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Minimum 2-percolating sets in 2-connected, diameter 2 graphs

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Abstract

The r-neighbor bootstrap percolation process is a process defined on a graph which begins with an initial set of infected vertices. In each subsequent round, an uninfected vertex becomes infected if it is adjacent to at least r previously infected vertices. If, starting with A as an initially infected set of vertices, every vertex of the graph eventually becomes infected, then we say that A percolates. The set of infected vertices at the end of the percolation process (whether or not all of $V(G)$ becomes infected) is called the closure of A and is denoted $\langle A\rangle$. In this article, we investigate 2-neighbor percolation in 2-connected, diameter 2 graphs. This process is equivalent to P_3-convexity in graphs. We begin with some sufficient conditions on a special class of graphs in which every pair of vertices percolates. We then present some necessary conditions on a maximal closure when $|A| \geq 3$. We determine the minimum cardinalities for 2-percolating sets in three infinite families of graphs and end with a discussion on strongly regular graphs.

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1 Introduction

The r-neighbor bootstrap percolation process is a process defined on a graph, G. The process begins with an initial set of infected vertices $A_0 \subseteq V(G)$. In each subsequent round, an uninfected vertex, v, becomes infected if v is adjacent to at least r previously infected vertices. Once infected, vertices remain infected. We use A_t to denote the set of all infected vertices as of round t. Symbolically,

$$A_t = A_{t-1} \cup \{v \in V(G) : |N(v) \cap A_{t-1}| \geq r\}.$$

The parameter r is called the percolation threshold. If G is a finite graph, then after a finite number of rounds, either all vertices of G become infected or the infection stops at some proper subset of $V(G)$. The set of infected vertices after the percolation process finishes is called the closure of A_0, denoted $\langle A_0 \rangle$. If $\langle A_0 \rangle = V(G)$, then we say that A_0 is contagious or A_0 percolates.

Bootstrap percolation was introduced by Chalupa et al. [20]. One model that has received much attention is when the vertices of A_0 are selected randomly; each vertex is selected independently and every vertex of G has probability p of being initially selected. After the initial step, the infection proceeds deterministically. This model has been studied extensively, for example in [2, 3, 5, 6, 7, 39].

Another area of study is extremal problems. The minimum size of a percolating set in a graph G with percolation threshold r is called the r-percolation number of G and denoted $m(G, r)$. When $|V(G)|$ is at least r, $m(G, r) \geq r$ (if $m(G, r)$ was less than r, no additional vertices could become infected).

A natural problem is determining the r-percolation number of particular graph classes. The d-dimensional lattice on n^d vertices, denoted $[n]^d$, has been studied in [4, 6, 34, 46, 47].

Minimum 3-percolating sets in grids have been investigated in [9, 26, 36, 49]. Dreyer and Roberts determined the value of $m(G, 4)$ for the grid and toroidal grid graphs [25]. Minimum percolation sets have also been investigated for trees in [8, 48]. Bidgoli et al. studied minimum percolating sets of Hamming graphs and line graphs of both complete and complete bipartite graphs [10].

Percolation in graph products has been discussed in [12, 13, 22]. Bushaw et al. [15] studied graphs for which the 2-percolation number is equal to 2, calling such graphs 2-bootstrap good or 2-BG for short. The 2-percolation number is also referred to as the P_3 hull number and has been studied in this context in [18, 31, 37, 43].

Cappelle et al. [18] studied 2-percolation in diameter 2 graphs. This is a natural area of investigation because when $r = 1$, any vertex in a connected graph percolates, and when the diameter of a graph is 1, the graph is complete and any set of r vertices is an r-percolating set. Hence, $r = 2$ and diameter 2 are the smallest nontrivial parameters for percolation threshold and diameter. Cappelle et al. proved that if G is diameter 2 and G contains a cut vertex, then that cut vertex, v, is the unique cut vertex of G and $m(G, 2) = cc(G)$, where $cc(G)$ is the number of components of $G - v$. The proof of this fact is straightforward. However, the case for 2-connected, diameter

2 graphs is more difficult. Cappelle et al. proved that for 2-connected, diameter 2 graphs the 2-percolation number is at most $\lceil \log(\Delta(G)+1) \rceil + 1$, where $\Delta(G)$ is the maximum degree of G. In this article, we continue this line of investigation, focusing on 2-percolation in diameter 2, 2-connected graphs.

We start with some results on a special class of diameter 2, 2-connected graphs, the strongly 2-BG graphs. These are the graphs for which every pair of vertices percolates. Next, we prove a result on some properties of maximal closures of an initial set of infected vertices, A, where $|A| \geq 3$. After this, we continue Cappelle et al.'s investigation into 2-percolation in strongly regular graphs. In particular, we determine the 2-percolation number of one infinite family of strongly regular graphs and two other infinite families which are built up from the first family of graphs. We conclude with a discussion of our results and some further lines of research.

Before presenting our results, we begin with some definitions. The *diameter* of a graph G, denoted $diam(G)$, is the greatest distance between a pair of vertices of G. The *open neighborhood* of a vertex v is the set of vertices adjacent to v, denoted by $N(v)$, and the *closed neighborhood* of v, the set vertices adjacent to v along with v itself, denoted by $N[v]$. We will denote the *mutual open neighborhood* of u and v by $N(u, v)$ and the *mutual closed neighborhood* of u and v by $N[u, v]$. The *second neighborhood* of v is the vertices at distance 2 from v, denoted $N_2(v)$. The i^{th} *neighborhood* of v is the vertices at distance i from v and denoted $N_i(v)$.

