{B}| \geq |A \cap T|$. Applying (4) with A and B replaced by $\bar{A}$ and $\bar{B}$, respectively, we also get $|A \cap T| \geq |S \cap \bar{B}|$. Hence $|S'| = 5$ by (2). Similarly $|S''| = 5$. Consequently, the desired conclusion follows from (3). To prove (6), suppose that $(A \cap B)_x = \emptyset$ and $A \cap B \neq \emptyset$. By (1), $|S'| \geq 5$. Suppose that $|S'| = 5$. Then, by (3), $N_G(A \cap B) = S'$. Since $x \in S'$, it follows that $(A \cap B)_x \neq \emptyset$, which contradicts the assumption that $(A \cap B)_x = \emptyset$. Thus $|S'| \geq 6$, and the

desired conclusion follows from (2) and (4). Statement (7) follows from (6). We show (8). If $y \in A$, then since $N_G(y) \subseteq S \cup A$ and $|N_G(y)| \geq 5$, it follows that $|(S \cup A) - N_G(y)| = |S| + |A| - |N_G(y)| \leq 5 + |A| - 5 = |A|$, which proves (8). □

Lemma 3.3 *Let G be a 5-connected graph and let $x \in V(G)$. Let S and T be 5-separators of G such that $x \in S \cap T$. Let A and B be an S-fragment and a T-fragment, respectively, and suppose that $|\bar{A}| \geq 2$, $|\bar{B}| \geq 2$ and $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \geq 6$. Then $\bar{A} \cap B \neq \emptyset$ and $A \cap \bar{B} \neq \emptyset$.*

Proof. By Lemma 3.2 (2), (4), $|A \cap T| > |S \cap \bar{B}|$, $|S \cap B| > |\bar{A} \cap T|$ and $\bar{A} \cap \bar{B} = \emptyset$. Suppose that $\bar{A} \cap B = \emptyset$. Then $\bar{A} = \bar{A} \cap T$, which implies $|\bar{A} \cap T| = |\bar{A}| \geq 2$. Since $|S \cap B| > |\bar{A} \cap T|$, we have $|S \cap B| \geq 3$, and it follows that $|S \cap \bar{B}| = 5 - |S \cap T| - |S \cap B| \leq 1$. Consequently $|S \cap \bar{B}| < |\bar{A} \cap T|$. Applying Lemma 3.2 (4) with the roles of A and $\bar{A}$ replaced by each other, we get $A \cap \bar{B} = \emptyset$. Therefore $\bar{B} = S \cap \bar{B}$, which implies $|\bar{B}| = |S \cap \bar{B}| \leq 1$. This contradicts the assumption that $|\bar{B}| \geq 2$. Thus $\bar{A} \cap B \neq \emptyset$. By the symmetry of A and B, we also obtain $A \cap \bar{B} \neq \emptyset$, as desired. □

Lemma 3.4 *Let G be a 5-connected graph. Let $x \in V(G)$, let S be a 5-separator with $x \in S$, and let A be an S-fragment. Let T be a 5-separator with $\{x\} \cup A_x \subseteq T$, and let B be a T-fragment. Suppose that $|B| = 1$, and write $B = \{y\}$. Then $y \in S_x \cap V_5(G)$ and $T = N_G(y)$.*

Proof. Since $T = N_G(B)$, we have $T = N_G(y)$. Hence $y \in V_5(G)$, $y \in N_G(x)$ and $y \in N_G(A_x)$. From $y \in N_G(A_x)$, we get $y \in S \cup A$, which implies $y \in (S \cup A)_x$. Since $A_x \subseteq T$ and $y \notin T$, we see that $y \notin A_x$. Consequently $y \in S_x$, as desired. □

Lemma 3.5 *Let G be a 5-connected graph. Let $x \in V(G)$ and let S be a 5-separator such that $x \in S$. Let A be an S-fragment of G such that $|A| = 2$. Let T be a 5-separator such that $\{x\} \cup A_x \subseteq T$ and let B be a T-fragment of G. Then the following hold.*

(1) *$A \subseteq T$.*
(2) *If $|B|, |\bar{B}| \geq 2$, then $|S \cap B| = |S \cap \bar{B}| = 2$ and $S \cap T = \{x\}$.*

Proof. (1) If $|A_x| = 2$, then we have $A = A_x \subseteq T$. Thus we may assume that $|A_x| = 1$, say $A_x = \{y\}$. Let $A = \{y, u\}$. We show $u \in A \cap T$. Suppose that $u \notin A \cap T$, say $u \in A \cap B$, which implies $A \cap B = \{u\}$, $A \cap T = \{y\}$ and $A \cap \bar{B} = \emptyset$. Since $A_x = \{y\}$, we observe that $(A \cap B)_x = \emptyset$. Hence, by Lemma 3.2 (6), $\bar{A} \cap \bar{B} = \emptyset$ and $|S \cap \bar{B}| < |A \cap T|$, which implies $S \cap \bar{B} = \emptyset$ because $|A \cap T| = 1$. Consequently $\bar{B} = \emptyset$. This contradicts the fact that B is a fragment, and this contradiction shows $u \in A \cap T$.

(2) Assume $|B|, |\bar{B}| \geq 2$. We show $|S \cap B| \geq 2$. Suppose that $|S \cap B| \leq 1$. Since $|A \cap T| = 2$, applying Lemma 3.2 (4), we see that $\bar{A} \cap B = \emptyset$. Since $A \cap B = \emptyset$, we observe that $B = S \cap B$ and $|B| = |S \cap B| \leq 1$, which contradicts the assumption that $|B| \geq 2$. This contradiction shows $|S \cap B| \geq 2$. By symmetry, we also obtain $|S \cap \bar{B}| \geq 2$. Since $x \in S \cap T$, we have $|S \cap B| = |S \cap \bar{B}| = 2$ and $S \cap T = \{x\}$. □

The following lemma summarizes important properties of 5-separators in 5-connected graphs.

Lemma 3.6 *Let G be a 5-connected graph and let $x \in V(G)$. Let S be a 5-separator of G which contains x and let A be an S-fragment such that $|\bar{A}| \geq 2$. Suppose that G has a 5-separator T which contains $\{x\} \cup A_x$, and let B be a T-fragment of G.*

(1) *If $|A| \geq 3$, then $|B| = 1$ or $|\bar{B}| = 1$. Further if $|B| = 1$ and if we write $B = \{y\}$, then $y \in S_x \cap V_5(G)$ and $T = N_G(y)$.*
(2) *If $|A| \geq 3$, then $|S \cap T| \leq 2$. Further if $|S \cap T| = 2$, then $\bar{A}_x - T \neq \emptyset$.*
(3) *If $|A| \geq 2$ and $|B|, |\bar{B}| \geq 2$, then $|A| = 2$ and $S \cap T = \{x\}$.*

Proof. Since T contains $\{x\} \cup A_x$, we observe that $(A \cap B)_x = (A \cap \bar{B})_x = \emptyset$.

Claim 3.6.1 Suppose that $|B|, |\bar{B}| \geq 2$. Then $A \cap B = A \cap \bar{B} = \emptyset$.

Proof. By way of contradiction, suppose that $A \cap B \neq \emptyset$. Then by Lemma 3.2 (6), $\bar{A} \cap \bar{B} = \emptyset$ and $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \geq 6$. Hence by Lemma 3.3, $A \cap \bar{B} \neq \emptyset$. Applying Lemma 3.2 (6) and Lemma 3.3 with the roles of B and $\bar{B}$ replaced by each other, we now get $\bar{A} \cap \bar{B} \neq \emptyset$, which contradicts the assertion that $\bar{A} \cap \bar{B} = \emptyset$. Thus $A \cap B = \emptyset$. By the symmetry of B and $\bar{B}$, we also obtain $A \cap \bar{B} = \emptyset$. □

We show (1). Thus assume that $|A| \geq 3$. In view of Lemma 3.4, it suffices to show that we have $|B| = 1$ or $|\bar{B}| = 1$. Suppose that $|B| \geq 2$ and $|\bar{B}| \geq 2$. Then Claim 3.6.1 assures us that $A \cap B = A \cap \bar{B} = \emptyset$, which implies $A = A \cap T$ and $|A \cap T| = |A| \geq 3$. Since $x \in S \cap T$, we observe that $|\bar{A} \cap T| = |T| - |S \cap T| - |A \cap T| \leq 1$. Since $x \in S \cap T$, we also observe that either $|S \cap B| \leq 2$ or $|S \cap \bar{B}| \leq 2$, say $|S \cap B| \leq 2$. Since $|A \cap T| \geq 3$, applying Lemma 3.2 (4), we see that $\bar{A} \cap B = \emptyset$. Consequently $B = S \cap B$ and $|S \cap B| = |B| \geq 2$. Since $|\bar{A} \cap T| \leq 1$, applying Lemma 3.2 (4), we see that $\bar{A} \cap \bar{B} = \emptyset$. Since $\bar{A} \cap B = \emptyset$, we see that $\bar{A} = \bar{A} \cap T$ and $|\bar{A}| = |\bar{A} \cap T| \leq 1$, which contradicts the assumption that $|\bar{A}| \geq 2$. This contradiction shows that if $|A| \geq 3$, then $|B| = 1$ or $|\bar{B}| = 1$.

Next we show (2). Thus again assume that $|A| \geq 3$. By (1), we may assume that $|B| = |S \cap B| = 1$. Then $A \cap B = \bar{A} \cap B = \emptyset$. We show that $|S \cap T| \leq 2$. We may assume that $|S \cap T| \geq 2$. Note that if $\bar{A} \cap T = \emptyset$, then $|\bar{A} \cap T| < 1 = |S \cap B|$, and hence $\bar{A} \cap \bar{B} = \emptyset$ by Lemma 3.2 (4), which implies $\bar{A} = \emptyset$, contradicting the definition of a fragment. Hence $\bar{A} \cap T \neq \emptyset$. Since $|S \cap T| \geq 2$, it follows that $|A \cap T| = |T| - |S \cap T| - |\bar{A} \cap T| \leq 2$. Since $A \cap B = \emptyset$ and $|A| \geq 3$, we observe that $A \cap \bar{B} \neq \emptyset$. Since $(A \cap \bar{B})_x = \emptyset$, applying Lemma 3.2 (6), we see that $|A \cap T| \geq |S \cap B| + 1 = 2$. Since $\bar{A} \cap T \neq \emptyset$, we get $|S \cap T| = |T| - |A \cap T| - |\bar{A} \cap T| \leq 2$. Now the first assertion of (2) is shown.

To show the second assertion of (2), assume that $|S \cap T| = 2$. Then by the argument in the preceding paragraph, $|\bar{A} \cap T| = 1$. Since $|\bar{A}| \geq 2$ and $\bar{A} \cap B = \emptyset$, it follows that $\bar{A} \cap \bar{B} \neq \emptyset$. If $(\bar{A} \cap \bar{B})_x = \emptyset$, then by Lemma 3.2 (6), $|\bar{A} \cap T| > |S \cap B|$, which contradicts the fact that $|\bar{A} \cap T| = 1 = |S \cap B|$. Thus $(\bar{A} \cap \bar{B})_x \neq \emptyset$, which implies $\bar{A}_x - T \neq \emptyset$. This shows the second assertion of (2).

We show (3). Thus assume that $|A| \geq 2$ and $|B|, |\bar{B}| \geq 2$. Then (1) assures us that $|A| \leq 2$, which implies $|A| = 2$. Hence $S \cap T = \{x\}$ by Lemma 3.5 (2), and (3) is shown. □

Lemma 3.6 has the following consequence.

Lemma 3.7 *Let G be a 5-connected graph and let $x \in V(G)$. Let S be a 5-separator of G which contains x and let A be an S-fragment such that $|\bar{A}| \geq 2$. Suppose $|A_x| = 1$, say $A_x = \{y\}$. Further suppose that xy is 5-noncontractible.*

(1) *If $|A| \geq 3$, then $S_x \cap N_G(y) \cap V_5(G) \neq \emptyset$.*
(2) *If $|A| \geq 3$ and $z \in S_x \cap N_G(y) \cap V_5(G)$, then $|S \cap N_G(z)| \leq 2$. Further if $|N_G(z) \cap S| = 2$, then $\bar{A}_x - N_G(z) \neq \emptyset$.*
(3) *If $S - N_G(y) \neq \emptyset$ and there exist a 5-separator T and a T-fragment B such that $x, y \in T$, $|B| \geq 2$ and $|\bar{B}| \geq 2$, then $|A| = 2$, $S \cap T = \{x\}$, $y \in V_5(G)$ and $\Delta(G_y) = 3$.*

Proof. Since xy is 5-noncontractible, there is a 5-separator T which contains $\{x, y\}$. Since $A_x = \{y\}$ and $\{y\} \subseteq T$, we see that T contains $\{x\} \cup A_x$.

First we show (1). Assume $|A| \geq 3$. Applying Lemma 3.6 (1), we see that there is a vertex $z \in S_x \cap V_5(G)$ such that $T = N_G(z)$. Since $y \in T$, we observe that $z \in N_G(y)$. Now (1) is shown.

Next we show (2). Assume that $|A| \geq 3$ and $z \in S_x \cap N_G(y) \cap V_5(G)$. Then $N_G(z)$ is a 5-separator containing $\{x, y\}$. Thus we may assume that $T = N_G(z)$. Hence applying Lemma 3.6 (2), we see that $|S \cap N_G(z)| \leq 2$ and, if $|S \cap N_G(z)| = 2$, then $\bar{A}_x - N_G(z) \neq \emptyset$. Now (2) is shown.

Finally we show (3). Assume that $S - N_G(y) \neq \emptyset$ and B is a T-fragment with $|B|, |\bar{B}| \geq 2$. Since $S - N_G(y) \neq \emptyset$, Lemma 3.2 (8) assures us that $|A| \geq 2$. Applying Lemma 3.6 (3), we see that $S \cap T = \{x\}$ and $|A| = 2$, say $A = \{y, u\}$. Since $S - N_G(y) \neq \emptyset$, $|N_G(y)| \leq |S \cup A| - |\{y\}| - |S - N_G(y)| \leq 5$, which implies $y \in V_5(G)$. Hence, to show (3), it suffices to prove $\Delta(G_y) = 3$. Since $A_x = \{y\}$, we observe that $xu \notin E(G)$. Let $S = \{x, v_1, v_2, v_3, v_4\}$. Since $N_G(u) \subseteq S \cup A$, it follows that $N_G(u) = \{y, v_1, v_2, v_3, v_4\}$. In view of Lemma 3.5 (2), we may assume that $S \cap B = \{v_1, v_2\}$ and $S \cap \bar{B} = \{v_3, v_4\}$. Recall that $S - N_G(y) \neq \emptyset$. By the symmetry of B and $\bar{B}$, we may assume $yv_4 \notin E(G)$. Then $N_G(y) = \{x, u, v_1, v_2, v_3\}$. Since $xu \notin E(G_y)$, we get $\deg_{G_y}(x) \leq 3$. Since $uv_1, uv_2, uv_3 \in E(G_y)$, we also get $\deg_{G_y}(u) = 3$. Since $E_G(B, \bar{B}) = \emptyset$, we have $E_{G_y}(v_3, \{v_1, v_2\}) = \emptyset$, which implies $\deg_{G_y}(v_i) \leq 3$ for each $i \in \{1, 2, 3\}$. Consequently $\Delta(G_y) = 3$ and (3) is shown. □

4 Properly liftable vertices

In this section, we introduce some more notations concerning the neighborhood of a degree 5 vertex of a 5-connected graph. We also give some sufficient conditions for a

degree 5 vertex to be properly 5-liftable. The contents of this section are of technical nature, but important results are Lemma 4.3, Lemma 4.4 and Lemma 4.5.

Let G be a 5-connected graph and let $x \in V_5(G)$. Recall a lifting on $x \in V(G)$ is an operation in which, we (I) delete x from G, (II) add a perfect pseudo-matching M on $N_G(x)$, and (III) replace each resulting pair of multiple edges by a simple edge. Also recall that $G^{(x;M)}$ denotes the resulting graph. Let $N_G(x) = \{y_1, y_2, y_3, y_4, y_5\}$. Let $G_x^{(x;M)}$ denotes $G^{(x;M)}[N_G(x)]$. A perfect pseudo-matching $M = \{y_1y_2, y_2y_3, y_4y_5\}$ is said to be a *perfect non-edge pseudo-matching on* $N_G(x)$ if $\{y_1y_2, y_2y_3, y_4y_5\} \cap E(G) = \emptyset$. If there is no danger of confusion, we write $\{y_1y_2y_3, y_4y_5\}$ for $\{y_1y_2, y_2y_3, y_4y_5\}$. We see that a lifting on x with M is proper if M is a perfect non-edge pseudo-matching on $N_G(x)$.

Let M be a perfect non-edge pseudo-matching on $N_G(x)$. Let $\mathscr{S}(G^{(x;M)})$ be the set of separating triplets $(A; S'; B)$ of $G^{(x;M)}$ with $|S'| = 4$. Note that if $(A; S'; B) \in \mathscr{S}(G^{(x;M)})$, then $B = G^{(x;M)} - A - N_{G^{(x;M)}}(A) = \bar{A}$. Let $\mathscr{S}_x(G)$ be the set of separating triplets $(A; S; B)$ of G with $|S| = 5$ and $x \in S$. Note that if $(A; S; B) \in \mathscr{S}_x(G)$, then $B = G - A - N_G(A) = \bar{A}$ (if $(A; S'; B) \in \mathscr{S}(G^{(x;M)})$, then $(A; S' \cup \{x\}; B) \in \mathscr{S}_x(G)$ and $G^{(x;M)} - A - N_{G^{(x;M)}}(A) = G - A - N_G(A)$, and thus the notation $\bar{A}$ will not cause any confusion). Let $\mathscr{S}_x(G; M) = \{(A; S' \cup \{x\}; B) | (A; S'; B) \in \mathscr{S}(G^{(x;M)})\}$. Note that $\mathscr{S}_x(G; M) \subseteq \mathscr{S}_x(G)$.

For $\mathfrak{X} \subseteq \mathscr{S}_x(G)$, we say that $\mathfrak{X}$ is *nontrivial* if $|A| \geq 2$ and $|B| \geq 2$ for every $(A; S; B) \in \mathfrak{X}$. Note that if $\mathfrak{X} = \emptyset$, then $\mathfrak{X}$ is nontrivial by definition. We remark that even if $\mathfrak{X} \subseteq \mathscr{S}_x(G)$ is nontrivial, it is possible that S is trivial for some $(A; S; B) \in \mathfrak{X}$.

Note that $V(G_x^{(x;M)}) = N_G(x)$. Recall that $\mathscr{T}(G_x^{(x;M)})$ is the set of separating triplets in $G_x^{(x;M)}$. Let $(C; X; C') \in \mathscr{T}(G_x^{(x;M)})$. We say that a separating triplet $(A; S'; B) \in \mathscr{S}(G^{(x;M)})$ *admits* $(C; X; C')$ if $S'_x = X$, $A_x = C$ and $B_x = C'$ (or $S'_x = X$, $A_x = C'$ and $B_x = C$; recall that $(C; X; C')$ and $(C'; X; C)$, as well as $(A; S'; B)$ and $(B; S'; A)$, are regarded as the same). Let $\mathscr{S}((C; X; C'); G^{(x;M)})$ be the set of separating triplets in $\mathscr{S}(G^{(x;M)})$ which admit $(C; X; C')$. Note that since G_x is a spanning subgraph of $G_x^{(x;M)}$, we have $\mathscr{T}(G_x^{(x;M)}) \subseteq \mathscr{T}(G_x)$. Let now $(C; X; C') \in \mathscr{T}(G_x)$. We say that a separating triplet $(A; S; B) \in \mathscr{S}_x(G)$ *admits* $(C; X; C')$ if $S_x = X$, $A_x = C$ and $B_x = C'$. Let $\mathscr{S}_x((C; X; C'); G)$ be the set of separating triplets in $\mathscr{S}_x(G)$ which admit $(C; X; C')$. If there is no ambiguity, we write $\mathscr{S}_x((C; X; C'))$ or $\mathscr{S}_x(C; X; C')$ for $\mathscr{S}_x((C; X; C'); G)$. For $\mathscr{Y} \subseteq \mathscr{T}(G_x)$, we let $\mathscr{S}_x(\mathscr{Y}) = \cup_{(C;X;C') \in \mathscr{Y}} \mathscr{S}_x(C; X; C')$.