The *connectivity* of G, denoted $\kappa(G)$, is the minimum number of vertices needed to remove from G to disconnect G. If $\kappa(G) > k - 1$, then G is k-connected. A *dominating set* of G is a subset, D, of $V(G)$ such that every vertex of G is either in D or adjacent to a member of D. If a graph G contains a dominating set consisting of a single vertex, then we say such a vertex is a *dominating vertex*.

An *r-forbidden subgraph* of G is a subgraph H of G such that every vertex of H is joined to $G - H$ by less than r edges. If A is a percolating set of G, then each r-forbidden subgraph, H, of G must contain at least one vertex of A (see Lemma 3.3). If not, no vertex of H could become infected from $G - H$ and A would not be a percolating set. Let $\mathcal{F}$ be the set of r-forbidden subgraphs of G. The complement of each r-forbidden subgraph is the closure of some set of vertices of G. Let $\mathcal{C}$ be the set of closures of G when $r = 2$.

When $r = 2$, the set of closures of G, $\mathcal{C}$, is identical to the collection of convex sets in the P_3 convexity. A graph convexity is a collection, S, of subsets of $V(G)$ such that both $V(G)$ and $\emptyset$ are elements of S and S is closed under intersection. The sets in S are the convex sets of G. The P_3-interval of a set $T \subseteq V(G)$, denoted $I[T]$, is T along with all vertices which have at least two neighbors in T. When $I[T] = T$, T is P_3-convex. If $I[T] \neq T$, we can take the interval set of $I[T]$, denoted $I[I[T]]$ or $I^2[T]$, and we can repeat this process until $I^k[T]$ is convex (the process must terminate in a P_3-convex set because $V(G)$ is P_3-convex). The P_3-convex sets are all the sets formed in this manner. The P_3-convex hull of T is the smallest P_3-convex set containing T, denoted $H(T)$. When $H(T) = V(G)$, we say that T is a P_3 hull set of G. The P_3 hull number of G, denoted $h_{P_3}(G)$, is the cardinality of a minimum hull set of G.

The P_3 hull number is equivalent to the 2-pecolation number. Although we do not use the framework of P_3 convexity in this article, we mention the connection because some papers have investigated the 2-percolation number from the perspective of P_3 convexity, for example [14, 18, 31].

Although the closure of a set A under r percolation is a set of vertices, we shall occasionally refer to $G[\langle A\rangle]$, the subgraph induced by the closure of A, as the closure of A.

2 Strongly 2-BG Graphs

An r-BG graph is a graph for which $m(G,r) = r$. We define a strongly r-BG graph as a graph in which every set of r vertices percolates. In this section, we investigate r-BG and strongly r-BG graphs with a particular focus on the case when $r = 2$.

The following three conditions are necessary for a graph of order at least 3 to be strongly 2-BG: G must be diameter 2, every edge of G must be contained in at least one triangle, and G must be 2-connected. If the diameter of G exceeds 2, then we could select a pair of vertices at distance at least 3, which cannot percolate. If there is an edge of G which is not contained in a triangle, then we could infect the vertices incident with that edge and the infection would stop. Lastly, if G contained a cut vertex, one could infect the cut vertex v and a vertex in some component, C, of $G - v$. It is possible that all of $V(C)$ might become infected, but no vertex in $G - V(C)$ would be infected.

These conditions are not sufficient, however: all three hold for the $n \times m$ rook's graph (provided that both $n, m \geq 3$), yet while every pair of non-adjacent vertices percolates in these graphs, no pair of adjacent vertices percolates. The $n \times m$ rook's graph is the graph whose vertex set is a rectangular $n \times m$ chessboard and where two squares are adjacent if a rook on one square can attack a rook on another square. The rook's graph is also a Cartesian product. The Cartesian product of graphs G and H, denoted $G \,\square\, H$, is the graph whose vertex set is $V(G) \times V(H)$ and where the vertices $(g_1, h_1), (g_2, h_2)$ are adjacent if $g_1 = g_2$ and h_1 is adjacent to h_2 (in H) or $h_1 = h_2$ and g_1 is adjacent to g_2 (in G). The $n \times m$ rook's graph is isomorphic to $K_n \,\square\, K_m$.

We will begin our investigation by looking at three sufficient conditions from [15] for a graph to be 2-BG. First, we describe the class of graphs these results are concerned with. A *block* of a graph G is a maximal 2-connected subgraph of G. In [15], it was shown that a graph with more than 2 blocks cannot be 2-BG. All graphs on less than 3 vertices are trivially 2-BG and no disconnected graph on more than 2 vertices is 2-BG. Let $\mathcal{G}$ be the class of connected graphs on at least 3 vertices with at most 2 blocks.

Each of the sufficient conditions concerns a particular class of graphs. The standard definition of a *cograph* is any graph which can be recursively built up from other cographs by complementation and disjoint union and where the initial cograph is K_1. Cographs can additionally be characterized as the graphs which do not contain an induced P_4. A graph is *locally connected* if the open neighborhood of every vertex induces a connected graph. A graph is *chordal* if it does not contain any induced cycles on more than 3 vertices.

We have the following theorems from [15]:

Theorem 2.1. *If $G \in \mathcal{G}$ is a cograph, then G is 2-BG.*

Theorem 2.2. *If $G \in \mathcal{G}$ is locally connected, then G is 2-BG.*

Theorem 2.3. *If $G \in \mathcal{G}$ and G is chordal, then G is 2-BG.*

Cographs in general are not strongly 2-BG, but in 2-connected cographs, every pair of non-adjacent vertices percolates.

Theorem 2.4. *If G is a cograph and G is 2-connected, then every pair of non-adjacent vertices percolates.*

Proof. The diameter of a P_4-free graph is at most 2. This is because if $x, y \in G$ and $d(x, y) \geq 3$, then a shortest x, y-path contains an induced P_4. Clearly, every pair of vertices percolates in complete graphs, so we may assume the diameter of G is 2.

Let u, v be a pair of non-adjacent vertices of a 2-connected graph. Our proof will use the contrapositive. We will show that if $\langle\{u, v\}\rangle \neq V(G)$, then G must contain an induced P_4. Let $H = \langle\{u, v\}\rangle$ and suppose $H \neq V(G)$. Then, because G is connected, there is some vertex, a, in H such that a has a neighbor, x, outside H. We now have two cases

Case 1: a is distance 2 from some vertex, c, of H.

Any common neighbor, b, of a and c is adjacent to two vertices of H, so b must also be contained in H. Since x can only be adjacent to one vertex inside of H, the set of vertices $\{x, a, b, c\}$ induces a P_4.

Case 2: Only dominating vertices of H have neighbors outside H.

Since G is 2-connected, there must be more than one vertex with a neighbor outside H. If there was only a single such vertex, then that vertex would be a cut vertex of G, since its removal would separate H and $G - H$. Let a, b be two such vertices and let x, y be their respective neighbors outside H (we know that x and y are distinct because any vertex outside H can only be adjacent to a single vertex in H). If x, y are not adjacent, then because a, b are dominating vertices, a, b are adjacent, so $\{x, a, b, y\}$ induces a P_4. Since G has diameter 2 and our initial infected vertices u, v were assumed to be non-adjacent, u and v must have at least one common neighbor. Hence, $|H| \geq 3$. Thus, there must exist some $c \in H$, such that c is adjacent to both a and b. Since $x, y \notin H$ and both x and y are already adjacent to a vertex of H, neither x nor y is adjacent to c. In this case, either $\{x, y, b, c\}$ or $\{y, x, a, c\}$ induces a P_4.

These are all the possibilities, so u, v must percolate. A diagram of Case 1 is shown on the left of Figure 1 and Case 2 is shown on the right. □

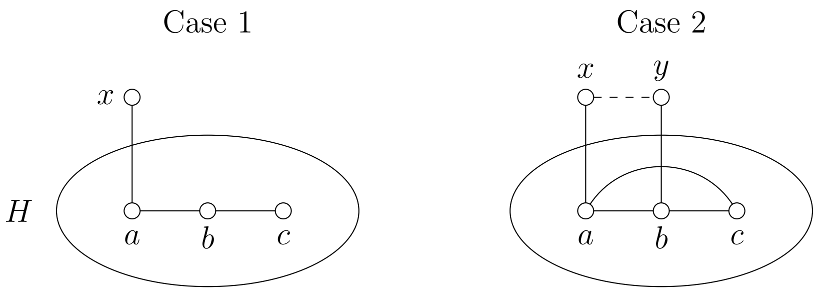

Figure 1

Theorem 2.2 showed that not only are locally connected graphs 2-BG, but in fact, any pair of adjacent vertices percolates in a locally connected graph.

From this we have the following corollary:

Corollary 2.5. *If G is diameter 2 and locally connected, then G is strongly 2-BG.*

Proof. Suppose G is locally connected, v is a vertex of G, and $w_0 \in N(v)$. Because $N(v)$ is connected, there is some $w_1 \in N(v)$ such that w_1 is adjacent to w_0. Since w_1 is adjacent to both v and w_0, w_1 becomes infected. If $N(v) \neq \{w_0, w_1\}$, then there is some other $w_2 \in N(v)$ which is adjacent to w_1. This vertex likewise becomes infected. Eventually, all of $N(v)$ will become infected. For any $w_i \in N(v)$, $N(w_i)$ contains both w_i and some other w_j from $N(v)$ (by the connectedness of $N(v)$). By the same reasoning as with $N(v)$, $N(w_i)$ eventually becomes infected. From this, we can see that eventually all of $N_2(v)$ will become infected and by induction, $N_i(v)$ will eventually become infected for all i.

We have two choices for an initial pair of infected vertices: either u, v are adjacent or u, v are non-adjacent. If u, v are adjacent, then by the above reasoning, u, v percolate. If u, v are non-adjacent, then because G is diameter 2, u, v have at least one common neighbor. Let w be a common neighbor of u and v. Either of u, w or v, w are a pair of infected adjacent vertices, so the infection must percolate to all of G. □

If G is a 2-connected chordal graph, then G is locally connected. We have the following lemma:

Lemma 2.6. *Let G be a 2-connected graph. If G is chordal, then G is locally connected.*

Proof. We will use the contrapositive. Suppose G fails to be locally connected. This implies that there is some $v \in G$ such that $G[N(v)]$, the graph induced by the open neighborhood of v, contains at least two components, C_1 and C_2. Since G is 2-connected $N[v] \neq V(G)$ (otherwise, v would be a cut vertex). Since v is not a cut vertex, there must some path from C_1 to C_2 which does not contain v. Let P be a shortest such path. Since there is no edge from C_1 to C_2, there must be some vertex of $u \in C_1$ such that every vertex on P between u and the first vertex of C_2 on P

must be from $G - N[v]$. Let $w \in C_2$ be the first vertex of C_2 on P and let Q be the section of P between u and w.