Lemma 4.1 *Let G be a 5-connected graph and let $x \in V_5(G)$.*

(1) *Every 5-separator S of G with $x \in S$ divides G_x.*
(2) *If M is a perfect pseudo-matching on $N_G(x)$, then every 4-separator of $G^{(x;M)}$ divides $G_x^{(x;M)}$.*

Proof. Let S be a 5-separator of G with $x \in S$. Let A be an S-fragment of G. Then $S = N_G(A)$. Since $x \in S$, this implies $A_x \neq \emptyset$. Since A is arbitrary, this means that S divides G_x. This proves (1).

Let now M be a perfect pseudo-matching on $N_G(x)$, and let S' be a 4-separator of $G^{(x;M)}$. Then $S' \cup \{x\}$ is a 5-separator of G, and every S'-fragment of $G^{(x;M)}$ is an $(S' \cup \{x\})$-fragment of G. Thus (2) follows from (1). □

Lemma 4.2 *Let M be a perfect non-edge pseudo-matching on $N_G(x)$, and let $(C; X; C') \in \mathscr{T}(G_x^{(x;M)})$. Then*

$$\mathscr{S}_x(C; X; C') = \{(A; S' \cup \{x\}; B) \mid (A; S'; B) \in \mathscr{S}((C; X; C'); G^{(x;M)})\}.$$

Proof. Let $(A; S; B) \in \mathscr{S}_x(C; X; C')$. Then $E_G(A, B) = \emptyset$. By the definition of $G^{(x;M)}$, it follows that $E_{G^{(x;M)}}(A - A_x, B - B_x) = \emptyset$, $E_{G^{(x;M)}}(A - A_x, B_x) = \emptyset$ and $E_{G^{(x;M)}}(A_x, B - B_x) = \emptyset$. Since $A_x = C$ and $B_x = C'$ and $(C; X; C') \in \mathscr{T}(G_x^{(x;M)})$, we also get $E_{G^{(x;M)}}(A_x, B_x) = \emptyset$. Hence $E_{G^{(x;M)}}(A, B) = \emptyset$. This means that for every $(A; S; B) \in \mathscr{S}_x(C; X; C')$, if we let $S' = S - \{x\}$, then $(A; S'; B) \in \mathscr{S}((C; X; C'); G^{(x;M)})$ and $(A; S; B) = (A; S' \cup \{x\}; B)$. On the other hand, for every $(A; S'; B) \in \mathscr{S}((C; X; C'); G^{(x;M)})$, we clearly have $(A; S' \cup \{x\}; B) \in \mathscr{S}_x(C; X; C')$. □

Lemma 4.3 *Let G be a 5-connected graph and let $x \in V_5(G)$. Let M be a perfect non-edge pseudo-matching on $N_G(x)$.*

(1) *The graph $G^{(x;M)}$ is 5-connected if and only if $\mathscr{S}_x(G; M)$ is empty.*
(2) $\mathscr{S}(G^{(x;M)}) = \cup_{(C;X;C') \in \mathscr{T}(G_x^{(x;M)})} \mathscr{S}((C; X; C'); G^{(x;M)})$.
(3) $\mathscr{S}_x(G; M) = \mathscr{S}_x(\mathscr{T}(G_x^{(x;M)}))$.

Proof. (1) Note that the existence of M implies $|V(G)| \geq 8$. Hence $G^{(x;M)}$ is 5-connected if and only if $\mathscr{S}(G^{(x;M)}) = \emptyset$. By the definition of $\mathscr{S}_x(G; M)$, $\mathscr{S}_x(G; M) = \emptyset$ if and only if $\mathscr{S}(G^{(x;M)}) = \emptyset$.

(2) Let $(A; S'; B) \in \mathscr{S}(G^{(x;M)})$. Since S' divides $G_x^{(x;M)}$ by Lemma 4.1 (2), neither A_x nor B_x is empty. This implies that $(A_x; S'_x; B_x) \in \mathscr{T}(G_x^{(x;M)})$ and $(A; S'; B) \in \mathscr{S}((A_x; S'_x; B_x); G^{(x;M)})$. On the other hand, if $(C; X; C') \in \mathscr{T}(G_x^{(x;M)})$ and $(A; S'; B) \in \mathscr{S}((C; X; C'); G^{(x;M)})$, then $(A; S'; B) \in \mathscr{S}(G^{(x;M)})$ by definition.

Since

$$\begin{aligned}
&\mathscr{S}_x(\mathscr{T}(G_x^{(x;M)})) \\
&\quad = \cup_{(C;X;C') \in \mathscr{T}(G_x^{(x;M)})} \mathscr{S}_x(C; X; C') \\
&\quad = \cup_{(C;X;C') \in \mathscr{T}(G_x^{(x;M)})} \{(A; S' \cup \{x\}; B) | (A; S'; B) \in \mathscr{S}((C; X; C'); G^{(x;M)})\} \\
&\quad = \{(A; S' \cup \{x\}; B) | (A; S'; B) \in \mathscr{S}(G^{(x;M)})\}
\end{aligned}$$

by Lemma 4.2 and (2), (3) follows from the definition of $\mathscr{S}_x(G; M)$. □

Lemma 4.3 (1) shows that $x \in V_5(G)$ is properly 5-liftable if and only if there is a perfect non-edge pseudo-matching M on $N_G(x)$ such that $\mathscr{S}_x(G; M) = \emptyset$.

Lemma 4.4 *Let G be a 5-connected graph and let $x \in V_5(G)$. Let M be a perfect non-edge pseudo-matching on $N_G(x)$.*

(1) *For every* $(A; S'; B) \in \mathscr{S}(G^{(x;M)})$, S' *is nontrivial.*
(2) *For every* $(A; S; B) \in \mathscr{S}_x(G; M)$, *we have* $|A|, |B| \geq 2$*; that is to say,* $\mathscr{S}_x(G; M)$ *is nontrivial.*

Proof. Statement (1) follows from the fact that $\delta(G^{(x;M)}) \geq 5$. To prove (2), let $(A; S; B) \in \mathscr{S}_x(G; M)$. By the definition of $\mathscr{S}_x(G; M)$, $(A; S - \{x\}; B) \in \mathscr{S}(G^{(x;M)})$. Hence $|A|, |B| \geq 2$ by (1). □

Lemma 4.5 *Let* G *be a* 5*-connected graph and let* $x \in V_5(G)$. *Let* $(C; X; C')$ *and* $(D; Y; D')$ *be distinct separating triplets of* G_x. *Suppose both* $\mathscr{S}_x(C; X; C')$ *and* $\mathscr{S}_x(D; Y; D')$ *are nontrivial. Suppose further that one of the following four conditions holds:*

(1) $N_G(x) = X \cup Y$, *and either* $X \cap Y \neq \emptyset$ *or* $|V(G)| \geq 10$*;*
(2) $|X| = |Y| = 3$*;*
(3) $|X| = 3$, $|Y| = 2$, $|D| = 1$ *and, either* $C = D$ *and* $D' \subseteq X$ *or* $X \cap D' = \emptyset$*; or*
(4) $|X| = |Y| = 2$, $|C| = |D| = 1$, $C \subseteq D'$, $D \subseteq C'$, $|N_G(x) \cap N_G(C)| = |N_G(x) \cap N_G(D)| = 1$, *and there is a vertex* $y \in X \cap Y \cap V_5(G)$ *such that* $|N_G(x) \cap N_G(y)| = 3$.

Then $\mathscr{S}_x(C; X; C')$ or $\mathscr{S}_x(D; Y; D')$ is empty.

Proof. Suppose that neither $\mathscr{S}_x(C; X; C')$ nor $\mathscr{S}_x(D; Y; D')$ is empty; say $(A; S; \bar{A}) \in \mathscr{S}_x(C; X; C')$ and $(B; T; \bar{B}) \in \mathscr{S}_x(D; Y; D')$. Then $x \in S \cap T$ and $X = S_x$ and we may assume that $C = A_x$ and $C' = \bar{A}_x$ (recall that $(C; X; C')$ and $(C'; X; C)$ are regarded as the same), and $Y = T_x$ and we may assume that $D = B_x$ and $D' = \bar{B}_x$. Since $\mathscr{S}_x(C; X; C')$ and $\mathscr{S}_x(D; Y; D')$ are nontrivial, we have $|A|, |\bar{A}|, |B|, |\bar{B}| \geq 2$. Hence $|V(G)| = |A| + |S| + |\bar{A}| \geq 9$.

(1) Suppose that $N_G(x) = S_x \cup T_x$, and $S_x \cap T_x \neq \emptyset$ or $|V(G)| \geq 10$. From $N_G(x) = S_x \cup T_x$, we get $(A \cap B)_x = (A \cap \bar{B})_x = (\bar{A} \cap B)_x = (\bar{A} \cap \bar{B})_x = \emptyset$. Note that $|S \cap T| \geq |(S_x \cap T_x) \cup \{x\}| = |S_x \cap T_x| + 1$. If $S_x \cap T_x \neq \emptyset$, then $|S \cup T| \leq |S| + |T| - |S \cap T| \leq 8 < |V(G)|$; if $S_x \cap T_x = \emptyset$, then $|V(G)| \geq 10$ by assumption, and hence $|S \cup T| \leq |S| + |T| - |S \cap T| \leq 9 < |V(G)|$. In either case, we have $V(G) \neq S \cup T$. We may assume that $A \cap B \neq \emptyset$. Since $(A \cap B)_x = \emptyset$, it follows from Lemma 3.2 (6) and Lemma 3.3 that $\bar{A} \cap B \neq \emptyset$ and $A \cap \bar{B} \neq \emptyset$. Hence we see from Lemma 3.2 (5) that $|(S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)| = 5$ and $A \cap \bar{B}$ is an $((S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T))$-fragment. Consequently $N_G(A \cap \bar{B}) = (S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)$, which contradicts the fact that $(A \cap \bar{B})_x = \emptyset$. This contradiction shows that $\mathscr{S}_x(C; X; C')$ or $\mathscr{S}_x(D; Y; D')$ is empty.

(2) Suppose that $|S_x| = |T_x| = 3$. Since $A_x, \bar{A}_x, B_x, \bar{B}_x \neq \emptyset$, we observe that $|A_x| = |\bar{A}_x| = |B_x| = |\bar{B}_x| = 1$, say $A_x = \{y\}$, $\bar{A}_x = \{y'\}$, $B_x = \{z\}$, $\bar{B}_x = \{z'\}$. We also have $S_x \cap T_x \neq \emptyset$, which implies $S \cap T \neq \{x\}$. Assume for the moment that $\{y, y'\} \cap \{z, z'\} = \emptyset$. Since $\{z, z'\} = B_x \cup \bar{B}_x$, it follows that $\{y, y'\} \subseteq N_G(x) - (B_x \cup \bar{B}_x) = T_x$. Since $N_G(x) - \{y, y'\} = S_x$, we see that $N_G(x) = S_x \cup T_x$. Since $S_x \cap T_x \neq \emptyset$, this implies that condition (1) is satisfied, and hence the desired conclusion holds.

Thus we may assume that $\{y, y'\} \cap \{z, z'\} \neq \emptyset$, say $y = z$. Then $B_x = A_x = \{y\}$. Note that this implies $y' \notin B_x$. Since $(A_x; S_x; \bar{A}_x)$ and $(B_x; T_x; \bar{B}_x)$ are distinct, we observe that $\bar{B}_x \neq \bar{A}_x = \{y'\}$, which implies that $y' \notin \bar{B}_x$. Consequently $y' \in T_x$. Hence $\{x\} \cup \bar{A}_x = \{x, y'\} \subseteq T$. Since $x \in S \cap T$ and $|A|, |\bar{A}|, |B|, |\bar{B}| \geq 2$, Lemma 3.6 (3) assures us that $S \cap T = \{x\}$, which contradicts the earlier assertion. This contradiction proves that $\mathscr{S}_x(C; X; C')$ or $\mathscr{S}_x(D; Y; D')$ is empty.

(3) Suppose that $|S_x| = 3$, $|B_x| = 1$, $|T_x| = 2$, and either $A_x = B_x$ and $\bar{B}_x \subseteq S_x$ or $S_x \cap \bar{B}_x = \emptyset$. Since $|S_x| = 3$, we observe that $|A_x| = |\bar{A}_x| = 1$, say $A_x = \{y\}$, $\bar{A}_x = \{y'\}$. Since $|B_x| = 1$ and $|T_x| = 2$, we observe that $|\bar{B}_x| = 2$, say $B_x = \{z\}$ and $\bar{B}_x = \{z', z''\}$.

First assume $A_x = B_x$ and $\bar{B}_x \subseteq S_x$, namely, $y = z$ and $\{z', z''\} \subseteq S_x$. Since $y = z$, we see that $S_x \subseteq T_x \cup \{z', z''\}$. Since $|T_x| = 2$ and $|S_x| = 3$, it follows that $(S \cap T)_x \neq \emptyset$, which implies $S \cap T \neq \{x\}$. On the other hand, since $\{x\} \cup \bar{B}_x \subseteq \{x\} \cup S_x \subseteq S$, $x \in T$ and $|A|, |\bar{A}|, |B|, |\bar{B}| \geq 2$, Lemma 3.6 (3) assures us that $S \cap T = \{x\}$, which is a contradiction.

Next assume $S_x \cap \bar{B}_x = \emptyset$. Then $S_x \subseteq B_x \cup T_x$. Since $|S_x| = 3 = |B_x| + |T_x|$, it follows that $S_x = B_x \cup T_x$. Hence $(S \cap T)_x = T_x \neq \emptyset$, which implies $S \cap T \neq \{x\}$. On the other hand, since $\{x\} \cup B_x \subseteq \{x\} \cup S_x \subseteq S$, $x \in T$ and $|A|, |\bar{A}|, |B|, |\bar{B}| \geq 2$, Lemma 3.6 (3) assures us that $S \cap T = \{x\}$, which is a contradiction.

Now in both cases, we have a contradiction, and it is shown that $\mathscr{S}_x(C; X; C')$ or $\mathscr{S}_x(D; Y; D')$ is empty.

(4) Suppose that $|S_x| = |T_x| = 2$ and $|A_x| = |B_x| = 1$. Write $A_x = \{y'\}$ and $B_x = \{z\}$, and suppose that $y' \in \bar{B}_x$, $z \in \bar{A}_x$ and $|N_G(x) \cap N_G(y')| = |N_G(x) \cap N_G(z)| = 1$. Further let $y \in S_x \cap T_x \cap V_5(G)$, and suppose that $|N_G(x) \cap N_G(y)| = 3$. Since $y' \in \bar{B}_x$ and $z \in \bar{A}_x$, we observe that $(A \cap \bar{B})_x = \{y'\}$ and $(\bar{A} \cap B)_x = \{z\}$. Since neither $A \cap \bar{B}$ nor $\bar{A} \cap B$ is empty, it follows from Lemma 3.2 (5) that $A \cap \bar{B}$ and $\bar{A} \cap B$ are fragments with $N_G(A \cap \bar{B}) = (S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)$ and $N_G(\bar{A} \cap B) = (\bar{A} \cap T) \cup (S \cap T) \cup (S \cap B)$. Since $A_x = \{y'\}$, $(A \cap T)_x = \emptyset$. Hence $T_x = N_G(x) \cap ((\bar{A} \cap T) \cup (S \cap T))$, which implies $|N_G(x) \cap ((\bar{A} \cap T) \cup (S \cap T) \cup (S \cap B))| \geq |T_x| = 2$. By symmetry, we also have $|N_G(x) \cap ((S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T))| \geq 2$. Since $|N_G(x) \cap N_G(y)| = 3$ and $y \in V_5(G)$, we see that $|N_G(y) - (N_G(x) \cup \{x\})| = 1$, say $N_G(y) - (N_G(x) \cup \{x\}) = \{w\}$. Since $(\bar{A} \cap B) \cap (A \cap \bar{B}) = \emptyset$, without loss of generality, we may assume that $w \notin \bar{A} \cap B$. Then $N_G(y) \cap (\bar{A} \cap B) \subseteq N_G(x) \cap (\bar{A} \cap B) = \{z\}$. On the other hand, since $N_G(\bar{A} \cap B) = (\bar{A} \cap T) \cup (S \cap T) \cup (S \cap B)$, we have $N_G(y) \cap (\bar{A} \cap B) \neq \emptyset$. Hence $N_G(y) \cap (\bar{A} \cap B) = \{z\}$. Consequently $N_G(\{x, y\}) \cap (\bar{A} \cap B) = \{z\}$. Applying Lemma 2.2 with A and U replaced by $\bar{A} \cap B$ and $\{x, y\}$, respectively, we see that $\bar{A} \cap B = \{z\}$, which implies that $N_G(z) = (\bar{A} \cap T) \cup (S \cap T) \cup (S \cap B)$. Since $|N_G(x) \cap ((\bar{A} \cap T) \cup (S \cap T) \cup (S \cap B))| \geq 2$, we obtain $|N_G(x) \cap N_G(z)| \geq 2$, which contradicts the assumption that $|N_G(x) \cap N_G(z)| = 1$. This contradiction shows that $\mathscr{S}_x(C; X; C')$ or $\mathscr{S}_x(D; Y; D')$ is empty.

Now the proof of Lemma 4.5 is completed. □

5 Graphs in $\mathscr{B}(5)$

Throughout the rest of this paper, we consider 5-connected graphs. Let G be a 5-connected graph. Recall that $E_n(G)$, $E^{**}(G)$ and $\mathscr{L}^*(G)$ denote the set of 5-noncontractible edges, semi-properly 5-contractible edges and properly 5-liftable vertices, respectively. Recall that $\mathscr{B}(5)$ is the set of connected non-tree graphs on five vertices whose maximum degree is three.

In this section, we show that if G is a minimally 5-connected graph with $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$ and $x \in V_5(G)$, then $G_x \notin \mathscr{B}(5)$. Let $\mathscr{B}'(5) = \{2K_1 + (K_1 \cup K_2), K_{2,3}, \tilde{P}_5, \widetilde{Ch}, Ch\}$. We show that $G_x \notin \mathscr{B}'(5)$ in Lemma 5.1 and we show that $G_x \notin \mathscr{B}(5) - \mathscr{B}'(5) = \{\tilde{D}_{1,2}, Bu\}$ in Lemma 5.3.

Lemma 5.1 *Let G be a minimally 5-connected graph G with $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$. If $x \in V_5(G)$, then G_x is not isomorphic to any of the graphs in $\mathscr{B}'(5)$.*

Proof. Let $x \in V_5(G)$ and let $N_G(x) = \{y_1, y_2, y_3, y_4, y_5\}$. By way of contradiction, suppose that G_x is isomorphic to $2K_1 + (K_1 \cup K_2)$, $K_{2,3}$, $\tilde{P}_5$, $\widetilde{Ch}$ or Ch.

If $G_x \cong 2K_1 + (K_1 \cup K_2)$, say $E(G_x) = \{y_1y_2, y_2y_3, y_3y_4, y_4y_5, y_5y_1, y_2y_4, y_3y_5\}$, then $M = \{y_3y_1y_4, y_2y_5\}$ is a perfect non-edge pseudo-matching on $N_G(x)$, $G_x^{(x;M)} \cong K_5$ and $\mathscr{T}(G_x^{(x;M)}) = \emptyset$, and hence $\mathscr{S}_x(G; M) = \emptyset$ by Lemma 4.3 (3) and, applying Lemma 4.3 (1), we see that x is properly 5-liftable, which contradicts the assumption that $\mathscr{L}^*(G) = \emptyset$. Hence $G_x \ncong 2K_1 + (K_1 \cup K_2)$. Thus $G_x \cong K_{2,3}$, $\tilde{P}_5$, $\widetilde{Ch}$ or Ch.