Since P is a shortest C_1, C_2 path, Q cannot contain any chords. Hence, u, v, w, Q induces a cycle on at least 4 vertices and G fails to be chordal. □

From this and Corollary 2.5 we have the following result:

Corollary 2.7. *If G is a 2-connected, diameter 2 chordal graph, then G is strongly 2-BG.*

Proof. Since G is 2-connected, Lemma 2.6 implies that G is locally connected. G is diameter 2, so Corollary 2.5 implies that G is strongly 2-BG. □

Although all 2-connected chordal graphs are locally connected, the converse is not true. The wheel graphs, denoted W_n, consist of an outer cycle on $n - 1$ vertices and a central vertex which is adjacent to every vertex of the outer cycle. The wheel graphs are 2-connected, diameter 2, and locally connected, but fail to be chordal when $n \geq 5$ because the outer vertices induce a cycle on more than 3 vertices. The wheel graphs are also strongly 2-BG. If we infect the central vertex and a vertex on the outer cycle, then these two infect the two neighbors of the outer vertex. The infection then spreads across the outer cycle. If we infect two outer vertices, these infect the central vertex and then the infection continues from there. W_5 is shown on the right in Figure 2.

It is also possible for a graph to fail to be locally connected, but to be strongly 2-BG. The graph on the left of Figure 2 is such a graph.

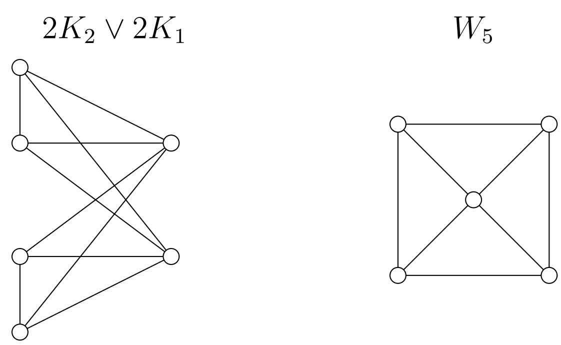

Figure 2: $2K_2 \vee 2K_1$ is strongly 2-BG but not locally connected, while W_5 is strongly 2-BG but not chordal

The *join* of two graphs G and H, denoted $G \vee H$, is the graph formed by the disjoint union of G and H along with an edge between every vertex of G and every vertex of H. The graph on the left in Figure 2 is a join and the wheel graphs can also be described as joins: $W_n = C_{n-1} \vee K_1$. Using joins, it is possible to show that there can be no forbidden induced subgraph characterization of strongly 2-BG graphs and, indeed for strongly r-BG graphs.

Theorem 2.8. *Let H be a graph. There exists a strongly r-BG graph containing H as an induced subgraph.*

Proof. Form the graph $G = H \vee K_r$. We show that G is strongly r-BG. Infect k vertices in H and $r - k$ vertices in the K_r. If $k = 0$, then the K_r infects the rest of G in the next round. If $k \neq 0$, then the vertices in H together with the vertices in the K_r infect the remaining vertices of the K_r and then the vertices of the K_r infect the rest of the graph. □

Since all strongly r-BG graphs are also r-BG, note that this implies that there can be no forbidden induced subgraph characterization of r-BG graphs. Alternatively, we can show this directly by forming the graph $H \vee rK_1$, where the r copies of K_1 form a percolating set of size r.

3 Maximal k-closures

Lemma 3.1 ([17, Lemma 2.2], [18, Lemma 4])**.** *Let G be a 2-connected graph with diameter 2 and $A \subseteq V(G)$. If $\langle A \rangle$ is a dominating set, then $m(G, 2) \leq |A| + 1$.*

A set of cardinality k is a k-set. If there is a k-set of vertices which percolates, then $m(G, 2) \leq k$. However, if no k-set percolates the graph, then it may be useful to define a notion of maximal in relation to closures. We may gain some insight into the structure of G through the closure of k-sets of vertices which are not properly contained in the closure of another k-set.

The closure of a k-set $A \subseteq G$ is a *maximal k-closure* if there is no other k-set $A' \subseteq V(G)$ such that $\langle A \rangle \subset \langle A' \rangle$.

Maximal k-closures (or maximal closure if k is clear from the context) are a natural object to define and study. The following Lemma 3.4 allows us to narrow down the structure of the subgraphs induced by maximal k-closures, along with their interaction with the rest of the graph. We begin with an observation.

Observation 3.2. *If G has diameter 2, then $G[\langle A \rangle]$ has diameter at most 2 for any k-set $A \subseteq V(G)$ with $k \geq 2$.*

If $x, y \in \langle A \rangle$ are not adjacent, then because G is diameter 2, x and y must have a common neighbor z. Since z is adjacent to both x and y, it must be that $z \in \langle A \rangle$. Thus every pair of vertices in $\langle A \rangle$ are adjacent or share a common neighbor.

Lemma 3.3 ([36])**.** *Let $A \subseteq V(G)$ such that $\langle A \rangle = V(G)$. If H is an r-forbidden subgraph of G, then $V(H)$ must contain at least one vertex of A.*

A set $I \subseteq V(G)$ is *independent* if no two vertices in I are adjacent. Let $\alpha(G)$ denote the cardinality of the largest independent set in G.

Lemma 3.4. *Let G be a 2-connected graph with diameter 2, let $A \subseteq V(G)$ be a maximal k-closure with $\langle A \rangle \neq V(G)$ and $k \geq 3$, and let $H = G[\langle A \rangle]$. Then*

(i) H is 2-connected.
(ii) Every vertex in $V(H)$ has at least one neighbor in $G - H$.
(iii) A is an independent set.
(iv) Every vertex in $V(H) - A$ has at most two neighbors in A.