First suppose that $G_x \cong K_{2,3}$, say $E(G_x) = \{y_1y_3, y_1y_4, y_1y_5, y_2y_3, y_2y_4, y_2y_5\}$. We consider two perfect non-edge pseudo-matchings on $N_G(x)$, $M_1 = \{y_3y_4y_5, y_1y_2\}$ and $M_2 = \{y_3y_5y_4, y_1y_2\}$. We observe that $\mathscr{T}(G_x^{(x;M_1)}) = \{(\{y_3\}; \{y_1, y_2, y_4\}; \{y_5\})\}$ and $\mathscr{T}(G_x^{(x;M_2)}) = \{(\{y_3\}; \{y_1, y_2, y_5\}; \{y_4\})\}$. Thus, Lemma 4.3 (3) implies that $\mathscr{S}_x(G; M_1) = \mathscr{S}_x(\{y_3\}; \{y_1, y_2, y_4\}; \{y_5\})$ and $\mathscr{S}_x(G; M_2) = \mathscr{S}_x(\{y_3\}; \{y_1, y_2, y_5\}; \{y_4\})$. By Lemma 4.4 (2), both $\mathscr{S}_x(G; M_1)$ and $\mathscr{S}_x(G; M_2)$ are nontrivial. Consequently applying Lemma 4.5 (2), we see that $\mathscr{S}_x(G; M_1)$ or $\mathscr{S}_x(G; M_2)$ is empty. By Lemma 4.3 (1), this implies that x is properly 5-liftable, which is a contradiction.

Next suppose that $G_x \cong \tilde{P}_5$, say $E(G_x) = \{y_1y_2, y_1y_3, y_2y_3, y_3y_4, y_4y_5, y_5y_2\}$. We consider two perfect non-edge pseudo-matchings on $N_G(x)$, $M_1 = \{y_1y_4y_2, y_3y_5\}$ and $M_2 = \{y_1y_5y_3, y_2y_4\}$. We observe that $\mathscr{T}(G_x^{(x;M_1)}) = \{(\{y_1\}; \{y_2, y_3, y_4\}; \{y_5\})\}$ and $\mathscr{T}(G_x^{(x;M_2)}) = \{(\{y_1\}; \{y_2, y_3, y_5\}; \{y_4\})\}$. Thus, Lemma 4.3 (3) implies that $\mathscr{S}_x(G; M_1) = \mathscr{S}_x(\{y_1\}; \{y_2, y_3, y_4\}; \{y_5\})$ and $\mathscr{S}_x(G; M_2) = \mathscr{S}_x(\{y_1\}; \{y_2, y_3, y_5\}; \{y_4\})$. By Lemma 4.4 (2), both $\mathscr{S}_x(G; M_1)$ and $\mathscr{S}_x(G; M_2)$ are nontrivial. Consequently, applying Lemma 4.5 (2), we see that $\mathscr{S}_x(G; M_1)$ or $\mathscr{S}_x(G; M_2)$ is empty. By Lemma 4.3 (1), this implies that x is properly 5-liftable, which is a contradiction.

Suppose now that $G_x \cong \widetilde{Ch}$, say $E(G_x) = \{y_1y_2, y_2y_3, y_3y_4, y_3y_5, y_4y_5\}$. We consider two perfect non-edge pseudo-matchings on $N_G(x)$, $M_1 = \{y_3y_1y_4, y_2y_5\}$ and $M_2 = \{y_3y_1y_5, y_2y_4\}$. We have $\mathscr{T}(G_x^{(x;M_1)}) = \{(\{y_1\}; \{y_2, y_3, y_4\}; \{y_5\}), (\{y_2\}; \{y_1, y_3, y_5\}; \{y_4\})\}$. Hence Lemma 4.3 (3) implies that

$$\mathscr{S}_x(G; M_1) = \mathscr{S}_x(\{y_1\}; \{y_2, y_3, y_4\}; \{y_5\}) \cup \mathscr{S}_x(\{y_2\}; \{y_1, y_3, y_5\}; \{y_4\}).$$

Similarly, $\mathscr{S}_x(G;M_2) = \mathscr{S}_x(\{y_1\};\{y_2,y_3,y_5\};\{y_4\}) \cup \mathscr{S}_x(\ \{y_2\};\{y_1,y_3,y_4\};\{y_5\})$. By Lemma 4.4 (2), both $\mathscr{S}_x(G;M_1)$ and $\mathscr{S}_x(G;M_2)$ are nontrivial. Now since x is not properly 5-liftable by assumption, it follows from Lemma 4.3 (1) that there exist $\Sigma \in \mathscr{T}(G_x^{(x;M_1)})$ and $\Sigma' \in \mathscr{T}(G_x^{(x;M_2)})$ such that $\mathscr{S}_x(\Sigma) \neq \emptyset$ and $\mathscr{S}_x(\Sigma') \neq \emptyset$. On the other hand, since $\mathscr{S}_x(\Sigma)$ and $\mathscr{S}_x(\Sigma')$ are nontrivial, we see from Lemma 4.5 (2) that $\mathscr{S}_x(\Sigma)$ or $\mathscr{S}_x(\Sigma')$ is empty. This is a contradiction.

Finally suppose that $G_x \cong Ch$, say $E(G_x) = \{y_1y_2, y_2y_3, y_3y_4, y_4y_5, y_5y_2\}$. We consider two perfect non-edge pseudo-matchings on $N_G(x)$, $M_1 = \{y_3y_1y_5, y_2y_4\}$ and $M_2 = \{y_1y_4y_2, y_3y_5\}$. We have

$$\mathscr{T}(G_x^{(x;M_1)}) = \{(\{y_1\};\{y_2,y_3,y_5\};\{y_4\}), (\{y_3\};\{y_1,y_2,y_4\};\{y_5\})\}.$$

Hence Lemma 4.3 (3) implies that

$$\mathscr{S}_x(G;M_1) = \mathscr{S}_x(\{y_1\};\{y_2,y_3,y_5\};\{y_4\}) \cup \mathscr{S}_x(\{y_3\};\{y_1,y_2,y_4\};\{y_5\}).$$

Similarly, $\mathscr{S}_x(G;\ M_2) = \mathscr{S}_x(\{y_1\};\{y_2,y_4\};\{y_3,y_5\}) \cup \mathscr{S}_x(\ \{y_1\};\{y_2,y_3,y_4\};\{y_5\}) \cup \mathscr{S}_x(\{y_1\};\{y_2,y_4,y_5\};\{y_3\})$. By Lemma 4.4 (2), both $\mathscr{S}_x(G;M_1)$ and $\mathscr{S}_x(G;M_2)$ are nontrivial. Now since x is not properly 5-liftable, we see from Lemma 4.3 (1) that there exist $\Sigma \in \mathscr{T}(G_x^{(x;M_1)})$ and $\Sigma' \in \mathscr{T}(G_x^{(x;M_2)})$ such that $\mathscr{S}_x(\Sigma) \neq \emptyset$ and $\mathscr{S}_x(\Sigma') \neq \emptyset$. Note that $\mathscr{S}_x(\Sigma)$ and $\mathscr{S}_x(\Sigma')$ are nontrivial. Hence if

$$\Sigma' \in \{(\{y_1\};\{y_2,y_3,y_4\};\{y_5\}), (\{y_1\};\{y_2,y_4,y_5\};\{y_3\})\},$$

then it follows from Lemma 4.5 (2) that $\mathscr{S}_x(\Sigma)$ or $\mathscr{S}_x(\Sigma')$ is empty, a contradiction. Thus $\Sigma' = (\{y_1\};\{y_2,y_4\};\{y_3,y_5\})$. Take $(B;T;\bar{B}) \in \mathscr{S}_x(\Sigma')$ with $B_x = \{y_1\}$ and $\bar{B}_x = \{y_3,y_5\}$. Also take $(A;S;\bar{A}) \in \mathscr{S}_x(\Sigma)$. Suppose that $\Sigma = (\{y_1\};\{y_2,y_3,y_5\};\{y_4\})$. We may assume that $A_x = \{y_1\}$. Then we observe that $|S_x| = 3$, $|B_x| = 1$, $|T_x| = 2$, $A_x = B_x$ and $\bar{B}_x \subseteq S_x$. Consequently applying Lemma 4.5 (3), we see that $\mathscr{S}_x(\Sigma)$ or $\mathscr{S}_x(\Sigma')$ is empty, a contradiction. Thus $\Sigma = (\{y_3\};\{y_1,y_2,y_4\};\{y_5\})$. We observe that $|S_x| = 3$, $|B_x| = 1$, $|T_x| = 2$ and $S_x \cap \bar{B}_x = \emptyset$. Therefore applying Lemma 4.5 (3), we see that $\mathscr{S}_x(\Sigma)$ or $\mathscr{S}_x(\Sigma')$ is empty. This is a contradiction, which concludes the discussion for the case $G_x \cong Ch$.

Now the proof of Lemma 5.1 is completed. □

Before proceeding to the analysis of the remaining two graphs in $\mathscr{B}(5)$, we need to prove the following technical lemma.

Lemma 5.2 *Let G be a minimally 5-connected graph with $E^{**}(G) = \emptyset$. Let $x \in V_5(G)$ and let $(A;S;\bar{A}) \in \mathscr{S}_x(G)$ with $|A|, |\bar{A}| \geq 2$. Suppose that $|A_x| = 1$, say $A_x = \{y\}$. Further suppose that $|N_G(y) \cap N_G(x)| = 1$, say $N_G(y) \cap N_G(x) = \{y'\}$. Then the following hold.*

(1) *Either $|S_x| \leq 2$ or $|N_G(y') \cap N_G(x)| \leq 2$.*
(2) *Suppose that $|S_x| = 2$, say $S_x = \{y', y''\}$. If $y'y'' \in E(G)$ and $N_G(x) \subseteq N_G(y') \cup N_G(y'')$, then $y' \in V_5(G)$.*

Proof. Since $E^{**}(G) = \emptyset$ and $|N_G(y) \cap N_G(x)| = 1$, we observe that $xy \in E_n(G)$. Since $|A_x| = 1$, we have $y' \in S$, i.e., $N_G(y) \cap S_x = \{y'\}$.

(1) Suppose that $|S_x| \geq 3$ and $|N_G(y') \cap N_G(x)| \geq 3$. Since $|S_x| \geq 3$ and $|N_G(y) \cap N_G(x)| = 1$, we see that $|S - N_G(y)| \geq |S_x - N_G(y)| \geq 2$, which implies $|A| \geq 3$ by Lemma 3.2 (8). Hence applying Lemma 3.7 (1), we see that $S_x \cap N_G(y) \cap V_5(G) \neq \emptyset$, which implies $y' \in S_x \cap N_G(y) \cap V_5(G)$ because $N_G(y) \cap S_x = \{y'\}$. Applying Lemma 3.7 (2), we see that $|N_G(y') \cap S| \leq 2$, which implies $|N_G(y') \cap S_x| \leq 1$ because $x \in N_G(y') \cap S$. Since $|N_G(y') \cap A_x| = |A_x| = |\{y\}| = 1$, $|N_G(y') \cap S_x| \leq 1$ and $|N_G(y') \cap N_G(x)| \geq 3$, we observe that $|N_G(y') \cap \bar{A}_x| \geq 1$. On the other hand, since $A_x = \{y\}$ and $|S_x| \geq 3$, we see that $|\bar{A}_x| = 1$. Therefore $|N_G(y') \cap \bar{A}_x| = |\bar{A}_x| = 1$ and $|N_G(y') \cap S_x| = 1$. From $|N_G(y') \cap \bar{A}_x| = |\bar{A}_x|$, we get $\bar{A}_x - N_G(y') = \emptyset$. From $|N_G(y') \cap S_x| = 1$, we get $|N_G(y') \cap S| \geq 2$, and hence it follows from Lemma 3.7 (2) that $\bar{A}_x - N_G(y') \neq \emptyset$. This is a contradiction. This contradiction shows that either $|S_x| \leq 2$ or $|N_G(y') \cap N_G(x)| \leq 2$ and (1) is proved.

(2) Suppose that $S_x = \{y', y''\}$, $y'y'' \in E(G)$, $N_G(x) \subseteq N_G(y') \cup N_G(y'')$ and $y' \in V_{\geq 6}(G)$. Since $xy \in E_n(G)$, there is a 5-separator T which contains $\{x, y\}$. Let B be a T-fragment of G. If $|B| = 1$, say $B = \{z\}$, then $z \in V_5(G)$ and $zx, zy \in E(G)$, which contradicts the assumption that $N_G(y) \cap N_G(x) = \{y'\}$ and $y' \in V_{\geq 6}(G)$. Thus $|B| \geq 2$. Similarly $|\bar{B}| \geq 2$. Consequently $|A| = 2$ by Lemma 3.6 (1). Hence by Lemma 3.5 (2), we have $|S \cap B| = |S \cap \bar{B}| = 2$ and $S \cap T = \{x\}$. Since $\{y', y''\} \subseteq S$ and $y'y'' \in E(G)$, without loss of generality, we may assume that $S \cap B = \{y', y''\}$. Since $\{y', y''\} \subseteq B$, we observe that $N_G(y') \cup N_G(y'') \subseteq B \cup T$. Since $N_G(x) \subseteq N_G(y') \cup N_G(y'')$, it follows that $N_G(x) \subseteq B \cup T$, which implies $\bar{B}_x = \emptyset$. This contradicts the fact that $\bar{B}$ is a T-fragment, and this contradiction shows $y' \in V_5(G)$. □

Lemma 5.3 *Let G be a minimally 5-connected graph G with $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$. If $x \in V_5(G)$, then G_x is not isomorphic to any of the graphs in $\mathscr{B}(5)$.*

Proof. Let $x \in V_5(G)$. Note that $\mathscr{B}(5) - \mathscr{B}'(5) = \{\tilde{D}_{1,2}, Bu\}$. In view of Lemma 5.1, it suffices to show that G_x is isomorphic to neither $\tilde{D}_{1,2}$ nor Bu. Let $N_G(x) = \{y_1, y_2, y_3, y_4, y_5\}$.

First suppose that $G_x \cong \tilde{D}_{1,2}$, say $E(G_x) = \{y_1y_2, y_2y_3, y_3y_4, y_4y_5, y_5y_2, y_3y_5\}$. Then $M = \{y_3y_1y_5, y_2y_4\}$ is a perfect non-edge pseudo-matching on $N_G(x)$. We observe that $\mathscr{T}(G_x^{(x;M)}) = \{(\{y_1\}; \{y_2, y_3, y_5\}; \{y_4\})\}$. By Lemma 4.3 (3), we have $\mathscr{S}_x(G; M) = \mathscr{S}_x(\{y_1\}; \{y_2, y_3, y_5\}; \{y_4\})$. Since x is not properly 5-liftable, we have $\mathscr{S}_x(G; M) \neq \emptyset$ by Lemma 4.3 (1), say $(A; S; \bar{A}) \in \mathscr{S}_x(G; M)$. We may assume that $A_x = \{y_1\}$ and $\bar{A}_x = \{y_4\}$. By Lemma 4.4 (2), $|A|, |\bar{A}| \geq 2$. Since $N_G(y_1) \cap N_G(x) = \{y_2\}$, applying Lemma 5.2 (1), we see that either $|S_x| \leq 2$ or $|N_G(y_2) \cap N_G(x)| \leq 2$, which contradicts the fact that $S_x = \{y_2, y_3, y_5\}$ and $N_G(y_2) \cap N_G(x) = \{y_1, y_3, y_5\}$, completing the discussion for the case where $G_x \cong \tilde{D}_{1,2}$.

Next suppose that $G_x \cong Bu$, say $E(G_x) = \{y_1y_2, y_2y_3, y_1y_3, y_2y_4, y_3y_5\}$. We consider two perfect non-edge pseudo-matchings on $N_G(x)$, $M_1 = \{y_1y_4y_3, y_2y_5\}$ and $M_2 = \{y_1y_5y_2, y_3y_4\}$.

Claim 5.3.1 $\mathscr{S}_x(G; M_1) = \mathscr{S}_x(\{y_5\}; \{y_2, y_3\}; \{y_1, y_4\})$.

Proof. We have $\mathscr{T}_x(G_x^{(x,M_1)}) = \{(\{y_5\}; \{y_2, y_3\}; \{y_1, y_4\}), (\{y_5\}; \{y_1, y_2, y_3\}; \{y_4\}), (\{y_5\}; \{y_2, y_3, y_4\}; \{y_1\})\}$. Hence by Lemma 4.3 (3), $\mathscr{S}_x(G; M_1) = \mathscr{S}_x(\{y_5\}; \{y_2, y_3\}; \{y_1, y_4\}) \cup \mathscr{S}_x(\{y_5\}; \{y_1, y_2, y_3\}; \{y_4\}) \cup \mathscr{S}_x(\{y_5\}; \{y_2, y_3, y_4\}; \{y_1\})$. Thus it suffices to show that $\mathscr{S}_x(\{y_5\}; \{y_1, y_2, y_3\}; \{y_4\}) = \mathscr{S}_x(\{y_5\}; \{y_2, y_3, y_4\}; \{y_1\}) = \emptyset$. Suppose that $(A; S; \bar{A}) \in \mathscr{S}_x(\{y_5\}; \{y_1, y_2, y_3\}; \{y_4\}) \cup \mathscr{S}_x(\{y_5\}; \{y_2, y_3, y_4\}; \{y_1\})$. We may assume that $A_x = \{y_5\}$. By Lemma 4.4 (2), $|A|, |\bar{A}| \geq 2$. Since $N_G(y_5) \cap N_G(x) = \{y_3\}$, applying Lemma 5.2 (1), we see that either $|S_x| \leq 2$ or $|N_G(y_3) \cap N_G(x)| \leq 2$, which contradicts the fact that $S_x = \{y_1, y_2, y_3\}$ or $S_x = \{y_2, y_3, y_4\}$ and $N_G(y_3) \cap N_G(x) = \{y_1, y_2, y_5\}$. This contradiction shows that $\mathscr{S}_x(\{y_5\}; \{y_1, y_2, y_3\}; \{y_4\}) \cup \mathscr{S}_x(\{y_5\}; \{y_2, y_3, y_4\}; \{y_1\}) = \emptyset$. □

By Claim 5.3.1, we know that $\mathscr{S}_x(G; M_1) = \mathscr{S}_x(\{y_5\}; \{y_2, y_3\}; \{y_1, y_4\})$. By symmetry, we have $\mathscr{S}_x(G; M_2) = \mathscr{S}_x(\{y_4\}; \{y_2, y_3\}; \{y_1, y_5\})$. Since x is not properly 5-liftable, we have $\mathscr{S}_x(\{y_5\}; \{y_2, y_3\}; \{y_1, y_4\}) \neq \emptyset$ and $\mathscr{S}_x(\{y_4\}; \{y_2, y_3\}; \{y_1, y_5\}) \neq \emptyset$ by Lemma 4.3 (1). By Lemma 4.4 (2), both $\mathscr{S}_x(\{y_5\}; \{y_2, y_3\}; \{y_1, y_4\})$ and $\mathscr{S}_x(\{y_4\}; \{y_2, y_3\}; \{y_1, y_5\})$ are nontrivial. Now fix $(A; S; \bar{A}) \in \mathscr{S}_x(\{y_5\}; \{y_2, y_3\}; \{y_1, y_4\})$ and $(B; T; \bar{B}) \in \mathscr{S}_x(\{y_4\}; \{y_2, y_3\}; \{y_1, y_5\})$ with $A_x = \{y_5\}$, $\bar{A}_x = \{y_1, y_4\}$ and $B_x = \{y_4\}$, $\bar{B}_x = \{y_1, y_5\}$.

Claim 5.3.2 $y_3 \in V_5(G)$.

Proof. Since $\mathscr{S}_x(\{y_5\}; \{y_2, y_3\}; \{y_1, y_4\})$ is nontrivial, $N_G(y_5) \cap N_G(x) = \{y_3\}$, $S_x = \{y_2, y_3\}$, $y_2y_3 \in E(G)$ and $N_G(x) \subseteq N_G(y_3) \cup N_G(y_2)$, applying Lemma 5.2 (2), we see that $y_3 \in V_5(G)$. □

Now since $\mathscr{S}_x(\{y_5\}; \{y_2, y_3\}; \{y_1, y_4\})$ and $\mathscr{S}_x(\{y_4\}; \{y_2, y_3\}; \{y_1, y_5\})$ are nontrivial, $|S_x| = |T_x| = 2$, $|N_G(x) \cap N_G(y_5)| = |N_G(x) \cap N_G(y_4)| = 1$, $y_5 \in \bar{B}_x$, $y_4 \in \bar{A}_x$ and $|N_G(x) \cap N_G(y_3)| = 3$, and since $y_3 \in S_x \cap T_x \cap V_5(G)$ by Claim 5.3.2, applying Lemma 4.5 (4), we see that $\mathscr{S}_x(\{y_5\}; \{y_2, y_3\}; \{y_1, y_4\})$ or $\mathscr{S}_x(\{y_4\}; \{y_2, y_3\}; \{y_1, y_5\})$ is empty, which contradicts the fact that $(A; S; \bar{A}) \in \mathscr{S}_x(\{y_5\}; \{y_2, y_3\}; \{y_1, y_4\})$ and $(B; T; \bar{B}) \in \mathscr{S}_x(\{y_4\}; \{y_2, y_3\}; \{y_1, y_5\})$, completing the discussion for the case where $G_x \cong \tilde{B}u$. □

6 Fragments of order two

In the following three sections, we consider minimally 5-connected graphs G with $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$.