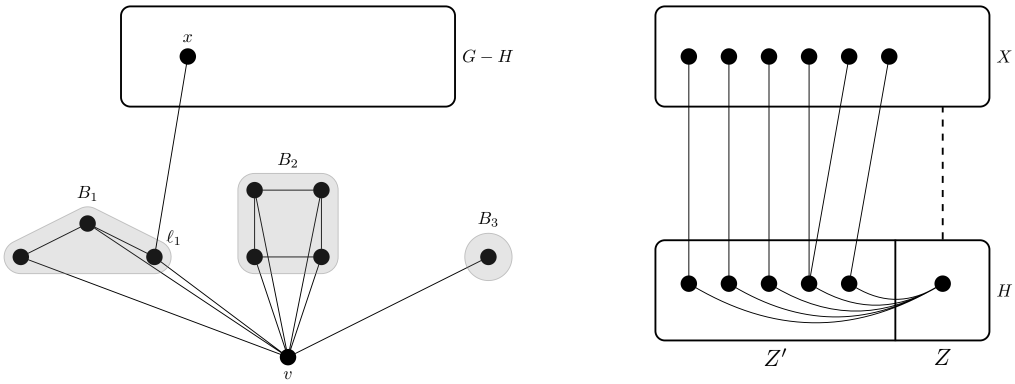

Figure 3: Cases in the proof of Lemma 3.4. Left: Case (i), blocks are highlighted without their common cut-vertex v. Right: Case (ii).

Proof. (i) Suppose toward a contradiction that H is not 2-connected. By Observation 3.2, the diameter of H is 2. Thus H must have a unique cut-vertex v with at least two blocks mutually intersecting at v, and v must be dominating [50, Exercise 2.1.44]. Since each block minus v induces a 2-forbidden subgraph of H, by Lemma 3.3 each block contains at least one vertex from A. Since there are at least two blocks and v dominates H, the vertices from A will also infect v. Thus, since $\langle A\rangle$ is a maximal k-closure, $v \notin A$. As v is dominating, each block along with v induces a graph which is locally connected with diameter 2 and thus strongly 2-BG by Corollary 2.5. That is, each block only requires exactly one vertex $\ell \in A$ to become completely infected from ℓ and v. In other words, if a block contained more than one vertex from A, then $\langle A\rangle$ would not be not a maximal k-closure, a contradiction. Thus each block contains exactly one vertex of A, and so there are k blocks. Let $V(H) = \{v\} \cup \bigcup_{i=1}^{k} B_i$ where B_i is the set of vertices in the i^{th} block not including v (see Figure 3.) Let $A = \{\ell_1, \ell_2, \dots, \ell_k\}$ with $\ell_i \in B_i$.

We claim that $\langle A\rangle$ is not a maximal k-closure. Indeed, if every vertex of $H - v$ has no neighbor in $G - H$, then v is a cut-vertex of G, a contradiction. So without loss of generality, we let $\ell_1 \in B_1$ be a vertex with a neighbor x in $G - H$. Since $k \geq 3$, $\ell_2, \dots, \ell_k$ would infect v, and in turn v and x would then infect ℓ_1. This implies that $\langle\{x, \ell_2, \dots, \ell_k\}\rangle$ contains A (and thus $\langle A\rangle$). This shows $\langle A\rangle$ is not a maximal k-closure, a contradiction. So, H must be 2-connected. If $k = 2$, this argument does not hold, as v would not become infected.

(ii) Let $X \subseteq V(G - H)$ be the vertices with a neighbor in H. Note since $G - H$ is 2-forbidden, vertices in X have exactly one neighbor in H. Let $V(H) = Z \cup Z'$ such that vertices in Z have no neighbor in $G - H$, and vertices in Z' have at least one neighbor in $G - H$.

Since G has diameter 2, each $z \in Z$ has a common neighbor with every vertex in X, and since each $z \in Z$ has no neighbors in $G - H$, the common neighbor must be in Z'. Notice that for $z', z'' \in Z'$, we have $N_X(z') \cap N_X(z'') = \emptyset$ (otherwise there is $x \in X$ where x has two neighbors in Z'.) So, for any $z \in Z$, it must be that z dominates Z'.

If $|Z| \geq 2$, then we claim any two vertices percolate H and thus $\langle A \rangle$ is not a maximal k-closure. If $|Z'| = 1$ then the lone vertex in Z' is a cut-vertex of G (disconnecting Z and $G - Z$). So $|Z'| \geq 2$. So infecting any two vertices of Z will infect all of Z', and in turn the infected vertices of Z' will infect any uninfected vertices remaining in Z. This contradicts the fact that $\langle A \rangle$ is a maximal k-closure, as $|A| \geq 3$.

If $Z = \{z\}$ then by (i) notice $H - z$ is connected. Since z dominates $H - z$ and H is 2-connected, H must be locally connected. By Observation 3.2, the diameter of H is 2. Thus by Corollary 2.5 $\langle \{x, z\} \rangle$ contains H for any $x \in H - z$, a contradiction to the assumption that $\langle A \rangle$ is a maximal k-closure.

(iii) Suppose two vertices $x, y \in A$ are adjacent. By (ii), every vertex in A has a neighbor in $G - H$. Let z be the neighbor of x in $G - H$ and let $A' = \{z\} \cup A \setminus \{x\}$. Notice that infecting A' results in x becoming infected from z and y. Then $\langle A' \rangle$ contains A and thus $\langle A \rangle$ is not a maximal k-closure, a contradiction.

(iv) Suppose there is a vertex $x \in V(H) - A$ with at least three neighbors in A. Let $a \in A$ be a neighbor of x. By (ii) a has a neighbor $z \in G - H$, and similar to the proof of (iii), we let $A' = \{z\} \cup A \setminus \{a\}$. Then x becomes infected by $A \setminus \{a\}$, and a becomes infected by x and z. That is, $\langle A' \rangle$ contains $\langle A \rangle$, a contradiction. □

Theorem 3.5 (Cappelle et al. [18]). *Let G be 2-connected and diameter 2. If G is C_6-free then $m(G, 2) \leq 4$*

Using the idea of maximal k-closures, we can improve the upper bound here by one.

Theorem 3.6. *Let G be 2-connected and diameter 2. If G is C_6-free then $m(G, 2) \leq 3$.*

Proof. First assume $\alpha(G) \leq 2$. Any maximal independent set is a dominating set, so by Lemma 3.1 we have $m(G, 2) \leq \alpha(G) + 1 \leq 3$. So we may assume $\alpha(G) \geq 3$.

Let $A = \{v_1, v_2, v_3\}$ be a 3-set of vertices in G such that $\langle A \rangle$ is a maximal 3-closure. If $\langle A \rangle = V(G)$ then we are done. Otherwise, $\langle A \rangle \neq V(G)$ and by Lemma 3.4 we may assume A is an independent set of size three. Since G has diameter 2, each pair of vertices $v_i, v_j \in A$ must have a common neighbor $v_{ij} \in V(G) - A$ (e.g. v_1 and v_2 have common neighbor v_{12}.) In particular $v_{ij} \in \langle A \rangle - A$. Let $A' = \{v_{12}, v_{13}, v_{23}\}$. By Lemma 3.4 (iv) since $\langle A \rangle$ is a maximal closure, there is no vertex in A' adjacent to all of A. This implies that the vertices of A' are distinct; for any $x, y \in A'$ we have $x \neq y$. Notice since $\langle A \rangle$ is a maximal closure and $A \subseteq \langle A' \rangle$, we have $\langle A \rangle = \langle A' \rangle$. This implies that $\langle A' \rangle$ is a maximal closure, and thus an independent set. But then $A \cup A'$ induces a copy of C_6 in G, a contradiction. □

4 Strongly Regular Graphs

4.1 Background and definitions

Continuing our study of 2-percolation in diameter 2, 2-connected graphs, we now investigate a special class of diameter 2 graphs, the strongly regular graphs. A *strongly regular graph*, $\mathrm{srg}(n, k, \lambda, \mu)$, is a k-regular graph of order n, where every pair of adjacent vertices has λ common neighbors and every pair of non-adjacent vertices has μ common neighbors. If G is a strongly regular graph with $\mu = 0$, then G must be a disjoint union of copies of $K_{\lambda+2}$. If G is such a graph, then clearly $m(G, 2) = 2t$, where t is the number of copies of $K_{\lambda+2}$.

Since μ is the number of common neighbors between non-adjacent vertices, when G is a strongy regular graph with $\mu \neq 0$, every pair of non-adjacent vertices has at least one common neighbor, hence G has diameter 2. We claim that when $\mu \neq 0$, strongly regular graphs are 2-connected. Suppose G is a diameter 2, k-regular graph with a cut vertex, c. Let B be a block of G and suppose u is a vertex where $u \in B$, but $u \neq c$. Since c must be a dominating vertex of G, $\deg(c) > |B|$ (because there is at least one other block of G), but $\deg(u) \leq |B|$, contradicting the assumption that G is k-regular.

Cappelle et al. studied bootstrap percolation in strongly regular graphs as part of their investigation of 2-percolation in diameter 2, 2-connected graphs [18]. They gave two bounds on the 2-percolation number of strongly regular graphs. When G is a strongly regular graph with parameters (n, k, λ, μ), then $m(G, 2) \leq \lceil \frac{k}{\lambda+1} \rceil + 1$ and $m(G, 2) \leq \lceil \log_{\mu+1}(k\mu + 1) \rceil + 1$. Cappelle et al. also included a table of strongly regular graphs comparing their bounds with the 2-percolation number of the graphs found by computation.

We continue the investigation of special classes of diameter 2, 2-connected graphs by determining the 2-percolation number of three infinite families of graphs. Before introducing these families, we begin with some basic facts about strongly regular graphs. For a reference, see [30].

The adjacency matrix of a diameter 2 graph has at least 3 eigenvalues and the strongly regular graphs can be characterized as those diameter 2 graphs which have exactly 3 eigenvalues. One of the eigenvalues is the degree, k, which always has multiplicity 1, while the other two eigenvalues, θ_1 and θ_2, are dependent on λ, μ, and k. When θ_1 and θ_2 have the same multiplicities, the resulting graph is called a conference graph.