In this section we show that if a minimally 5-connected graph G with $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$ has a fragment A such that $|A| = 2$ and $A \subseteq V_5(G)$, then there exists $x \in A$ such that $\Delta(G_x) = 4$ (see Lemma 6.5).

Before investigating fragments of order two, we prove the following two lemmas.

Lemma 6.1 *Let G be a minimally 5-connected graph with $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$ and let $x \in V_5(G)$. Suppose that $\Delta(G_x) \leq 3$. Then the following hold.*

(1) *There is a perfect non-edge pseudo-matching M on $N_G(x)$.*
(2) *If M is a perfect non-edge pseudo-matching on $N_G(x)$, then $\mathscr{S}(G^{(x;M)}) \neq \emptyset$, and for every $(B; T'; \bar{B}) \in \mathscr{S}(G^{(x;M)})$, T' is $G_x^{(x;M)}$-dividing and nontrivial.*
(3) *If M is a perfect non-edge pseudo-matching on $N_G(x)$, then $\mathscr{S}_x(G; M) \neq \emptyset$, and for every $(B; T; \bar{B}) \in \mathscr{S}_x(G; M)$, T is G_x-dividing and $|B|, |\bar{B}| \geq 2$.*

Proof. (1) Let $N_G(x) = \{u_1, u_2, \cdots, u_5\}$. We show that G_x has no K_4. Suppose that $K_4 \subseteq G_x$, say $G[\{u_1, u_2, u_3, u_4\}] \cong K_4$. If $xu_5 \in E_n(G)$, then there exists a 5-separator S with $x, u_5 \in S$ and, since $G[\{u_1, u_2, u_3, u_4\}] \cong K_4$, letting A be an S-fragment, we have $\{u_1, u_2, u_3, u_4\}] \subseteq S \cup A$ or $\{u_1, u_2, u_3, u_4\}] \subseteq S \cup \bar{A}$, which implies $\bar{A}_x = \emptyset$ or $A_x = \emptyset$, a contradiction. Thus $xu_5 \in E_c(G)$. Also since $\Delta(G_x) \leq 3$, we observe that $N_G(u_5) \cap \{u_1, u_2, u_3, u_4\} = \emptyset$. Hence we have $xu_5 \in E^{**}(G)$, which contradicts the assumption that $E^{**}(G) = \emptyset$. This contradiction shows that G_x has no K_4.

Assume for the moment that G_x has a triangle, say $G[\{u_1, u_2, u_3\}] \cong K_3$. Since $\Delta(G_x) \leq 3$, we observe that $N_G(u_4) \cap N_G(u_5) \cap \{u_1, u_2, u_3\} = \emptyset$. Since G_x has no K_4, we observe that $|N_G(u_i) \cap \{u_1, u_2, u_3\}| \leq 2$ for $i = 4, 5$. Thus we may assume that $N_G(u_4) \cap \{u_1, u_2, u_3\} \subset \{u_1\}$ and $N_G(u_5) \cap \{u_1, u_2, u_3\} \subset \{u_2, u_3\}$. We now find a perfect non-edge pseudo-matching $M = \{u_2u_4u_3, u_1u_5\}$. Thus we may assume that G_x has no triangle. Then we see that for some four vertices in $\{u_1, u_2, \cdots, u_5\}$, say u_1, u_2, u_3, u_4, $G[\{u_1, u_2, u_3, u_4\}]$ has a perfect non-edge matching. We may assume that $u_1u_2, u_3u_4 \notin E(G)$. Since $\Delta(G_x) \leq 3$, $|N_G(u_5) \cap \{u_1, u_2, u_3, u_4\}| \leq 3$, say $u_5u_1 \notin E(G)$. We now see that G_x has a perfect non-edge pseudo-matching $\{u_2u_1u_5, u_3u_4\}$, as desired. This proves (1).

(2) Since x is not 5-liftable, Lemma 4.3 (1) together with the definition of $\mathscr{S}_x(G; M)$ shows that $\mathscr{S}(G^{(x;M)}) \neq \emptyset$, and it follows from Lemma 4.1 (2) and Lemma 4.4 (1) that T' is $G_x^{(x;M)}$-dividing and nontrivial for every $(B; T'; \bar{B}) \in \mathscr{S}(G^{(x;M)})$.

(3) Since x is not 5-liftable, Lemma 4.3 (1) shows that $\mathscr{S}_x(G; M) \neq \emptyset$, and it follows from Lemma 4.1 (1) and Lemma 4.4 (2) that T is G_x-dividing and $|B|, |\bar{B}| \geq 2$ for every $(B; T; \bar{B}) \in \mathscr{S}_x(G; M)$. □

Lemma 6.2 *Let G be a 5-connected graph. Let S be a 5-separator of G and let A be an S-fragment of G such that $|A| = 2$. Let $y_1, y_2 \in S$. Let T_1 and T_2 be 5-separators of G such that $A \cup \{y_i\} \subseteq T_i$, and let B_i be a T_i-fragment of G for $i = 1, 2$. If $|S \cap B_i| = |S \cap \bar{B}_i| = 2$ for $i = 1, 2$, then neither $N_G(y_1) \cap (S - \{y_1, y_2\})$ nor $N_G(y_2) \cap (S - \{y_1, y_2\})$ is empty.*

Proof. Let $S = \{y_1, y_2, y_3, y_4, y_5\}$. Suppose that $|S \cap B_i| = |S \cap \bar{B}_i| = 2$ for $i = 1, 2$. For each $i \in \{1, 2\}$, since $S \cap T_i = \{y_i\}$, we may assume that $y_{3-i} \in S \cap B_i$ by the symmetry of B_i and $\bar{B}_i$. Thus $y_1 \in S \cap B_2$ and $y_2 \in S \cap B_1$. Let $S \cap B_1 = \{y_2, y\}$ and let $S \cap B_2 = \{y_1, y'\}$. Assume for the moment that $y = y'$, say $y = y' = y_3$. Then, $A \subseteq T_1 \cap T_2$, $y_3 \in B_1 \cap B_2$, $y_1 \in T_1 \cap B_2$, $y_2 \in B_1 \cap T_2$

and $\{y_4, y_5\} \subseteq \bar{B}_1 \cap \bar{B}_2$. Hence, by Lemma 3.2 (5), $B_1 \cap B_2$ is a fragment of G with $N_G(B_1 \cap B_2) = (T_1 \cap B_2) \cup (T_1 \cap T_2) \cup (B_1 \cap T_2) \supseteq \{y_1, y_2\} \cup A$. Since $A \subseteq N_G(B_1 \cap B_2)$, $N_G(A) \cap (B_1 \cap B_2) = S \cap (B_1 \cap B_2) = \{y_3\}$ and $|A| = 2$, applying Lemma 2.2 with U and A replaced by A and $B_1 \cap B_2$, respectively, we see that $B_1 \cap B_2 = \{y_3\}$. Since $\{y_1, y_2\} \subseteq N_G(B_1 \cap B_2)$, we get $\{y_1y_3, y_2y_3\} \subseteq E(G)$, which implies that $N_G(y_i) \cap (S - \{y_1, y_2\}) \neq \emptyset$ for $i = 1, 2$, as desired. Thus we may assume that $y \neq y'$, say $y = y_3$ and $y' = y_4$. Then, $A \subseteq T_1 \cap T_2$, $y_3 \in B_1 \cap \bar{B}_2$, $y_4 \in \bar{B}_1 \cap B_2$, $y_1 \in T_1 \cap B_2$, $y_2 \in B_1 \cap T_2$ and $y_5 \in \bar{B}_1 \cap \bar{B}_2$. Hence by Lemma 3.2 (5), $B_1 \cap \bar{B}_2$ is a fragment of G with $N_G(B_1 \cap \bar{B}_2) \supseteq A \cup \{y_2\}$. Since $A \subseteq N_G(B_1 \cap \bar{B}_2)$, $N_G(A) \cap (B_1 \cap \bar{B}_2) = \{y_3\}$ and $|A| = 2$, applying Lemma 2.2 with U and A replaced by A and $B_1 \cap \bar{B}_2$, respectively, we see that $B_1 \cap \bar{B}_2 = \{y_3\}$. Since $y_2 \in N_G(B_1 \cap \bar{B}_2)$, we get $y_2y_3 \in E(G)$. By symmetry, we have $y_1y_4 \in E(G)$, which implies that $N_G(y_i) \cap (S - \{y_1, y_2\}) \neq \emptyset$ for $i = 1, 2$. This proves Lemma 6.2. □

Lemma 6.3 *Let G be a minimally 5-connected graph with $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$. Let A be a fragment of G with $|A| = 2$ and $E(A) = \emptyset$. Then $\Delta(G[N_G(A)]) = 4$.*

Proof. Write $A = \{x_1, x_2\}$. Then $x_1, x_2 \in V_5(G)$ and $N_G(A) = N_G(x_1) = N_G(x_2)$. Let $S = N_G(A)$ and write $S = \{y_1, y_2, y_3, y_4, y_5\}$. Note that $G_{x_i} = G[S]$ for $i = 1, 2$.

Claim 6.3.1 *Let $x \in \{x_1, x_2\}$ and suppose that S has a vertex y such that $xy \in E_n(G)$. Let T be a 5-separator of G which contains $\{x, y\}$ and let B be a T-fragment. If $N_G(y) \cap S = \emptyset$, then $A \subseteq T$ and $|S \cap B| = |S \cap \bar{B}| = 2$.*

Proof. We may assume that $x = x_1$. We show that $x_2 \in T$. Suppose that $x_2 \notin T$. We may assume that $x_2 \in B$. Then $N_G(x_2) \cap \bar{B} = \emptyset$. Since $N_G(x_1) = N_G(x_2)$, it follows that $N_G(x_1) \cap \bar{B} = \emptyset$. Since $x_1 \in T$, this contradicts the fact that B is a T-fragment. Thus $x_2 \in T$. Hence $A \subseteq T$. Now if $|B| = 1$, then applying Lemma 3.4 with x replaced by y, we see that the vertex of B is in $N_G(y) \cap S$, which contradicts the assumption that $N_G(y) \cap S = \emptyset$. Thus $|B| \geq 2$. By symmetry, we have $|\bar{B}| \geq 2$. Consequently $|S \cap B| = |S \cap \bar{B}| = 2$ by Lemma 9. □

Claim 6.3.2 *The graph $G[S]$ has at most one isolated vertex.*

Proof. Suppose that $G[S]$ has two or more isolated vertices. We may assume $N_G(y_i) \cap S = \emptyset$ for $i = 1, 2$. If $y_1x_1 \in E_c(G)$, then since $N_G(y_1) \cap N_G(x_1) = N_G(y_1) \cap S = \emptyset$, we get $y_1x_1 \in E^{**}(G)$, a contradiction. Thus $y_1x_1 \in E_n(G)$. Let T_1 be a 5-separator of G which contains $\{x_1, y_1\}$ and let B_1 be a T_1-fragment of G. By Claim 6.3.1, $\{y_1\} \cup A \subseteq T_1$ and $|S \cap B_1| = |S \cap \bar{B}_1| = 2$. Similarly there exists a 5-separator T_2 with $\{y_2\} \cup A \subseteq T_2$ such that for a T_2-fragment B_2, we have $|S \cap B_2| = |S \cap \bar{B}_2| = 2$. Hence $N_G(y_1) \cap S \neq \emptyset$ by Lemma 6.2, which contradicts the choice of y_1. This contradiction proves Claim 6.3.2. □

Claim 6.3.3 *If there is a vertex $y \in S$ such that $N_G(y) \cap S = \emptyset$, then $G[S] \cong K_1 \cup 2K_2$.*

Proof. Suppose that $y \in S$ and $N_G(y) \cap S = \emptyset$. Since $E^{**}(G) = \emptyset$, we see that xy is 5-noncontractible for $x \in \{x_1, x_2\}$. Let T be a 5-separator which contains $\{x, y\}$ and let B be a T-fragment of G. By Claim 6.3.1, $|S \cap B| = |S \cap \bar{B}| = 2$. Since y

is the only isolated vertex of $G[S]$ by Claim 6.3.2, this forces $G[S \cap B] \cong K_2$ and $G[S \cap \bar{B}] \cong K_2$. Consequently $G[S] \cong K_1 \cup 2K_2$, as desired. □

Claim 6.3.4 *Let M be a perfect non-edge pseudo-matching on S and let $x \in \{x_1, x_2\}$. Then $G^{(x;M)}[S]$ does not contain C_5.*

Proof. Suppose that $C_5 \subseteq G^{(x;M)}[S]$. Since $\mathscr{L}^*(G) = \emptyset$, $\mathscr{S}_x(G; M) \neq \emptyset$ by Lemma 6.1 (3). Take $(B; T; \bar{B}) \in \mathscr{S}_x(G; M)$. By Lemma 6.1 (3), $|B|, |\bar{B}| \geq 2$. By the definition of $\mathscr{S}_x(G; M)$, $(B; T - \{x\}; \bar{B}) \in \mathscr{S}(G^{(x;M)})$. By Lemma 6.1 (2), $T - \{x\}$ divides $G_x^{(x;M)} = G^{(x;M)}[S]$. Since $C_5 \subseteq G^{(x;M)}[S]$, it follows that $|S \cap T| = |S \cap (T - \{x\})| \geq 2$. By the definition of $\mathscr{S}_x(G; M)$, $x \in T$. Since $N_G(x_1) = N_G(x_2)$, arguing as in the proof of Claim 6.3.1, we get $A \subseteq T$. Therefore, applying Lemma 3.5 (2) with x replaced by a vertex in $S \cap T$, we obtain $|S \cap T| = 1$, which contradicts the earlier assertion that $|S \cap T| \geq 2$. □

We are in a position to complete the proof of Lemma 6.3. By way of contradiction, suppose that $\Delta(G[S]) \leq 3$. If $G[S]$ has an isolated vertex, then by Claim 6.3.3, $G[S] \cong K_1 \cup 2K_2$. If $G[S]$ has no isolated vertex and $G[S]$ is disconnected, we have either $G[S] \cong K_2 \cup K_3$ or $G[S] \cong K_2 \cup P_3$. In each case, we can find a perfect non-edge pseudo-matching M on S such that $C_5 \subseteq G^{(x;M)}[S]$, which contradicts Claim 6.3.4. Thus $G[S]$ is connected. If $G[S] \cong C_5$, then by Lemma 6.1 (1), we get a contradiction to Claim 6.3.4. Thus $G[S] \ncong C_5$. Since $\Delta(G[S]) \leq 3$, we now see from Lemma 5.3 and Lemma 3.1 that either $G[S] \cong P_5$ or $G[S] \cong D_{1,2}$ (see the paragraph following the proof of Lemma 5). In each case, we can again find a perfect non-edge pseudo-matching M on S such that $C_5 \subseteq G^{(x;M)}[S]$, which contradicts Claim 6.3.4, and this contradiction completes the proof of Lemma 6.3. □

Before proceeding to the analysis of connected fragments of order two, we prove the following lemma.

Lemma 6.4 *Let G be a minimally 5-connected graph with $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$. Suppose that G has a fragment A with $|A| = 2$, say $A = \{x_1, x_2\}$ and write $N_G(A) = \{y_1, y_2, y_3, y_4, y_5\} = S$. Moreover suppose that $N_G(x_i) = \{x_{3-i}, y_i, y_3, y_4, y_5\}$ and $\Delta(G_{x_i}) \leq 3$ for $i = 1, 2$.*

(1) *Let $i \in \{1, 2\}$ and $j \in \{3, 4, 5\}$ and suppose that $y_i y_j \in E(G)$. Write $\{3, 4, 5\} = \{j, j', j''\}$. Then $G_{x_i} \cong D_{1,2}$ and G has a 5-separator T and a T-fragment B such that* (i) *$A \subseteq T$,* (ii) *$|S \cap B| = |S \cap \bar{B}| = 2$, and* (iii) *either $S \cap T = \{y_i\}$ and T separates $\{y_{3-i}, y_{j'}\}$ and $\{y_j, y_{j''}\}$, or $S \cap T = \{y_j\}$ and T separates $\{y_i, y_{j'}\}$ and $\{y_{3-i}, y_{j''}\}$.*
(2) *$(N_G(y_1) \cap N_G(y_2)) \cap \{y_3, y_4, y_5\} = \emptyset$.*

Proof. (1) Let i, j, j', j'' be as in (1). Since $y_i y_j, y_j x_{3-i}, x_{3-i} y_{j'}, x_{3-i} y_{j''} \in E(G)$, $D_{1,2} \subseteq G_{x_i}$. In view of Lemma 5.3, it follows that $E(G_{x_i}) = \{y_i y_j, y_j x_{3-i}, x_{3-i} y_{j'}, x_{3-i} y_{j''}\}$ and $G_{x_i} \cong D_{1,2}$ (see the paragraph following the proof of Lemma 5). Hence $M = \{x_{3-i} y_i y_{j'}, y_j y_{j''}\}$ is a perfect non-edge pseudo-matching on G_{x_i}. By Lemma 6.1 (3), $\mathscr{S}_{x_i}(G; M) \neq \emptyset$. Take $(B; T; \bar{B}) \in \mathscr{S}_{x_i}(G; M)$. By Lemma 6.1 (3),

$|B|, |\bar{B}| \geq 2$. Set $T' = T - \{x_i\}$. We have $(B; T'; \bar{B}) \in \mathscr{S}(G^{(x_i;M)})$ by definition. By Lemma 6.1 (2), T' is $G_{x_i}^{(x_i;M)}$-dividing. Since $\{y_i, y_j, y_{j'}, y_{j''}\} \subseteq N_{G_{x_i}^{(x_i;M)}}(x_{3-i})$ and $G_{x_i}^{(x_i;M)}[\{y_i, y_j, y_{j'}, y_{j''}\}]$ is a path $y_{j'}y_iy_jy_{j''}$, it follows that $x_{3-i} \in T'$ and $\{y_i, y_j\} \cap T' \neq \emptyset$. Hence $A \subseteq T$ and $\{y_i, y_j\} \cap T \neq \emptyset$. Applying Lemma 9(2) with x replaced by a vertex in $\{y_i, y_j\} \cap T$, we get $|S \cap B| = |S \cap \bar{B}| = 2$ and we have $S \cap T = \{y_i\}$ or $S \cap T = \{y_j\}$. Thus (i) and (ii) are shown. Since $G_{x_i}^{(x_i;M)}[\{y_i, y_j, y_{j'}, y_{j''}\}] = y_{j'}y_iy_jy_{j''}$ and $S = \{y_{3-i}, y_i, y_j, y_{j'}, y_{j''}\}$, we see that if $S \cap T = \{y_i\}$, then T separates $\{y_{3-i}, y_{j'}\}$ and $\{y_j, y_{j''}\}$, and that if $S \cap T = \{y_j\}$, then T separates $\{y_i, y_{j'}\}$ and $\{y_{3-i}, y_{j''}\}$. Now (iii) is shown.