In particular, we note the following:

Lemma 4.1 ([30]). *Conference graphs have parameters*

$$(n, (n-1)/2, (n-5)/4, (n-1)/4).$$

Our first infinite family of graphs is the conference graphs. We present some context for our study of the 2-percolation number of these graphs. Let $\sigma_2(G)$ denote the minimum sum of degrees over all pairs of non-adjacent vertices of G. If G has order n and $\sigma_2(G) \geq n$, then G satisfies *Ore's condition.* Freund et al. proved the following result in 2018:

Theorem 4.2 ([28])**.** *If G is a graph of order n and $\sigma_2(G) \geq n$, then $m(G, 2) = 2$.*

Dairyko et al. extended Theorem 4.2:

Theorem 4.3 ([23])**.** *Suppose G is a graph of order $n \geq 2$ such that G is not in $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$, or $\mathcal{X}$, where $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are infinite families of exceptional graphs and $\mathcal{X}$ is a finite family of exceptional graphs. Then, $\sigma_2(G) \geq n-2$ implies $m(G, 2) = 2$.*

Dairyko et al. noted the following corollary of their result:

Corollary 4.4 ([23])**.** *C_5 is the only graph with $\sigma_2(G) = n - 1$ and $m(G, 2) > 2$.*

From Lemma 4.1 all conference graphs have $\sigma_2(G) = n - 1$, so Corollary 4.4 implies the following result:

Theorem 4.5 ([23])**.** *If G is a conference graph, then $m(G, 2) = 2$ unless $G = C_5$, in which case $m(G, 2) = 3$.*

In the next subsection, we will provide an alternative proof of Theorem 4.5, based on a lemma concerning 2-percolation in strongly regular graphs.

Our second infinite family of graphs is built up from a particular class of conference graphs: the Paley graphs. Let q be a prime power congruent to 1 mod 4. The vertices of the Paley graph of order q are the elements of the finite field of order q and two vertices are adjacent if their difference is a quadratic residue. The Paley graph of order q is denoted $QR(q)$ (where QR stands for quadratic residue).

The *complement* of a graph, G, denoted $\overline{G}$ is the graph formed from deleting the edges between pairs of adjacent vertices of G and adding edges between non-adjacent vertices of G. A graph is *self-complementary* if $G \cong \overline{G}$. Paley graphs are self-complementary. A *complementary prism* (introduced in [35]), is a graph constructed by taking G and the complement of G and joining each vertex of G to the corresponding vertex of $\overline{G}$. The complementary prism of G is denoted $G\overline{G}$. We will determine the 2-percolation number of complementary prisms of Paley graphs:

Theorem 4.6. *If $G = QR(q)\overline{QR(q)}$ and q is a prime power congruent to $1 \pmod 4$ such that $q \neq 5, 9$ then $m(G, 2) = 2$. If $q = 5, 9$, then $m(G, 2) = 3$.*

Our third family of graphs is built up from multiple complementary prisms of Paley graphs. Before describing this graph class, we provide some further context. Given a graph, G, with maximum degree Δ and diameter d, the maximum number of vertices of G is $1 + \Delta \sum_{i=1}^{d} (\Delta - 1)^{i-1}$. This expression is known as the Moore bound and the graphs with order equal to the Moore bound are known as Moore graphs. Hoffman and Singleton introduced Moore graphs and showed that Moore graphs of diameter 2 can have degrees 2,3,7, or 57 [38]. In addition, Hoffman and Singleton showed that there is a unique Moore graph for degrees 3 and 7: the Petersen graph and Hoffman-Singleton graph, respectively (the proof of the uniqueness of the Moore graph of degree 2, the 5-cycle, is elementary). The existence of a Moore graph of degree 57 is unknown and such a hypothetical graph is known as the missing Moore graph. See [24] for a survey of properties of the missing Moore graph.

The Moore bound is related to two well-known problems in graph theory: the cage problem and the degree diameter problem. A (k, g)-*cage* is a k-regular graph of girth g with smallest possible order amongst all k-regular graphs of girth g. The Moore bound provides a lower bound for the order of a cage, so Moore graphs are also cages [21] (Chapter 12). The degree-diameter problem is to find the largest possible order of a k-regular graph of diameter d. The following surveys provide more information on these problems [27, 45].

McKay–Miller–Širáň graphs were introduced in the context of the degree diameter problem as a class of regular, diameter 2 graphs with large order [44]. McKay–Miller–Širáň graphs have order $2q^2$ where q is a prime power congruent to $3, 1$ or $0 \pmod 4$. The original construction used voltage graphs, but we will describe a different construction, due to Hafner [33], which illustrates how certain McKay–Miller–Širáň graphs are built up from Paley graphs.

Before presenting this construction, we provide an example. The cycle on 5 vertices is a Paley graph and the complementary prism of C_5 is the Petersen graph. Figure 4 shows the Petersen graph. Robertson constructed the Hoffman-Singleton graph using 5 pentagons and 5 pentagrams. Label 5 pentagons clockwise with $\{0, 1, 2, 3, 4\}$. Label 5 more pentagons in the same way, but then take their complements, keeping the same labeling. Label the 5 pentagrams F_1, F_2, F_3, F_4, F_5 and the 5 pentagons H_1, H_2, H_3, H_4, H_5. Join each pentagon to each pentagram with a perfect matching as follows: the vertex labeled i in F_j is adjacent to the vertex $i + jk \pmod 5$ in H_k [21] (Chapter 12). Each pentagon-pentagram pair forms a Petersen graph, so the Hoffman-Singleton graph can be viewed as being constructed from 25 copies of the Petersen graph.

Hafner generalized this construction for the McKay–Miller–Širáň graphs. Let q be a prime power and let $V_q = \mathbb{Z}_2 \times \mathbb{F}_q \times \mathbb{F}_q$, where $\mathbb{F}_q$ is the finite field of order q. The bipartite graph B_q has V_q as its vertex set where $(0, x, y)$ is adjacent to $(1, m, c)$ if and only if $y = mx + c$ [33].