(2) Suppose that $(N_G(y_1) \cap N_G(y_2)) \cap \{y_3, y_4, y_5\} \neq \emptyset$, say $y_3 \in N_G(y_1) \cap N_G(y_2)$. Applying (1) with $j = 3$, $j' = 4$ and $j'' = 5$, we see that, for $i = 1, 2$, $G_x \cong D_{1,2}$ and G has a 5-separator T_i and a T_i-fragment B_i such that (i) $A \subseteq T_i$, (ii) $|S \cap B_i| = |S \cap \bar{B}_i| = 2$, and either $S \cap T_i = \{y_i\}$ and T_i separates $\{y_{3-i}, y_4\}$ and $\{y_3, y_5\}$, or $S \cap T_i = \{y_3\}$ and T_i separates $\{y_i, y_4\}$ and $\{y_{3-i}, y_5\}$. By the symmetry of the B_i and $\bar{B}_i$, we may assume that either $S \cap T_i = \{y_i\}$, $S \cap B_i = \{y_{3-i}, y_4\}$ and $S \cap \bar{B}_i = \{y_3, y_5\}$, or $S \cap T_i = \{y_3\}$, $S \cap B_i = \{y_i, y_4\}$ and $S \cap \bar{B}_i = \{y_{3-i}, y_5\}$.

We divide the proof into three cases according to $S \cap T_1$ and $S \cap T_2$.

Case I : $S \cap T_1 = \{y_1\}$

In this case, $S \cap B_1 = \{y_2, y_4\}$ and $S \cap \bar{B}_1 = \{y_3, y_5\}$, and hence $y_2y_3 \notin E(G)$, which contradicts the assumption that $y_3 \in N_G(y_1) \cap N_G(y_2)$.

Case II : $S \cap T_1 = \{y_3\}$ and $S \cap T_2 = \{y_2\}$

In this case, $S \cap B_2 = \{y_1, y_4\}$ and $S \cap \bar{B}_2 = \{y_3, y_5\}$, and hence $y_1y_3 \notin E(G)$, which contradicts the assumption that $y_3 \in N_G(y_1) \cap N_G(y_2)$.

Case III : $S \cap T_1 = \{y_3\}$ and $S \cap T_2 = \{y_3\}$

In this case, $y_4 \in B_1 \cap B_2$, $y_1 \in B_1 \cap \bar{B}_2$, $y_2 \in \bar{B}_1 \cap B_2$, $\{x_1, x_2, y_3\} \subseteq T_1 \cap T_2$ and $y_5 \in \bar{B}_1 \cap \bar{B}_2$. Hence by Lemma 3.2 (5), $B_1 \cap B_2$ is a fragment of G with $N_G(B_1 \cap B_2) \supseteq \{x_1, x_2, y_3\}$. Since $N_G(\{x_1, x_2\}) \cap (B_1 \cap B_2) = \{y_4\}$, applying Lemma 2 with U and A replaced by $\{x_1, x_2\}$ and $B_1 \cap B_2$, respectively, we see that $B_1 \cap B_2 = \{y_4\}$, which implies $y_3y_4 \in E(G)$. This contradicts the fact that $G_{x_1} \cong D_{1,2}$ and this contradiction shows that Case III does not occur.

Consequently none of the three cases occurs and (2) is shown. □

Lemma 6.5 *Let G be a minimally 5-connected graph with $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$ and suppose that G has a fragment A with $|A| = 2$ and $A \subseteq V_5(G)$. Then there exists $x \in A$ such that $\Delta(G_x) = 4$.*

Proof. Suppose that G has a fragment A such that $|A| = 2$ and $A \subseteq V_5(G)$, say $A = \{x_1, x_2\}$. Let $S = N_G(A) = \{y_1, y_2, y_3, y_4, y_5\}$. If $x_1x_2 \notin E(G)$, then $\Delta(G_{x_1}) = \Delta(S) = 4$ by Lemma 6.3. Thus we may assume that $x_1x_2 \in E(G)$. By symmetry, we may assume that $N_G(x_1) = \{x_2, y_1, y_3, y_4, y_5\}$ and $N_G(x_2) = \{x_1, y_2, y_3, y_4, y_5\}$. Note that we have $x_1y_2 \notin E(G)$ and $x_2y_1 \notin E(G)$. By way of contradiction, suppose that $\Delta(G_{x_i}) \leq 3$ for $i = 1, 2$.

Claim 6.5.1 *Let $y \in \{y_1, y_2\}$. If $N_G(y) \cap \{y_3, y_4, y_5\} = \emptyset$, then there is a 5-separator T such that $A \cup \{y\} \subseteq T$. Moreover, there is a T-fragment B such that $|S \cap B| = |S \cap \bar{B}| = 2$.*

Proof. We may assume that $y = y_1$. If x_1y_1 is 5-contractible, then since $E^{**}(G) = \emptyset$, $|N_G(y_1) \cap N_G(x_1)| \geq 2$, which implies $N_G(y_1) \cap \{y_3, y_4, y_5\} \neq \emptyset$, a contradiction. Thus x_1y_1 is 5-noncontractible. Let T be a 5-separator of G which contains $\{x_1, y_1\}$ and let B be a T-fragment of G. Since $x_2y_1 \notin E(G)$, $A_{y_1} = \{x_1\}$. Hence applying Lemma 3.5 (1) with $x = y_1$, we obtain $A \subseteq T$. Suppose that $|B| = 1$, and write $B = \{z\}$. By Lemma 3.4, $z \in S_{y_1}$ and $T = N_G(z)$. Since $N_G(y_1) \cap \{y_3, y_4, y_5\} = \emptyset$, this forces $z = y_2$, and hence $T = N_G(y_2)$. Since $x_1 \in A \subseteq T$, this contradicts the fact that $x_1y_2 \notin E(G)$. Thus $|B| \geq 2$. Similarly $|\bar{B}| \geq 2$. Consequently it follows from Lemma 3.5 (2) that $|S \cap B| = |S \cap \bar{B}| = 2$, as desired.

Now Claim 6.5.1 is shown. □

Claim 6.5.2 *Either $N_G(y_1) \cap \{y_3, y_4, y_5\} \neq \emptyset$ or $N_G(y_2) \cap \{y_3, y_4, y_5\} \neq \emptyset$.*

Proof. Suppose that $N_G(y_1) \cap \{y_3, y_4, y_5\} = N_G(y_2) \cap \{y_3, y_4, y_5\} = \emptyset$. By Claim 6.5.1, there is a 5-separator T_i and a T_i-fragment B_i such that $A \cup \{y_i\} \subseteq T_i$ and $|S \cap B_i| = |S \cap \bar{B}_i| = 2$ for $i = 1, 2$. Hence, applying Lemma 6.2, we see that $N_G(y_1) \cap (S - \{y_1, y_2\}) \neq \emptyset$. This contradicts the assumption that $N_G(y_1) \cap \{y_3, y_4, y_5\} = \emptyset$ and this contradiction proves Claim 6.5.2. □

Claim 6.5.3 *Neither $N_G(y_1) \cap \{y_3, y_4, y_5\}$ nor $N_G(y_2) \cap \{y_3, y_4, y_5\}$ is empty.*

Proof. Suppose either $N_G(y_1) \cap \{y_3, y_4, y_5\}$ or $N_G(y_2) \cap \{y_3, y_4, y_5\}$ is empty. By Claim 6.5.2, we know that either $N_G(y_1) \cap \{y_3, y_4, y_5\} \neq \emptyset$ or $N_G(y_2) \cap \{y_3, y_4, y_5\} \neq \emptyset$, say $N_G(y_1) \cap \{y_3, y_4, y_5\} \neq \emptyset$ and $y_1y_3 \in E(G)$. Then $N_G(y_2) \cap \{y_3, y_4, y_5\} = \emptyset$. Applying Lemma 6.4(1) with $i = 1$ and $j = 3$, we see that G has a 5-separator T_1 and a T_1-fragment B_1 such that (i) $A \subseteq T_1$, (ii) $|S \cap B_1| = |S \cap \bar{B}_1| = 2$, and either $S \cap T_1 = \{y_1\}$ and $y_2y_3 \notin E(G)$, or $S \cap T_1 = \{y_3\}$ and $y_1y_2 \notin E(G)$. Applying Claim 6.5.1 with $y = y_2$, we see that G has a 5-separator T_2 and a T_2-fragment B_2 such that $A \cup \{y_2\} \subseteq T_2$ and $|S \cap B_2| = |S \cap \bar{B}_2| = 2$. If $S \cap T_1 = \{y_1\}$, then applying Lemma 6.2, we see that $N_G(y_2) \cap \{y_3, y_4, y_5\} \neq \emptyset$, which contradicts the assertion that $N_G(y_2) \cap \{y_3, y_4, y_5\} = \emptyset$. Thus $S \cap T_1 = \{y_3\}$ and $y_1y_2 \notin E(G)$. Hence applying Lemma 6.2, we see that $N_G(y_2) \cap \{y_1, y_4, y_5\} = N_G(y_2) \cap (S - \{y_2, y_3\}) \neq \emptyset$. Since $y_1y_2 \notin E(G)$, this implies that $N_G(y_2) \cap \{y_4, y_5\} \neq \emptyset$. This contradicts the assertion that $N_G(y_2) \cap \{y_3, y_4, y_5\} = \emptyset$, and this contradiction proves Claim 6.5.3. □

By Claim 6.5.3, neither $N_G(y_1) \cap \{y_3, y_4, y_5\}$ nor $N_G(y_2) \cap \{y_3, y_4, y_5\}$ is empty. By Lemma 6.4(2), $(N_G(y_1) \cap N_G(y_2)) \cap \{y_3, y_4, y_5\} = \emptyset$. Hence we may assume that $y_3 \in N_G(y_1)$ and $y_4 \in N_G(y_2)$. Applying Lemma 6.4(1) with $i = 1$, $j = 3$, $j' = 4$ and $j'' = 5$, we see that G has a 5-separator T_1 with $A \subseteq T_1$ and a T_1-fragment B_1 such that either $S \cap T_1 = \{y_1\}$, $S \cap B_1 = \{y_3, y_5\}$ and $S \cap \bar{B}_1 = \{y_2, y_4\}$, or $S \cap T_1 = \{y_3\}$, $S \cap B_1 = \{y_1, y_4\}$ and $S \cap \bar{B}_1 = \{y_2, y_5\}$. Again applying Lemma 6.4(1) with $i = 2$, $j = 4$, $j' = 3$ and $j'' = 5$, we see that G has a 5-separator T_2 with $A \subseteq T_2$ and a T_2-fragment B_2 such that either $S \cap T_2 = \{y_2\}$, $S \cap B_2 = \{y_4, y_5\}$ and $S \cap \bar{B}_2 = \{y_1, y_3\}$, or $S \cap T_2 = \{y_4\}$, $S \cap B_2 = \{y_2, y_3\}$ and $S \cap \bar{B}_2 = \{y_1, y_5\}$.

Claim 6.5.4 *$S \cap T_1 = \{y_1\}$ and $S \cap T_2 = \{y_2\}$.*

Proof. If $S \cap T_1 = \{y_3\}$, then $S \cap B_1 = \{y_1, y_4\}$ and $S \cap \bar{B}_1 = \{y_2, y_5\}$, and hence $y_2y_4 \notin E(G)$, which contradicts the assumption that $y_4 \in N_G(y_2)$. If $S \cap T_2 = \{y_4\}$, then $S \cap B_2 = \{y_2, y_3\}$ and $S \cap \bar{B}_2 = \{y_1, y_5\}$, and hence $y_1y_3 \notin E(G)$, which contradicts the assumption that $y_3 \in N_G(y_1)$. Thus we have $S \cap T_1 = \{y_1\}$ and $S \cap T_2 = \{y_2\}$. □

Claim 6.5.5

(1) *$\bar{B}_1 \cap \bar{B}_2 = \emptyset$, $T_1 \cap \bar{B}_2 = \{y_1\}$ and $\bar{B}_1 \cap T_2 = \{y_2\}$.*
(2) *$B_1 \cap \bar{B}_2 = \{y_3\}$ and $\bar{B}_1 \cap B_2 = \{y_4\}$.*
(3) *$|T_1 \cap T_2| = 3$, $|T_1 \cap B_2| = |B_1 \cap T_2| = 1$ and $B_1 \cap B_2 = \{y_5\}$.*

Proof. (1) By Claim 6.5.4, $S \cap T_1 = \{y_1\}$ and $S \cap T_2 = \{y_2\}$. Thus, $y_3 \in B_1 \cap \bar{B}_2$, $y_1 \in T_1 \cap \bar{B}_2$, $y_2 \in \bar{B}_1 \cap T_2$, $y_5 \in B_1 \cap B_2$, $\{x_1, x_2\} \subseteq T_1 \cap T_2$ and $y_4 \in \bar{B}_1 \cap B_2$. In particular, $(\bar{B}_1 \cap \bar{B}_2)_{x_1} = \emptyset$. Hence by Lemma 3.2 (7), $\bar{B}_1 \cap \bar{B}_2 = \emptyset$. Also it follows from Lemma 3.2 (5) that both $B_1 \cap \bar{B}_2$ and $\bar{B}_1 \cap B_2$ are fragments of G and $|(\bar{B}_1 \cap T_2) \cup (T_1 \cap T_2) \cup (T_1 \cap B_2)| = 5$. Applying Lemma 3.2 (2) with A and B replaced by $\bar{B}_1$ and B_2, respectively, we get $|T_1 \cap \bar{B}_2| = |\bar{B}_1 \cap T_2|$. On the other hand, since $B_1 \cap B_2 \neq \emptyset$, we see from Lemma 3.2 (1) that $|(B_1 \cap T_2) \cup (T_1 \cap T_2) \cup (T_1 \cap B_2)| \geq 5$. Applying Lemma 3.2 (2) with A and B replaced by B_1 and B_2, respectively, we get $|T_1 \cap \bar{B}_2| \leq |B_1 \cap T_2|$. Hence, applying Lemma 3.2 (2) with A and B replaced by $\bar{B}_1$ and $\bar{B}_2$, respectively, we obtain $|(\bar{B}_1 \cap T_2) \cup (T_1 \cap T_2) \cup (T_1 \cap \bar{B}_2)| \leq 5$. Since $|T_1 \cap \bar{B}_2| = |\bar{B}_1 \cap T_2| \geq 1$ and $|T_1 \cap T_2| \geq 2$, it follows that $|T_1 \cap \bar{B}_2| = |\bar{B}_1 \cap T_2| = 1$, which implies $T_1 \cap \bar{B}_2 = \{y_1\}$ and $\bar{B}_1 \cap T_2 = \{y_2\}$, and (1) is shown.

(2) Recall that both $B_1 \cap \bar{B}_2$ and $\bar{B}_1 \cap B_2$ are fragments of G. Since $N_G(\{x_1, x_2\}) \cap (B_1 \cap \bar{B}_2) = \{y_3\}$, applying Lemma 2 with U and A replaced by $\{x_1, x_2\}$ and $B_1 \cap \bar{B}_2$, respectively, we see that $B_1 \cap \bar{B}_2 = \{y_3\}$. Since $N_G(\{x_1, x_2\}) \cap (\bar{B}_1 \cap B_2) = \{y_4\}$, applying Lemma 2 with U and A replaced by $\{x_1, x_2\}$ and $\bar{B}_1 \cap B_2$, respectively, we see that $\bar{B}_1 \cap B_2 = \{y_4\}$. Now (2) is shown.

(3) By (1) and (2), $\bar{B}_2 = \{y_1, y_3\}$. Write $T_2 - \{y_2\} = \{x_1, x_2, u, w\}$. Since $y_1x_2 \notin E(G)$, $N_G(y_1) = (T_2 - \{x_2\}) \cup \{y_3\} = \{x_1, y_2, y_3, u, w\}$. By Lemma 3.2 (3), $N_G(y_3) = \{x_1, x_2, y_1, u, w\}$. In particular, $y_3y_2 \notin E(G)$ and $y_1, y_3 \in V_5(G)$. Since $u, w \notin S = N_G(\{x_1, x_2\})$, $ux_1, ux_2, wx_1, wx_2 \notin E(G)$. Since $y_3y_2, y_1x_2 \notin E(G)$, it follows that $\Delta(G_{y_1}) \leq 3$ and $\Delta(G_{y_3}) \leq 3$. Recall that $\bar{B}_2 = \{y_1, y_2\}$ is T_2-fragment, $T_2 = \{y_2, x_2, x_1, u, w\}$, and $y_1x_2, y_3y_2 \notin E(G)$. Therefore we can apply Claim 6.5.3 with A, S, x_1, x_2, y_1, y_2 replaced by $\bar{B}_2, T_2, y_1, y_3, y_2, x_2$, respectively, to see that $N_G(y_2) \cap \{x_1, u, w\} \neq \emptyset$. Since $y_2x_1 \notin E(G)$, we get $N_G(y_2) \cap \{u, w\} \neq \emptyset$. Thus we may assume that $u \in N_G(y_2) \cap T_2$. On the other hand, since $\bar{B}_1 \cap T_2 = \{y_2\}$, we also have $N_G(y_2) \cap T_2 = N_G(y_2) \cap (T_1 \cap T_2)$. Consequently, $\{x_1, x_2, u\} \subseteq T_1 \cap T_2$, which implies that $|T_1 \cap T_2| \geq 3$. Since $B_1 \cap B_2 \neq \emptyset$, $|B_1 \cap T_2| \geq |T_1 \cap \bar{B}_2| = 1$ and $|T_1 \cap B_2| \geq |\bar{B}_1 \cap T_2| = 1$, by Lemma 3.2 (4). Since $|B_1 \cap T_2| + |T_1 \cap T_2| + |\bar{B}_1 \cap T_2| = 5$ and $|T_1 \cap B_2| + |T_1 \cap T_2| + |T_1 \cap \bar{B}_2| = 5$, we obtain $|B_1 \cap T_2| = |T_1 \cap B_2| = 1$ and $|T_1 \cap T_2| = 3$. Since $B_1 \cap B_2 \neq \emptyset$, it follows from Lemma 3.2 (3) that $B_1 \cap B_2$ is a fragment of G with $N_G(B_1 \cap B_2) \supseteq \{x_1, x_2\}$. Since $N_G(\{x_1, x_2\}) \cap (B_1 \cap B_2) = \{y_5\}$,

applying Lemma 2 with U and A replaced by $\{x_1, x_2\}$ and $B_1 \cap B_2$, respectively, we see that $B_1 \cap B_2 = \{y_5\}$. Now (3) is shown. □

We are in a position to complete the proof of Lemma 6.5. Let u, w be as in the proof of Claim 6.5.5 and write $T_1 \cap B_2 = \{v\}$. Note that $B_1 \cap T_2 = \{w\}$. Since $wx_1, wx_2, \notin E(G)$ and $|N_G(w)| \geq 5$, we observe that $N_G(w) = \{y_1, y_3, u, v, y_5\}$. Since $N_G(y_1) \supseteq T_2 - \{x_2\}$ and $x_2x_1 \in E(G)$, we see that $\{x_2x_1, x_1y_1, y_1u, y_1w, uw\} \subseteq E(G)$. therefore G_{y_3} is connected and contains a cycle and $\Delta(G_{y_3}) \geq 3$. Since $\Delta(G_{y_3}) \leq 3$, this contradicts Lemma 5.3. This contradiction completes the proof of Lemma 6.5. □

7 Dominating vertex

In this section we show that every minimally 5-connected graph G with $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$ has a dominating vertex. Note that a vertex x in $V_5(G)$ is dominated by some other vertex if and only if $\Delta(G_x) = 4$.

Lemma 7.1 *Let G be a minimally 5-connected graph with $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$. Suppose that G has no dominating vertex.*

(1) *Let $x \in V_5(G)$. Then, there is a fragment $A \in \mathscr{F}_2(G)$ such that $x \in N_G(A)$; in particular, $|G| \geq 9$.*
(2) *Let $xy \in E(G)$ and $\{x, y\} \subseteq V_{\geq 6}(G)$. Then, there is a fragment $A \in \mathscr{F}_2(G)$ such that $\{x, y\} \cap N_G(A) \neq \emptyset$.*

Proof. (1) Let $x \in V_5(G)$. Since G has no vertex which dominates x, we have $\Delta(G_x) \leq 3$, and hence it follows from Lemma 6.1 (1) that there is a perfect non-edge pseudo-matching M on $N_G(x)$. By Lemma 6.1 (3), $\mathscr{S}_x(G; M) \neq \emptyset$. Take $(A; S; \bar{A}) \in \mathscr{S}_x(G; M)$. Then $A \in \mathscr{F}_2(G)$ by Lemma 6.1 (3) and $x \in N_G(A) = S$ by definition. We also have $|G| = |A| + |S| + |\bar{A}| \geq 9$.

(2) Since $V_5(G) \neq \emptyset$ by Lemma 2.1, $|G| \geq 9$ by (1). Since G is minimally 5-connected, $G - xy$ has a 4-separator S' which separates x and y. Let A' be an S'-fragment of $G - xy$ such that $x \in A'$ and $y \in \bar{A}'$. Since $x, y \in V_{\geq 6}(G)$, we observe that $x, y \in V_{\geq 5}(G - xy)$, which implies that $|A'|, |\bar{A}'| \geq 2$. Since $|G| \geq 9$ and $|S'| = 4$, we see that either $|A'| \geq 3$ or $|\bar{A}'| \geq 3$, say $|A'| \geq 3$. Let $S = S' \cup \{x\}$. Then S is a 5-separator of G and $A = A' - \{x\}$ is an S-fragment such that $|A| = |A'| - 1 \geq 2$ and $|\bar{A}| = |\bar{A}'| \geq 2$. Hence $A \in \mathscr{F}_2(G)$ and we have $x \in S = N_G(A)$. Now (2) is shown. □

Lemma 7.2 *Let G be a minimally 5-connected graph with $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$. Then G has a dominating vertex.*

Proof. Suppose that G has no dominating vertex. Then $\Delta(G_x) \leq 3$ for all $x \in V_5(G)$. By Lemma 2.1, $V_5(G)$ is not empty. Hence by Lemma 7.1 (1), $\mathscr{F}_2(G) \neq \emptyset$. Let A be a $\mathscr{F}_2(G)$-minimum fragment of G and let $S = N_G(A)$. Since $A \in \mathscr{F}_2(G)$, $|A| \geq 2$. We show that $|A| \geq 3$. Suppose that $|A| = 2$, say $A = \{x_1, x_2\}$. Since $\Delta(G_x) \leq 3$

for all $x \in V_5(G)$, it follows from Lemma 6.5 that $\{x_1, x_2\} \not\subseteq V_5(G)$. We may assume that $x_1 \in V_{\geq 6}(G)$. Then, $N_G(x_2) \cup \{x_2\} \subseteq N_G(A) \cup A = N_G(x_1) \cup \{x_1\}$ and hence x_1 dominates x_2, which is a contradiction. This contradiction shows that $|A| \geq 3$. Since A is $\mathscr{F}_2(G)$-minimum, we have $|\bar{A}| \geq 3$. Since A is $\mathscr{F}_2(G)$-minimum, A is connected. Applying (1) or (2) of Lemma 7.1 according as $A \cap V_5(G) \neq \emptyset$ or $A \subseteq V_{\geq 6}(G)$, we see that there is a fragment $B \in \mathscr{F}_2(G)$ such that $N_G(B) \cap A \neq \emptyset$. Let $T = N_G(B)$.

Claim 7.2.1 $|S \cap B| \geq |A \cap T|$, $|S \cap \bar{B}| \geq |A \cap T|$.

Proof. Suppose that $|S \cap \bar{B}| < |A \cap T|$. Then $\bar{A} \cap \bar{B} = \emptyset$ by Lemma 3.2 (4). Hence $|\bar{B}| = |S \cap \bar{B}| + |A \cap \bar{B}| < |A \cap T| + |A \cap \bar{B}| \leq |A|$, which contradicts the fact that A is $\mathscr{F}_2(G)$-minimum. Thus $|S \cap \bar{B}| \geq |A \cap T|$. By symmetry, we similarly get $|S \cap B| \geq |A \cap T|$. □

Claim 7.2.2 $|A \cap B|, |A \cap \bar{B}| \leq 1$ *and* $|A \cap T| \leq 2$.

Proof. Suppose $|A \cap B| \geq 2$. Then since $A \cap T \neq \emptyset$ and A is $\mathscr{F}_2(G)$-minimum, we see that $A \cap B$ is not a fragment of G, which implies that $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \geq 6$. Hence, $|S \cap \bar{B}| < |A \cap T|$ by Lemma 3.2 (2), which contradicts Claim 7.2.1. This contradiction shows $|A \cap B| \leq 1$. By the symmetry of $A \cap B$ and $A \cap \bar{B}$, we also get $|A \cap \bar{B}| \leq 1$. Since $|S| = 5$, we observe that either $|S \cap B| \leq 2$ or $|S \cap \bar{B}| \leq 2$, say $|S \cap B| \leq 2$. Thus $|A \cap T| \leq |S \cap B| \leq 2$ by Claim 7.2.1. □

Claim 7.2.2 assures us that $|A| \leq 4$.

Claim 7.2.3 $|A| \neq 4$.

Proof. Suppose that $|A| = 4$. By Claim 7.2.2, $|A \cap B| = |A \cap \bar{B}| = 1$ and $|A \cap T| = 2$, say $A \cap T = \{x_1, x_2\}$, $A \cap B = \{x_3\}$ and $A \cap \bar{B} = \{x_4\}$. By Claim 7.2.1, $|S \cap B| \geq |A \cap T|$. Since $A \cap \bar{B} \neq \emptyset$, $|A \cap T| \geq |S \cap B|$ by Lemma 3.2 (4). Hence $|S \cap B| = |A \cap T| = 2$. By the symmetry of $S \cap B$ and $S \cap \bar{B}$, we also have $|S \cap \bar{B}| = |A \cap T| = 2$. Consequently, $|S \cap T| = 1$. Write $S \cap B = \{y_1, y_2\}$, $S \cap \bar{B} = \{y_4, y_5\}$ and $S \cap T = \{y_3\}$. By Lemma 3.2 (3), $N_G(x_3) = \{x_1, x_2, y_1, y_2, y_3\}$ and $N_G(x_4) = \{x_1, x_2, y_3, y_4, y_5\}$, in particular, $x_3, x_4 \in V_5(G)$, $|N_G(x_3) \cap S| = |N_G(x_4) \cap S| = 3$, $N_G(x_3) \cup N_G(x_4) \supseteq S$ and $N_G(x_3) \cap N_G(x_4) \cap S = S \cap T$. Since A is $\mathscr{F}_2(G)$-minimum and $|A| = 4$, Lemma 2.3 (1) assures us that $|N_G(y_i) \cap A| \geq 2$ for $i = 1, 2, 4, 5$, and Lemma 2.3 (2) assures us that $|N_G(\{y_1, y_2\}) \cap A| \geq 3$ and $|N_G(\{y_4, y_5\}) \cap A| \geq 3$. Hence $|N_G(y_i) \cap \{x_1, x_2\}| \geq 1$ for $i \in \{1, 2, 4, 5\}$ and $N_G(\{y_1, y_2\}) \cap \{x_1, x_2\} = N_G(\{y_4, y_5\}) \cap \{x_1, x_2\} = \{x_1, x_2\}$. Thus, without loss of generality, we may assume $y_1x_1, y_4x_1, y_2x_2, y_5x_2 \in E(G)$. Since $x_3 \in V_5(G)$, it follows from Lemma 7.1 (1) that there is a fragment $B' \in \mathscr{F}_2(G)$ such that $x_3 \in N_G(B')$. Let $T' = N_G(B')$. Since $|A| = 4$, applying Claim 7.2.2 to B', we get $|A \cap B'| = |A \cap \bar{B}'| = 1$ and $|A \cap T'| = 2$. Since $G[A]$ has a 4-cycle $C_4 : x_3x_1x_4x_2x_3$ and $x_3 \in A \cap T'$, we see that $A \cap T' = \{x_3, x_4\}$. By symmetry, we may assume that $A \cap B' = \{x_1\}$ and $A \cap \bar{B}' = \{x_2\}$.

Applying the above argument with B replaced by B', we obtain $|S \cap T'| = 1$, $x_1, x_2 \in V_5(G)$, $|N_G(x_1) \cap S| = 3$, $|N_G(x_2) \cap S| = 3$, $N_G(x_1) \cup N_G(x_2) \supseteq S$ and $N_G(x_1) \cap N_G(x_2) \cap S = S \cap T'$. Suppose that $S \cap T' \neq \{y_3\}$. By the symmetry of $\{y_1, y_2\}$ and $\{y_4, y_5\}$ and that of y_1 and y_2, we may assume that $S \cap T' = \{y_1\}$. Since $x_1y_1, x_1y_4, x_2y_2, x_2y_5 \in E(G)$, this forces $N_G(x_1) \cap S = \{y_1, y_3, y_4\}$ and $N_G(x_2) \cap S =$

$\{y_1, y_2, y_5\}$, which implies $S \cap B' = \{y_3, y_4\}$ and $S \cap \bar{B}' = \{y_2, y_5\}$. Consequently $\{y_1, x_3\} \subseteq B \cap T'$, $y_2 \in B \cap \bar{B}'$, $y_4 \in \bar{B} \cap B'$, $x_2 \in T \cap \bar{B}'$ and $\{y_3, y_4, y_5, x_1, x_4\} \subseteq \bar{B} \cup B'$. Hence by Lemma 3.2 (5), $B \cap \bar{B}'$ is a fragment with $N_G(B \cap \bar{B}') \supseteq \{y_1, x_2, x_3\}$. Since $N_G(\{x_2, x_3\}) \cap (B \cap \bar{B}') = \{y_2\}$, we see from Lemma 2.2 that $B \cap \bar{B}' = \{y_2\}$, which implies $y_2y_1 \in E(G)$. Therefore $\{x_1y_1, x_1y_3, x_2y_1, x_2y_2, y_1y_2\} \subseteq E(G)$, which implies that G_{x_3} is connected and contains a cycle and $\Delta(G_{x_3}) \geq 3$. Since we also have $\Delta(G_{x_3}) \leq 3$, we obtain $G_{x_3} \in \mathscr{B}(5)$, which contradicts Lemma 5.3. Thus $S \cap T' = N_G(x_1) \cap N_G(x_2) \cap S = \{y_3\}$, which forces $N_G(x_1) \cap S = \{y_1, y_3, y_4\}$ and $N_G(x_2) \cap S = \{y_2, y_3, y_5\}$, and hence $S \cap B' = \{y_1, y_4\}$ and $S \cap \bar{B}' = \{y_2, y_5\}$. Consequently $y_1 \in B \cap B'$, $y_3 \in T \cap T'$, $y_5 \in \bar{B} \cap \bar{B}'$, $x_1 \in T \cap B'$, $x_3 \in B \cap T'$ and $\{y_2, y_4, y_5, x_2, x_4\} \subseteq \bar{B} \cup \bar{B}'$. Hence, by Lemma 3.2 (5), $B \cap B'$ is a fragment of G with with $N_G(B \cap B') \supseteq \{y_3, x_1, x_3\}$. Since $N_G(\{x_1, x_3\}) \cap (B \cap B') = \{y_1\}$, applying Lemma 2.2 with U and A replaced by $\{x_1, x_3\}$ and $B \cap B'$, respectively, we see that $B \cap B' = \{y_1\}$, which implies $y_1y_3 \in E(G)$. Therefore $\{x_1y_1, x_1y_3, x_2y_2, x_2y_3, y_1y_3\} \subseteq E(G)$, which implies $G_{x_3} \in \mathscr{B}(5)$. This contradicts Lemma 5.3 and this contradiction proves Claim 7.2.3. □

By Claim 7.2.3, $|A| = 3$.

Claim 7.2.4 $|A \cap T| = 2$.

Proof. By way of contradiction, suppose that $|A \cap T| = 1$.

Subclaim 7.2.4.1 $|A \cap B| = |A \cap \bar{B}| = 1$, $|S \cap B| = |S \cap \bar{B}| = 1$ and $|S \cap T| = 3$.

Proof. Since $|A| = 3$ and $|A \cap T| = 1$, $|A \cap B| = |A \cap \bar{B}| = 1$ by Claim 7.2.2. Since $A \cap \bar{B} \neq \emptyset$, it follows from Lemma 3.2 (4) that $|S \cap B| \leq |A \cap T| = 1$. Hence $|S \cap B| = 1$ by Claim 7.2.1. Similarly $|S \cap \bar{B}| = 1$, which implies $|S \cap T| = 3$. □

Write $A \cap T = \{x_1\}$, $A \cap B = \{x_2\}$, $A \cap \bar{B} = \{x_3\}$, $S \cap B = \{y_1\}$, $S \cap T = \{y_2, y_3, y_4\}$ and $S \cap \bar{B} = \{y_5\}$.

Subclaim 7.2.4.2

(1) $x_2x_3 \notin E(G)$.
(2) $x_2, x_3 \in V_5(G)$, $N_G(x_2) = \{x_1, y_1, y_2, y_3, y_4\}$ and $N_G(x_3) = \{x_1, y_2, y_3, y_4, y_5\}$.
(3) $x_2, x_3, y_1, y_5 \subseteq N_G(x_1)$.

Proof. Since $x_2 \in B$ and $x_3 \in \bar{B}$, (1) clearly holds.

Since $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| = 5$ by Subclaim 7.2.4.1, it follows from Lemma 3.2 (3) that $x_2 \in V_5(G)$ and $N_G(x_2) = \{x_1, y_1, y_2, y_3, y_4\}$. Similarly $x_3 \in V_5(G)$ and $N_G(x_3) = \{x_1, y_2, y_3, y_4, y_5\}$, which proves (2).

By Lemma 2.3 (1), $|N_G(y_1) \cap \{x_1, x_2, x_3\}| \geq 2$. Since $y_1 \in B$ and $x_3 \in \bar{B}$, this implies that $N_G(y_1) \cap \{x_1, x_2, x_3\} = \{x_1, x_2\}$. Hence $x_1y_1 \in E(G)$, and we similarly get $x_1y_5 \in E(G)$. In view of (2), this proves (3). □

Having Subclaim 7.2.4.2 (2) and Lemma 7.1 (1) in mind, take $B' \in \mathscr{F}_2(G)$ such that $x_2 \in N_G(B')$, and let $T' = N_G(B')$.

Subclaim 7.2.4.3

(1) $A \cap T' = \{x_2, x_3\}$.
(2) $x_1 \in V_5(G)$.

Proof. If $A \cap T' = \{x_2\}$, then applying Subclaim 7.2.4.2 (1) with x_1 replaced by x_2, we get $x_1x_3 \notin E(G)$, which contradicts Subclaim 7.2.4.2 (2). Thus $|A \cap T'| = 2$ by Claim 7.2.2. By symmetry, we may assume that $A \cap B' = \emptyset$ and $|A \cap \bar{B}'| = 1$. Write $A \cap \bar{B}' = \{x_i\}$ ($i \in \{1, 3\}$). By Lemma 3.2 (4), $|S \cap B'| \leq |A \cap T'|$, which implies that $|S \cap B'| = |A \cap T'|$ by Claim 7.2.1. Hence $|(S \cap \bar{B}') \cup (S \cap T') \cup (A \cap T')| = 5$ by Lemma 3.2 (2). Consequently it follows from Lemma 3.2 (3) that $x_i \in V_5(G)$ and $N_G(x_i) = (S \cap \bar{B}') \cup (S \cap T') \cup (A \cap T')$. In particular, $|N_G(x_i) \cap S| = |(S \cap \bar{B}') \cup (S \cap T')| = 3$. Hence $i \neq 3$ by Subclaim 7.2.4.2 (2). therefore $i = 1$, which implies $A \cap T' = \{x_2, x_3\}$ and $x_1 \in V_5(G)$, as desired. □

Since $N_G(x_1) \subseteq A \cup S$, it follows from Subclaim 7.2.4.2 (3) and 26.4.3 (2) that $N_G(x_1) = \{x_2, x_3, y_1, y_5, y_j\}$ for some $j \in \{2, 3, 4\}$. We may assume that $j = 3$. By Subclaim 7.2.4.2 (2), $y_1x_2, x_2y_3, y_3x_3, x_3y_5 \in E(G)$. Since $y_1y_5 \notin E(G)$, in view of Lemma 5.3, this implies $G_{x_1} \cong P_5$ and $E(G_{x_1}) = \{y_1x_2, x_2y_3, y_3x_3, x_3y_5\}$ (see the paragraph following the proof of Lemma 5). Hence $M = \{x_3x_2y_5, y_1y_3\}$ is a perfect non-edge pseudo-matching on $N_G(x_1)$ and x_2 has degree four in $G_{x_1}^{(x_1;M)}$. Having Lemma 6.1 (3) in mind, let $(B''; T''; \bar{B}'') \in \mathscr{S}_{x_1}(G; M)$. Then $(B''; T'' - \{x_1\}; \bar{B}'') \in \mathscr{S}(G^{(x_1;M)})$ by definition, and hence $T'' - \{x_1\}$ is $G_{x_1}^{(x_1;M)}$-dividing by Lemma 6.1 (2). Consequently $x_2 \in T'' - \{x_1\}$. Since $|B''|, |\bar{B}''| \geq 2$ by Lemma 6.1 (3), it follows that $B'' \in \mathscr{F}_2(G)$ and $x_1, x_2 \in T'' = N_G(B'')$. Since B' was an arbitrary member of $\mathscr{F}_2(G)$ with $x_2 \in N_G(B'')$, this contradicts Subclaim 7.2.4.3 (1), and this completes the proof of Claim 7.2.4. □

We proceed to the final stage of the proof of Lemma 7.2. Since $|A| = 3$, it follows from Claim 7.2.4 that we have either $A \cap B = \emptyset$ or $A \cap \bar{B} = \emptyset$, say $A \cap B = \emptyset$. Then $A \cap \bar{B} \neq \emptyset$. Hence by Lemma 3.2 (4), $|S \cap B| \leq |A \cap T| = 2$, which implies that $|S \cap B| = |A \cap T| = 2$ by Claim 7.2.1. Hence by Lemma 3.2 (2), $|(S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)| = 5$. Let $A \cap T = \{x_1, x_2\}$, $A \cap \bar{B} = \{x_3\}$, $S \cap B = \{y_1, y_2\}$, $S \cap (T \cup \bar{B}) = \{y_3, y_4, y_5\}$. Then $x_3 \in V_5(G)$ and $N_G(x_3) = (S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)$ by Lemma 3.2 (3). Since A is $\mathscr{F}_2(G)$-minimum and $|A| = 3$, Lemma 2.3 (1) assures us that $|N_G(y_i) \cap A| \geq 2$ for $i = 1, 2$, which implies that $\{y_1x_1, y_1x_2, y_2x_1, y_2x_2\} \subseteq E(G)$.

Since $x_3 \in V_5(G)$, G has a fragment $B' \in \mathscr{F}_2(G)$ such that $x_3 \in N_G(B')$ by Lemma 7.1 (1). Let $T' = N_G(B')$. Claim 7.2.4 assures us that $|A \cap T'| = 2$ and either $A \cap B' = \emptyset$ or $A \cap \bar{B}' = \emptyset$, say $A \cap B' = \emptyset$. Without loss of generality, we may assume that $A \cap T' = \{x_2, x_3\}$ and $A \cap \bar{B}' = \{x_1\}$. Then, applying the argument in the preceding paragraph with T replaced by T', we get $x_1 \in V_5(G)$ and $A \cap T' \subseteq N_G(x_1)$, which implies $x_1x_2 \in E(G)$. Since $\{y_1x_1, y_2x_1\} \subseteq E(G)$, we have $N_G(x_1) = \{x_2, x_3, y_1, y_2, y\}$, where $y \in \{y_3, y_4, y_5\}$. Since $\{y_1x_2, y_2x_2\} \subseteq E(G)$ and $N_G(x_3) = \{x_1, x_2, y_3, y_4, y_5\}$, we have $\{y_1x_2, y_2x_2, x_2x_3, x_3y\} \subseteq E(G)$, which implies that $D_{1,2} \subseteq G_{x_1}$. In view of Lemma 5.3, we see that $E(G_{x_1}) = \{y_1x_2, y_2x_2, x_2x_3, x_3y\}$

and $G_{x_1} \cong D_{1,2}$ (see the paragraph following the proof of Lemma 5). Hence $M = \{y_1x_3y_2, x_2y\}$ is a perfect non-edge pseudo-matching on $N_G(x_1)$, and x_2 and x_3 have degree four in $G_{x_1}^{(x_1;M)}$. Having Lemma 6.1 (3) in mind, let $(B''; T''; \bar{B}'') \in \mathscr{S}_{x_1}(G; M)$. Then, by Lemma 6.1 (2), $T'' - \{x_1\}$ is $G_{x_1}^{(x_1;M)}$-dividing in $G^{(x_1;M)}$. Consequently $x_2, x_3 \in T'' - \{x_1\}$. By Lemma 6.1 (3), $B'' \in \mathscr{F}_2(G)$ and $\{x_1, x_2, x_3\} \subseteq T'' = N_G(B'')$. This contradicts Claim 7.2.2. This is the final contradiction and the proof of Lemma 7.2 is completed. □

8 Graphs with a dominating vertex

In this section we consider minimally 5-connected graphs G with $|G| \geq 7$ and $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$ which have a dominating vertex. We show that if y dominates z, then y dominates every neighbor of z (see Lemma 8.4).

Lemma 8.1 *Let G be a minimally 5-connected graph with $|G| \geq 7$ and $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$. Suppose that G has three vertices x, y, z such that y dominates z, $xz \in E(G)$ and y does not dominate x. Then $E_G(x) \cap E_c(G) = \emptyset$.*

Proof. Since y dominates z and $xz \in E(G)$, we have $yz, xy \in E(G)$ by definition, and it follows from Lemma 2.4 (2) and (3) that $x, z \in V_5(G)$ and $y \in V_{\geq 6}(G)$. Since y does not dominate x by assumption, it follows from Lemma 2.4 (4) that no vertex in $V(G) - \{x\}$ dominates x. Hence $\Delta(G_x) \leq 3$. Let $N_G(x) = \{y, z, u, u_1, u_2\}$. Since $z \in V_5(G)$ and $xz, yz \in E(G)$, we see that $\deg_{G/xy}(z) = 4$, and hence $xy \in E_n(G)$. We show $xz \in E_n(G)$. Suppose that $xz \in E_c(G)$. Since $E^{**}(G) = \emptyset$, $|N_G(x) \cap N_G(z)| \geq 2$, say $u \in N_G(x) \cap N_G(z)$. Then, by Lemma 2.4 (2), $u \in V_5(G)$. Since $xu, zu \in E(G)$, it follows that $xz \in E_n(G)$. This contradicts the assumption that $xz \in E_c(G)$, and it is shown that $xz \in E_n(G)$. Let S be a 5-separator which contains $\{x, z\}$. By Lemma 2.4 (1), $y \in S$. Suppose that $E_G(x) \cap E_c(G) \neq \emptyset$, say $xu \in E_c(G)$. Since xu is 5-contractible, $u \notin S$. Let A be the S-fragment which contains u. Since $\{x, z, y\} \subseteq S$, $u \in A$ and $\bar{A}_x \neq \emptyset$, $\bar{A} \cap \{u_1, u_2\} \neq \emptyset$, say $u_1 \in \bar{A}$. Then $uu_1 \notin E(G)$. If $uz \in E(G)$, then, since $z \in V_5(G)$ and $xz \in E(G)$, it follows that $xu \in E_n(G)$, which contradicts the assumption that $xu \in E_c(G)$. Thus we have $uz \notin E(G)$. Since $xu \in E_c(G)$ and $E^{**}(G) = \emptyset$, $|N_G(x) \cap N_G(u)| \geq 2$. Consequently $\{x, y, u_2\} \subseteq N_G(u)$.

Claim 8.1.1 *The edge xu_1 is 5-noncontractible.*

Proof. Suppose that xu_1 is 5-contractible. Then, applying the argument just before Claim 8.1.1 with u replaced by u_1, we get $\{x, y, u_2\} \subseteq N_G(u_1)$. Consequently $\{yz, u_2u, uy, yu_1, u_2u_1\} \subseteq E(G_x)$, which implies $Ch \subseteq G_x$. Since $\Delta(G_x) \leq 3$, it follows that $G_x \in \mathscr{B}(5)$. This contradicts Lemma 5.3 and this contradiction shows xu_1 is 5-noncontractible. □

Since xu_1 is 5-noncontractible by Claim 8.1.1, G has a 5-separator T which contains $\{x, u_1\}$. Then $x \in S \cap T$ and $u_1 \in \bar{A} \cap T$. Since xu is 5-contractible, $u \notin T$. Let B be the T-fragment which contains u. Then $u \in A \cap B$. Since $\{y, u_2\} \subseteq N_G(u)$, we see that $\{y, u_2\} \cap (\bar{A} \cup \bar{B}) = \emptyset$. Since $x, u_1 \in T$ and $\bar{B}_x \neq \emptyset$, this implies $z \in \bar{B}$.

Hence $z \in S \cap \bar{B}$. Since $z \in \bar{B}$, $u \in B$ and $yz, yu \in E(G)$, we see that $y \in T$. Hence $y \in S \cap T$. Consequently $\{y, z, u, u_1\} \cap ((\bar{A} \cap B) \cup (\bar{A} \cap \bar{B}) \cup (A \cap \bar{B})) = \emptyset$. Since $u_2 \notin \bar{A} \cup \bar{B}$, we now obtain $(\bar{A} \cap B)_x = (A \cap \bar{B})_x = (\bar{A} \cap \bar{B})_x = \emptyset$.

Claim 8.1.2 *$zu_1 \in E(G)$.*

Proof. Since $A \cap B \neq \emptyset$ and $(\bar{A} \cap \bar{B})_x = \emptyset$, $\bar{A} \cap \bar{B} = \emptyset$ by Lemma 3.2 (7). Since $A \cap B \neq \emptyset$, we also get $|\bar{A} \cap T| \leq |S \cap B|$ and $|S \cap \bar{B}| \leq |A \cap T|$ by Lemma 3.2 (4). Since $|\bar{A} \cap T| + |S \cap B| + |S \cap \bar{B}| + |A \cap T| = (|T| - |S \cap T|) + (|S| - |S \cap T|) \leq 3 + 3$, this implies $|\bar{A} \cap T| + |S \cap \bar{B}| \leq 3$. Hence we have $|\bar{A} \cap T| \leq 1$ or $|S \cap \bar{B}| \leq 1$; that is to say, $\bar{A} \cap T = \{u_1\}$ or $S \cap \bar{B} = \{z\}$. Assume for the moment that $\bar{A} \cap T = \{u_1\}$. Then $|\bar{A} \cap T| \leq |S \cap \bar{B}|$. Since $(\bar{A} \cap B)_x = \emptyset$, applying Lemma 3.2 (6) with the roles of A and $\bar{A}$ replaced by each other, we see that $\bar{A} \cap B = \emptyset$. Since $\bar{A} \cap \bar{B} = \emptyset$, it follows that $\bar{A} = \bar{A} \cap T = \{u_1\}$. Consequently $N_G(u_1) = S$, which implies $u_1 z \in E(G)$. Thus we may assume that $S \cap \bar{B} = \{z\}$. Then since $(A \cap \bar{B})_x = \emptyset$, we similarly obtain $N_G(z) = T$, which implies $zu_1 \in E(G)$, as desired. □

We are in a position to complete the proof of Lemma 8.1. Since y dominates z, it follows from Claim 8.1.2 that $yu_1 \in E(G)$. Therefore $\{u_2u, uy, yz, yu_1, zu_1\} \subseteq E(G)$ and $\widetilde{Ch} \subseteq G_x$. Since $\Delta(G_x) \leq 3$, this implies $G_x \in \mathscr{B}(5)$. This contradicts Lemma 5.3 and this contradiction completes the proof of Lemma 8.1. □

Lemma 8.2 *Let G be a minimally 5-connected graph with $|G| \geq 7$ and $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$. Suppose that G has four vertices w, x, y, z such that y dominates z and $xw, xz \in E(G)$. Let M be a perfect non-edge pseudo-matching on $N_G(x)$. If $wz \in E(G^{(x;M)})$, then $G^{(x;M)}$ has no 4-separator separating y and w.*

Proof. Suppose that $G^{(x;M)}$ has a 4-separator S' separating y and w. Since $yz, wz \in E(G^{(x;M)})$, $z \in S'$. Let $S = S' \cup \{x\}$. Then S is a 5-separator of G separating y and w. Since $z \in S$, we see from Lemma 2.4 (1) that $y \in S$, which contradicts the fact that S separates y and w. This contradiction proves Lemma 8.2. □

Lemma 8.3 *Let G be a minimally 5-connected graph with $|G| \geq 7$ and $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$. Suppose that G has three vertices x, y, z such that y dominates z, $xz \in E(G)$ and y does not dominate x.*

(1) *Suppose that $N_G(x) \cap N_G(z) \neq \{y\}$, say $u \in N_G(x) \cap N_G(z) - \{y\}$. Then the following hold.* (i) *$G_x \ncong K_2 \cup K_3$.* (ii) *If $G_x \cong K_1 \cup (K_1 + (K_1 \cup K_2))$, then $\deg_{G_x}(u) \neq 3$.*

(2) *If $N_G(x) \cap N_G(z) = \{y\}$, then $G_x - \{y, z\} \ncong P_3$.*

Proof. By Lemma 2.4 (2), we have $x \in V_5(G)$. Let $N_G(x) = \{y, z, u, u_1, u_2\}$.

(1) Since y dominates z and $uz \in E(G)$, we have $uy \in E(G)$ and it follows from Lemma 2.4 (2) that $u \in V_5(G)$. Let $S = N_G(u)$ and let $A = \{u\}$. Then S is a 5-separator of G which contains $\{x, y, z\}$ and A is an S-fragment. We have $G[\{x, u, y, z\}] \cong K_4$.

First we show $G_x \ncong K_2 \cup K_3$. Suppose that $G_x \cong K_2 \cup K_3$. Since $G_x \cong K_2 \cup K_3$ and $G[\{u, y, z\}] \cong K_3$, we see that $u_1u_2 \in E(G)$ and $E_G(\{u, y, z\}, \{u_1, u_2\}) = \emptyset$. Let $M_i =$

$\{yu_iu, u_{3-i}z\}$ for $i = 1, 2$. Both M_1 and M_2 are perfect non-edge pseudo-matchings on $N_G(x)$. By Lemma 6.1 (2), $G^{(x;M_i)}$ has a $G_x^{(x;M_i)}$-dividing 4-separator T_i' for $i = 1, 2$. By inspection, we see that $\mathcal{T}(G_x^{(x;M_1)}) = \{(\{y\}; \{u_1, u, z\}; \{u_2\}), (\{y, u\}; \{u_1, z\}; \{u_2\}), (\{u\}; \{u_1, y, z\}; \{u_2\}), (\{u_1\}; \{y, u, u_2\}; \{z\})\}$. Since $zu_2 \in E(G^{(x;M_1)})$, Lemma 8.2 assures us that $\mathcal{S}_x(\{y\}; \{u_1, u, z\}; \{u_2\}) = \mathcal{S}_x(\{y, u\}; \{u_1, z\}; \{u_2\}) = \emptyset$. Consequently $\mathcal{S}_x(G; M_1) = \mathcal{S}_x(\{u\}; \{u_1, y, z\}; \{u_2\}) \cup \mathcal{S}_x(\{u_1\}; \{y, u, u_2\}; \{z\})$. Similarly, $\mathcal{S}_x(G; M_2) = \mathcal{S}_x(\{u\}; \{u_2, y, z\}; \{u_1\}) \cup \mathcal{S}_x(\{u_2\}; \{y, u, u_1\}; \{z\})$. Hence by Lemma 6.1 (3) and Lemma 4.5 (2), either $\mathcal{S}_x(G; M_1)$ or $\mathcal{S}_x(G; M_2)$ is empty. Therefore by Lemma 4.3 (1), we see that x is properly 5-liftable, which contradicts the assumption that $\mathcal{L}^*(G) = \emptyset$. This contradiction shows that $G_x \ncong K_2 \cup K_3$ and (1) (i) is shown.

Next suppose that $G_x \cong K_1 \cup (K_1 + (K_1 \cup K_2))$ and $\deg_{G_x}(u) = 3$. Recall that $G[\{u, y, z\}] \cong K_3$. Thus either u_1 or u_2, say u_1, is the unique isolated vertex of G_x. Then $E(G_x) = \{yz, zu, yu, uu_2\}$. Let $M_1 = \{zu_1u, yu_2\}$ and let $M_2 = \{yu_1u, zu_2\}$. Both M_1 and M_2 are perfect non-edge pseudo-matchings on $N_G(x)$. By Lemma 6.1 (2), $G^{(x;M_i)}$ has a nontrivial $G_x^{(x;M_i)}$-dividing 4-separator T_i' for $i = 1, 2$. Let B_i be a T_i'-fragment of $G^{(x;M_i)}$ for $i = 1, 2$. Since $\{y, z, u_1, u_2\} \subseteq N_{G^{(x;M_i)}}(u)$ and T_i' divides $G_x^{(x;M_i)}$, we observe that $u \in T_i'$ for $i = 1, 2$. Let $T_i = T_i' \cup \{x\}$ for $i = 1, 2$. Then $(B_i; T_i; \bar{B}_i) \in \mathcal{S}_x(G; M)$ by definition and, by Lemma 6.1 (3), T_i is a G_x-dividing 5-separator of G and $|B_i|, |\bar{B}_i| \geq 2$ for $i = 1, 2$.

Claim 8.3.1 *$z \notin T_i'$ for $i = 1, 2$.*

Proof. Suppose that $z \in T_1'$. Since y dominates z, we have $y, z \in T_1$ by Lemma 2.4 (1). Thus T_1 separates u_1 and u_2. We may assume $u_1 \in B_1$. Recall that $A = \{u\}$ and $S = N_G(u)$. We clearly have $A \cap B_1 = A \cap \bar{B}_1 = \emptyset$. Also we observe that $u_1 \in \bar{A} \cap B_1$, $u_2 \in S \cap \bar{B}_1$, $\{x, y, z\} \subseteq S \cap T_1$ and $u \in A \cap T_1$, which implies $(\bar{A} \cap \bar{B}_1)_x = \emptyset$. Since $\bar{A} \cap B_1 \neq \emptyset$, we see from Lemma 3.2 (4) that $|\bar{A} \cap T_1| \geq |S \cap \bar{B}_1| \geq 1$ and $|S \cap B_1| \geq |A \cap T_1| \geq 1$. Consequently $|S \cap T_1| = 3$ and $|A \cap T_1| = |\bar{A} \cap T_1| = |S \cap B_1| = |S \cap \bar{B}_1| = 1$; in particular, $|S \cap \bar{B}_1| = |A \cap T_1|$. Since $(\bar{A} \cap \bar{B}_1)_x = \emptyset$, applying Lemma 3.2 (6) with A and B replaced by $\bar{A}$ and $\bar{B}_1$, respectively, we get $\bar{A} \cap \bar{B}_1 = \emptyset$, which implies $\bar{B}_1 = S \cap \bar{B}_1 = \{u_2\}$. This contradicts the fact that $|\bar{B}_1| \geq 2$ and this contradiction shows that $z \notin T_1'$. Note that after we proved that $y, z \in T_1$, the roles of T_1 and T_2 become symmetric in the above argument. Thus we have $z \notin T_2'$. □

Recall that $T_i = T_i' \cup \{x\}$.

Note that $E(G_x^{(x;M_i)} - u) = \{u_iz, zy, yu_{3-i}\}$ and T_i' is $G_x^{(x;M_i)}$-dividing for $i = 1, 2$. Hence by Claim 8.3.1, $y \in T_i'$ and T_i' separates z and u_{3-i} for $i = 1, 2$. For $i = 1, 2$, we may assume that $z \in B_i$ and $u_{3-i} \in \bar{B}_i$. Since $zu_i \in E(G_x^{(x;M_i)})$, it follows that $u_i \in B_i \cup T_i$ for $i = 1, 2$. Thus $\{x, y, u\} \subseteq T_1 \cap T_2$, $z \in B_1 \cap B_2$, $u_2 \in \bar{B}_1 \cap (B_2 \cup T_2)$ and $u_1 \in (B_1 \cup T_1) \cap \bar{B}_2$, which implies $(\bar{B}_1)_x = \{u_2\}$ and $(\bar{B}_1 \cap \bar{B}_2)_x = \emptyset$.

Since $B_1 \cap B_2 \neq \emptyset$, it follows from Lemma 3.2 (7) that $\bar{B}_1 \cap \bar{B}_2 = \emptyset$.

Recall that $\{xu, yu, zu, uu_2\} \subseteq E(G)$, and write $N_G(u) = \{x, y, z, u_2, v\}$. Since $\{x, y, z, u_2\} \cap \bar{B}_2 = \emptyset$, we see that $v \in \bar{B}_2$. Since $v \in \bar{B}_2$ and $\bar{B}_1 \cap \bar{B}_2 = \emptyset$, it follows that $v \notin \bar{B}_1$, which implies that $N_G(u) \cap \bar{B}_1 = \{u_2\}$. Since $N_G(x) \cap \bar{B}_1 = \{u_2\}$,

applying Lemma 2.2 with U and A replaced by $\{x, u\}$ and $\bar{B}_1$, respectively, we now see that $\bar{B}_1 = \{u_2\}$, which contradicts the fact that $|\bar{B}_1| \geq 2$. This contradiction shows (1) (ii) and now (1) is shown.

We show (2). Suppose that $N_G(x) \cap N_G(z) = \{y\}$ and $G_x - \{y, z\} = G_x[\{u, u_1, u_2\}] \cong P_3$, say $E(G[\{u, u_1, u_2\}]) = \{uu_2, u_2u_1\}$. Since $N_G(x) \cap N_G(z) = \{y\}$, $N_G(z) \cap \{u, u_1, u_2\} = \emptyset$. Since x is not dominated by any vertex by Lemma 2.4 (4), $\Delta(G_x) \leq 3$.

Claim 8.3.2 *The graph G_x is isomorphic to one of the following three graphs; $K_2 \cup P_3, D_{1,2}$ and P_5.*

Proof. Since $N_G(z) \cap \{u, u_1, u_2\} = \emptyset$, $E(G_x) = \{uu_2, u_2u_1, yz\} \cup E_{G_x}(\{y\}, \{u, u_1, u_2\})$. If $E_{G_x}(\{y\}, \{u, u_1, u_2\}) = \emptyset$, then $G_x \cong K_2 \cup P_3$. Thus, we may assume that $E_{G_x}(\{y\}, \{u, u_1, u_2\}) \neq \emptyset$, which implies that G_x is connected. By Lemma 5.3 $G_x \notin \mathscr{B}(5)$. Since $\Delta(G_x) \leq 3$, it follows that $|E_G(\{y\}, \{u, u_1, u_2\})| = 1$, which implies that either $G_x \cong D_{1,2}$ or $G_x \cong P_5$. □

Claim 8.3.2 assures us that $G_x \cong K_2 \cup P_3$, $G_x \cong D_{1,2}$ or $G_x \cong P_5$. If $G_x \cong K_2 \cup P_3$, then $E(G_x) = \{zy, uu_2, u_2u_1\}$ and, if $G_x \cong D_{1,2}$, then $E(G_x) = \{zy, yu_2, uu_2, u_2u_1\}$. In the case where $G_x \cong P_5$, we may assume that $E(G_x) = \{zy, yu, uu_2, u_2u_1\}$. By Lemma 8.1, xz is 5-noncontractible. Let S be a 5-separator of G which contains $\{x, z\}$. By Lemma 2.4 (1), $y \in S$. Let A be an S-fragment of G. Note that neither A_x nor $\bar{A}_x$ is empty. Since $y, z \in S$ and $E(G[\{u, u_1, u_2\}]) = \{uu_2, u_2u_1\}$, we observe that $|\{u, u_1, u_2\} \cap A| = |\{u, u_1, u_2\} \cap S| = |\{u, u_1, u_2\} \cap \bar{A}| = 1$. Thus $u_2 \in S$ and, by symmetry, we may assume that $u \in A$ and $u_1 \in \bar{A}$. If A or $\bar{A}$ consists of a single vertex u' ($u' \in \{u_1, u\}$), then $z \in S = N_G(u')$, which contradicts the fact that $N_G(z) \cap \{u, u_1, u_2\} = \emptyset$. Thus $|A|, |\bar{A}| \geq 2$. Let $M = \{yu_1u, zu_2\}$. Since $\{yu_1, u_1u, zu_2\} \cap E(G_x) = \emptyset$, we see that M is a perfect non-edge pseudo-matching on $N_G(x)$. Lemma 6.1 (2) assures us that $G^{(x;M)}$ has a nontrivial $G_x^{(x;M)}$-dividing 4-separator T'. Let B be a T'-fragment of $G_x^{(x;M)}$. Let $T = T' \cup \{x\}$. Then $(B; T; \bar{B}) \in \mathscr{S}_x(G; M)$ by definition and, by Lemma 6.1 (3), T is a G_x-dividing 5-separator of G and $|B|, |\bar{B}| \geq 2$.

Claim 8.3.3 $z \notin T$.

Proof. Suppose that $z \in T$, then $z \in T'$. By Lemma 2.4 (1), we have $y \in T$, which implies $y \in T'$. Since $G^{(x;M)}[\{u, u_1, u_2\}] \cong K_3$, this contradicts the fact that T' is a $G_x^{(x;M)}$-dividing 4-separator. This contradiction shows that $z \notin T$. □

We may assume that $z \in B$ by Claim 8.3.3.

Claim 8.3.4 $y \in T \cup B$, $u, u_1 \in T \cup \bar{B}$ *and* $u_2 \in T$.

Proof. It suffices to show that $y \in T' \cup B$, $u, u_1 \in T' \cup \bar{B}$ and $u_2 \in T'$. Since $zy \in E(G_x^{(x;M)})$ and $z \in B$, we see that $y \in T' \cup B$. Let $u' \in \{u, u_1, u_2\}$. If $u' \in B$, then since $y, z \in T' \cup B$ and $G^{(x;M)}[\{u, u_1, u_2\}] \cong K_3$, it follows that $V(G_x^{(x;M)}) \subseteq T' \cup B$, which contradicts the fact that T' divides $G_x^{(x;M)}$. Consequently $u' \in T' \cup \bar{B}$. Thus $\{u, u_1, u_2\} \subseteq T' \cup \bar{B}$. Since $zu_2 \in E(G^{(x;M)})$ and $z \in B$, it follows that $u_2 \in T'$. □

Since $z \in B$, it follows from Claim 8.3.4 that either $B_x = \{z, y\}$ or $B_x = \{z\}$. In either case, $\{x\} \cup B_x \subseteq S$. Recall that $|A|, |B|, |\bar{A}|, |\bar{B}| \geq 2$, and we have $|S \cap T| \geq |\{x, u_2\}| = 2$ by Claim 8.3.4. Therefore we get a contradiction by applying Lemma 3.6 (3) with the roles of S and T replaced each other. This contradiction completes the proof of (2) and now Lemma 8.3 is proved. □

Lemma 8.4 *Let G be a minimally 5-connected graph with $|G| \geq 7$ and $E^{**}(G) = \mathcal{L}^*(G) = \emptyset$. If y dominates z, then y dominates every neighbor of z other than y.*

Proof. Suppose that there exists a vertex $x \in N_G(z) - \{y\}$ which is not dominated by y. We have $y \in N_G(x) \cap N_G(z)$. By Lemma 2.4 (2), (3), $y \in V_{\geq 6}(G)$ and $x, z \in V_5(G)$, say $N_G(x) = \{y, z, u, u_1, u_2\}$. Since x is not dominated by any vertex by Lemma 2.4 (4), $\Delta(G_x) \leq 3$.

Claim 8.4.1 $|N_G(x) \cap N_G(z)| \leq 2$.

Proof. Suppose that $|N_G(x) \cap N_G(z)| \geq 3$. Since $x, z \in V_5(G)$ and G is 5-connected, $|N_G(x) \cap N_G(z)| \leq 3$. Hence $|N_G(x) \cap N_G(z)| = 3$, say $N_G(x) \cap N_G(z) = \{y, u, u_1\}$. Since y dominates z, $uy, u_1y \in E(G)$. Hence if $N_G(u_2) \cap \{y, z, u, u_1\} \neq \emptyset$, then $G_x \in \mathscr{B}(5)$, which contradicts Lemma 5.3. Thus $N_G(u_2) \cap \{y, z, u, u_1\} = \emptyset$. By Lemma 8.1, G has a 5-separator S which contains $\{u_2, x\}$. Let A be an S-fragment. We show that $y \in S$. Suppose not, say $y \in A$. Then, since $\{u, u_1, z\} \subseteq N_G(y)$, $\bar{A}_x = \emptyset$, which contradicts the fact that A is an S-fragment of G. This contradiction shows that $y \in S$. By the same reason, $z \in S$. Since $A_x \neq \emptyset, \bar{A}_x \neq \emptyset$ and $\{y, z, u_2\} \subseteq S$, we have $|A_x| = |\bar{A}_x| = 1$, say $A_x = \{u\}$ and $\bar{A}_x = \{u_1\}$. Since $A_z \neq \emptyset, \bar{A}_z \neq \emptyset$ and $\{x, y\} \subseteq S$, we have $|A_z| = 1$ or $|\bar{A}_z| = 1$. By symmetry we may assume that $|A_z| = 1$. Since $u \in A$, it follows that $A_z = \{u\}$. Consequently $N_G(\{x, z\}) \cap A = \{u\}$. Applying Lemma 2.2 with U replaced by $\{x, z\}$, we see that $A = \{u\}$, which implies $uu_2 \in E(G)$. This contradicts the earlier assertion that $N_G(u_2) \cap \{y, z, u, u_1\} = \emptyset$. This contradiction proves Claim 8.4.1. □

Claim 8.4.1 assures us that either $|N_G(x) \cap N_G(z)| = 2$ or $|N_G(x) \cap N_G(z)| = 1$.

First we consider the case where $|N_G(x) \cap N_G(z)| = 2$, say $N_G(x) \cap N_G(z) = \{y, u\}$. Since y dominates z, $u \in V_5(G)$ by Lemma 2.4 (2), and $uy \in E(G)$ by definition, which implies that $G[\{x, u, y, z\}] \cong K_4$.

Claim 8.4.2 $u_1u_2 \notin E(G)$.

Proof. Suppose that $u_1u_2 \in E(G)$. Since $G[\{u, y, z\}] \cong K_3$, we see that $K_2 \cup K_3 \subseteq G_x$. In view of Lemma 5.3, this implies $G_x \cong K_2 \cup K_3$, which contradicts Lemma 8.3 (1) (i). This contradiction proves Claim 8.4.2. □

By Lemma 8.1, $xu_i \in E_n(G)$ for $i = 1, 2$. Hence G has a 5-separator S_i which contains $\{x, u_i\}$ for $i = 1, 2$. Let A_i be an S_i-fragment of G for $i = 1, 2$. Since $G[\{x, y, z, u\}] \cong K_4$, either $\{x, y, z, u\} \subseteq A_i \cup S_i$ or $\{x, y, z, u\} \subseteq \bar{A}_i \cup S_i$, say $\{x, y, z, u\} \subseteq \bar{A}_i \cup S_i$ for $i = 1, 2$. Thus we have $\{x, y, z, u\} \subseteq (\bar{A}_1 \cup S_1) \cap (\bar{A}_2 \cup S_2)$. Since neither $(A_1)_x$ nor $(A_2)_x$ is empty, we have $(A_1)_x = \{u_2\}$ and $(A_2)_x = \{u_1\}$, which implies $u_2 \in A_1 \cap S_2$ and $u_1 \in S_1 \cap A_2$ and, hence $(A_1 \cap A_2)_x = (\bar{A}_1 \cap A_2)_x = (A_1 \cap \bar{A}_2)_x = \emptyset$.

Claim 8.4.3

(1) $|S_1 \cap A_2|, |A_1 \cap S_2| \geq 2$.
(2) *Neither* $\bar{A}_1 \cap S_2$ *nor* $S_1 \cap \bar{A}_2$ *is empty.*
(3) *Either* $|\bar{A}_1 \cap S_2| = 1$ *or* $|S_1 \cap \bar{A}_2| = 1$.

Proof. (1) Suppose that $|S_1 \cap A_2| \leq 1$. Since $u_1 \in S_1 \cap A_2$, we have $S_1 \cap A_2 = \{u_1\}$. Since $u_2 \in A_1 \cap S_2$, we see that $|S_1 \cap A_2| \leq |A_1 \cap S_2|$. Since $(\bar{A}_1 \cap A_2)_x = \emptyset$, applying Lemma 3.2 (6) with $A = \bar{A}_1$ and $B = A_2$, we get $\bar{A}_1 \cap A_2 = \emptyset$. Suppose that $\bar{A}_1 \cap S_2 = \emptyset$. Then $|\bar{A}_1 \cap S_2| < |S_1 \cap A_2|$, which implies $\bar{A}_1 \cap \bar{A}_2 = \emptyset$ by Lemma 3.2 (4). Hence $\bar{A}_1 = \emptyset$, which contradicts the fact that A_1 is a fragment of G. This contradiction shows $\bar{A}_1 \cap S_2 \neq \emptyset$. Hence $|S_1 \cap A_2| \leq |\bar{A}_1 \cap S_2|$. Since $(A_1 \cap A_2)_x = \emptyset$, we get $A_1 \cap A_2 = \emptyset$ by Lemma 3.2 (6). Consequently $A_2 = S_1 \cap A_2 = \{u_1\}$, which implies $S_2 = N_G(u_1)$ and $u_1u_2 \in E(G)$. This contradicts Claim 8.4.2 and this contradiction shows $|S_1 \cap A_2| \geq 2$. By symmetry, $|A_1 \cap S_2| \geq 2$, and (1) is shown.

(2) Suppose that $\bar{A}_1 \cap S_2 = \emptyset$. Then $|\bar{A}_1 \cap S_2| < |S_1 \cap A_2|$, which implies $\bar{A}_1 \cap \bar{A}_2 = \emptyset$ by Lemma 3.2 (4). We also have $|\bar{A}_1 \cap S_2| \leq |S_1 \cap \bar{A}_2|$. Since $(\bar{A}_1 \cap A_2)_x = \emptyset$, we get $\bar{A}_1 \cap A_2 = \emptyset$ by Lemma 3.2 (6). Consequently $\bar{A}_1 = \emptyset$, which contradicts the fact that A_1 is a fragment of G. This contradiction shows $\bar{A}_1 \cap S_2 \neq \emptyset$. By symmetry, we obtain $S_1 \cap \bar{A}_2 \neq \emptyset$. Now (2) is shown.

(3) Suppose that $|\bar{A}_1 \cap S_2|, |S_1 \cap \bar{A}_2| \geq 2$. Since $x \in S_1 \cap S_2$, we see from (1) that $|\bar{A}_1 \cap S_2| = |S_1 \cap \bar{A}_2| = |S_1 \cap A_2| = |A_1 \cap S_2| = 2$ and $S_1 \cap S_2 = \{x\}$. Note that $\{x\} \cup (A_2)_x = \{x, u_1\} \subseteq S_1$. Hence by Lemma 3.6 (3) and Lemma 3.5 (1), $A_2 = S_1 \cap A_2$ and $|A_2| = |S_1 \cap A_2| = 2$, say $A_2 = \{u_1, v_1\}$. Since $xv_1 \notin E(G)$ and since $u_1u_2 \notin E(G)$ by Claim 8.4.2, it follows that $A_2 \subseteq V_5(G)$, $N_G(u_1) = ((A_1 \cap S_2) - \{u_2\}) \cup (\bar{A}_1 \cap S_2) \cup \{x, v_1\}$ and $N_G(v_1) = (A_1 \cap S_2) \cup (\bar{A}_1 \cap S_2) \cup \{u_1\}$. We clearly have $E_G(A_1 \cap S_2, \bar{A}_1 \cap S_2) = \emptyset$. Since $v_1x, u_1u_2 \notin E(G)$, we now obtain $\Delta(G_{u_1}) \leq 3$ and $\Delta(G_{v_1}) \leq 3$, which contradicts Lemma 6.5. In view of (2), this contradiction proves (3). □

We are in a position to complete the discussion for the case where $|N_G(x) \cap N_G(z)| = 2$. In view of Claim 8.4.3 (3), by the symmetry of $\bar{A}_1 \cap S_2$ and $S_1 \cap \bar{A}_2$, we may assume that $|\bar{A}_1 \cap S_2| = 1$, say $\bar{A}_1 \cap S_2 = \{w\}$. Then $|\bar{A}_1 \cap S_2| < |S_1 \cap A_2|$ by Claim 8.4.3 (1), and hence we have $\bar{A}_1 \cap \bar{A}_2 = \emptyset$ by Lemma 3.2 (4). By Claim 8.4.3 (2), we also have $|\bar{A}_1 \cap S_2| \leq |S_1 \cap \bar{A}_2|$. Since $(\bar{A}_1 \cap A_2)_x = \emptyset$, we get $\bar{A}_1 \cap A_2 = \emptyset$ by Lemma 3.2 (6). Consequently $\bar{A}_1 = \bar{A}_1 \cap S_2 = \{w\}$. Hence $N_G(w) = S_1$. In particular, $w \in N_G(x)$ and $wu_1 \in E(G)$. Since $u_1, u_2 \notin \bar{A}_1$, $w \in N_G(x) - \{u_1, u_2\} = \{u, y, z\}$. If $w = z$, then $zu_1 \in E(G)$, which contradicts the fact that $u_1 \notin N_G(x) \cap N_G(z)$. Thus $w \neq z$. Since $y \in V_{\geq 6}(G)$, $w \neq y$. Consequently $w = u$, i.e., $\bar{A}_1 = \bar{A}_1 \cap S_2 = \{u\}$ which, in particular, implies that $uu_1 \in E(G)$. If $|S_1 \cap \bar{A}_2| = 1$, then arguing as above, we get $S_1 \cap \bar{A}_2 = \{u\}$, which is absurd. Thus $|S_1 \cap \bar{A}_2| \geq 2$. Hence $S_1 \cap S_2 = \{x\}$ by Claim 8.4.3 (1). Recall that $\{x, y, z, u\} \subseteq (\bar{A}_1 \cup S_1) \cap (\bar{A}_2 \cup S_2)$. Since $\bar{A}_1 = \{u\}$ and $S_1 \cap S_2 = \{x\}$, we obtain $\{y, z\} \subseteq S_1 \cap \bar{A}_2$, which implies $u_1y, u_1z \notin E(G)$. Since $uu_1 \in E(G)$, it follows that $G[\{y, z, u, u_1\}] \cong K_1 + (K_1 \cup K_2)$ and $\deg_{G[\{y,z,u,u_1\}]}(u) = 3$. If $E_G(\{y, z, u, u_1\}, \{u_2\}) \neq \emptyset$, then $G_x \in \mathscr{B}(5)$, which contradicts Lemma 5.3. Thus

$E_G(\{y,z,u,u_1\},\{u_2\}) = \emptyset$. Therefore $G_x \cong K_1 \cup (K_1 + (K_1 \cup K_2))$ and $\deg_{G_x}(u) = 3$, which contradicts Lemma 8.3 (1) (ii). This contradiction completes the discussion for the case where $|N_G(x) \cap N_G(z)| = 2$.

Next we consider the case where $|N_G(x) \cap N_G(z)| = 1$. In this case $N_G(x) \cap N_G(z) = \{y\}$. Having Lemma 8.1 in mind, let S be a 5-separator of G which contains $\{x,z\}$ and let A be an S-fragment of G. By Lemma 2.4 (1), we see that $y \in S$. Let $S = \{x,y,z,w_1,w_2\}$. Since neither A_x nor $\bar{A}_x$ is empty and $\{y,z\} \subseteq S$, without loss of generality, we may assume that $A_x = \{u\}$ and $u_1 \in \bar{A}$. Having Lemma 8.1 in mind, let T be a 5-separator of G which contains $\{x,u\}$ and let B be a T-fragment of G. Then, since $A_x = \{u\}$, we have $\{x\} \cup A_x \subseteq T$.

Claim 8.4.4

(1) $A_x \cap A_z = \emptyset$, $\bar{A}_x \cap \bar{A}_z = \emptyset$.
(2) $|A|, |\bar{A}| \geq 2$.
(3) *Either* $|B| = 1$ *or* $|\bar{B}| = 1$.
(4) $u_2 \in S$ *and* $u_2u \in E(G)$.

Proof. (1) Recall that $S = \{x,y,z,w_1,w_2\}$. Since $N_G(x) \cap N_G(z) = \{y\}$, we see that $A_x \cap A_z = \emptyset$ and $\bar{A}_x \cap \bar{A}_z = \emptyset$.

(2) Since $A_x \neq \emptyset$ and $A_z \neq \emptyset$, we see from (1) that $|A| \geq |A_x| + |A_z| \geq 2$, and we similarly get $|\bar{A}| \geq 2$.

(3) Recall that $\{x\} \cup A_x \subseteq T$. Hence if $|A| \geq 3$, then the desired conclusion follows from (2) and Lemma 3.6 (1). Thus we may assume that $|A| = 2$. By way of contradiction, suppose that $|B|, |\bar{B}| \geq 2$. By Lemma 3.5, $A = A \cap T$, $|S \cap B| = |S \cap \bar{B}| = 2$ and $S \cap T = \{x\}$. Write $A = \{u,v\}$. Since $A_x = \{u\}$, we get $A_z = \{v\}$ by (1). Hence $xv, zu \notin E(G)$, which implies $u,v \in V_5(G)$, $N_G(u) = ((S \cap B) \cup (S \cap \bar{B}) - \{z\}) \cup \{x,v\}$ and $N_G(v) = ((S \cap B) \cup (S \cap \bar{B})) \cup \{u\}$. We have $E_G(S \cap B, S \cap \bar{B}) = \emptyset$. Since $vx, uz \notin E(G)$, it follows that $\Delta(G_u) \leq 3$ and $\Delta(G_v) \leq 3$, which contradicts Lemma 6.5, and this contradiction shows either $|B| = 1$ or $|\bar{B}| = 1$. Now (3) is shown.

(4) By (3), we may assume that $|B| = 1$, say $B = \{w\}$. By Lemma 3.4, $w \in S_x \cap V_5(G)$ and $wu \in E(G)$. Since $S_x = N_G(x) - (A_x \cup \bar{A}_x) \subseteq N_G(x) - \{u,u_1\}$, we have $w \in \{y,z,u_2\}$. Since $y \in V_{\geq 6}(G)$, $w \neq y$. If $w = z$, then $zu \in E(G)$, which contradicts the assumption that $N_G(x) \cap N_G(z) = \{y\}$. Thus $w \neq z$. Consequently $w = u_2$. Hence $u_2 \in S$ and $u_2u \in E(G)$ as desired, and (4) is shown, □

By Claim 8.4.4 (4), we know that $\bar{A}_x = \{u_1\}$. Thus we can apply Claim 8.4.4 (4) with A and u replaced by $\bar{A}$ and u_1, respectively, to obtain $u_2u_1 \in E(G)$. Since $u \in A$ and $u_1 \in \bar{A}$, $uu_1 \notin E(G)$. Therefore $G_x - \{y,z\} = G[\{u,u_2,u_1\}] \cong P_3$, which contradicts Lemma 8.3 (2). This contradiction completes the discussion for the case where $|N_G(x) \cap N_G(z)| = 1$.

Now the proof of Lemma 8.4 is completed. □

9 The proof of the Main Theorem

In this section we prove the Main Theorem.

Since "edge removal", "semi-proper edge-contraction" and "proper lifting" are the inverse operations of "edge addition", "semi-proper vertex-splitting" and "pseudo edge-binding", respectively, to prove the Main Theorem, it suffices to show that every minimally 5-connected graph G with $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$ is a member of $\mathscr{H}_5$. Thus let G denote a minimally 5-connected graph with $E^{**}(G) = \mathscr{L}^*(G) = \emptyset$. If $|G| = 6$, then $G \cong K_6 \in \mathscr{H}_5$. Thus we may assume that $|G| \geq 7$. By Lemma 7.2, G has a dominating vertex y. By definition, there is a vertex z' in $V(G) - \{y\}$ such that z' is dominated by y. We show that all vertices in $V(G) - \{y\}$ are dominated by y. Suppose that there is a vertex x' in $V(G) - \{y\}$ such that x' is not dominated by y. Since $G - \{y\}$ is connected, $G - \{y\}$ contains a path connecting z' and x'. Hence it follows that there exist $z, x \in V(G) - \{y\}$ with $zx \in E(G)$ such that z is dominated by y and x is not dominated by y. This contradicts Lemma 8.4 and this contradiction shows that all vertices in $V(G) - \{y\}$ are dominated by y. This implies $N_G(y) = V(G) - \{y\}$. By Lemma 2.4 (2), $V(G) - \{y\} \subseteq V_5(G)$, which implies that $G - y$ is 4-regular. Therefore $G \in \mathscr{H}_5$, as desired.

Now the Main Theorem is proved.

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