A *primitive element* of a finite field is an element that generates the field multiplicatively. Let α be a primitive element of $\mathbb{F}_q$. When $q \equiv 1 \bmod 4$, define

$X = \{1, \alpha^2, ..., \alpha^{q-3}\}$ and $X' = \{\alpha, \alpha^3, ..., \alpha^{q-2}\}$. From the construction of B_q, we can think of the vertex set as being partitioned into two rows of q^2 vertices where each row has the same first coordinate. Each row in turn is split into q independent sets of q vertices, where each set of q vertices has the same first and second coordinate. Pick one row and within each set of q vertices, add each element of X to the last coordinate of each vertex, adding a directed edge between each vertex and the vertex corresponding to the sum. Suppress any directed or multiple directed edges. We then

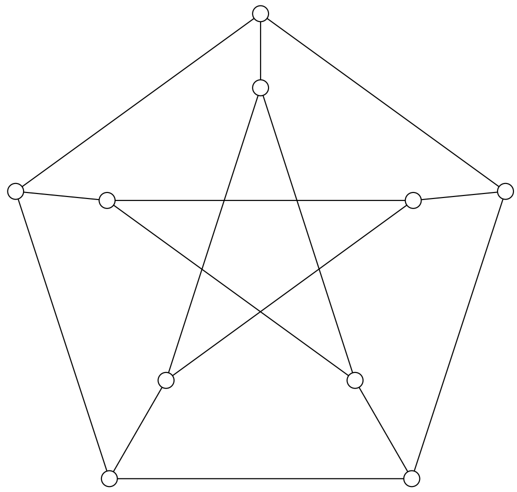

Figure 4: The Petersen graph

do the same thing with X' to each set of q vertices in the other row. This turns each such set of q vertices into a Paley graph or its complement [33]. Hafner also defines an X and X' for the cases when $q \equiv 0, 3 \bmod 4$, but we are only concerned with the 1 mod 4 case. The resulting graph consists of a row of q copies of $QR(q)$ and then a row of q copies of $\overline{QR}(q)$, where each $QR(q)$ is connected with each complement of $QR(q)$ by a perfect matching. See Figure 5 for a diagram.

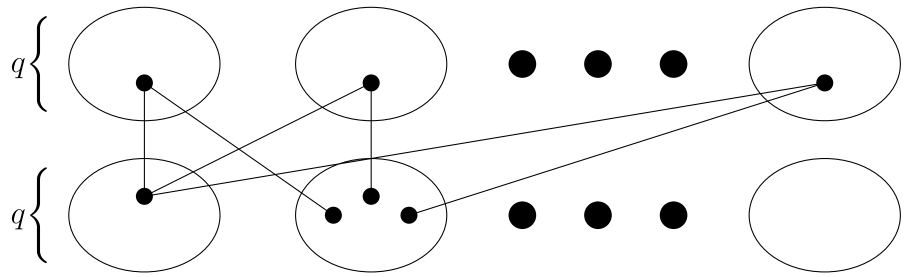

Figure 5

In Hafner's notation, a McKay–Miller–Širáň graph of order $2q^2$ is denoted H_q. The Hoffman-Singleton graph is H_5. We determine the 2-percolation number of every McKay–Miller–Širáň graph where $q \equiv 1 \bmod 4$, except for the Hoffman-Singleton graph.

Theorem 4.7. *If G is a McKay–Miller–Širáň of order $2q^2$ where q is a prime power congruent to* 1 (mod 4) *and G is not the Hoffman-Singleton graph, then $m(G, 2) = 3$.*

Cappelle et al. determined the 2-percolation number of the Hoffman-Singleton graph by computation, while the only graph in their table whose 2-percolation number was not determined by computation is the Games graph, the unique strongly regular graph with parameters (729,112,1,20) [11]. In the last subsection, we determine the 2-percolation of the Hoffman-Singleton graph by hand and sketch how to use our methods to show that the 2-percolation number of the Games graph is 2. We also use percolation to provide an alternative proof of a known result on the maximum number of induced Petersen graphs that can be contained in the missing Moore graph.

4.2 The 2-percolation number of three infinite families of graphs

We begin this subsection with a lemma about percolation in strongly regular graphs. Before presenting this lemma, we introduce some terminology from [29]: G is a (λ, μ)-graph if every pair of adjacent vertices of G has λ neighbors in common and every pair of non-adjacent vertices of G has μ neighbors in common. If G is not regular and G is a (λ, μ)-graph, then we say G is an irregular (λ, μ)-graph. If a graph, G, consists of k disjoint copies of H, then we may write G as kH.

Gera and Shen characterized irregular (λ, μ)-graphs:

Theorem 4.8 ([29],Theorem 1)**.** *If G is an irregular (λ, μ)-graph, then either $\mu = 0$ and G is a collection of m disjoint copies of $K_{\lambda+2}$ and t disjoint copies of K_1 (where $n = m(\lambda + 2) + t$), or $\mu = 1$ and $G = K_1 \vee mK_{\lambda+1}$, where $n = m(\lambda + 1) + 1$.*

In the second case, G is a generalization of the friendship graph, where a central vertex is adjacent to each vertex of m mutually disjoint copies of $K_{\lambda+1}$.

Lemma 4.9. *If G is an $srg(n, k, \lambda, \mu)$ and $A \subseteq V(G)$, then $\langle A \rangle$ under 2-percolation is either $K_{\lambda+2}$, a strongly regular graph with the same λ and μ as G, or an irregular (λ, μ)-graph.*

Proof. Suppose that A is an initial set of infected vertices of a strongly regular graph G and that $\langle A \rangle$ is the closure of A.

Since $\langle A \rangle$ is the closure of A, we know that for every pair of vertices $u, v \in \langle A \rangle$, the common neighbors of u, v must be contained in $\langle A \rangle$.

We have two cases:

Case 1: $\langle A \rangle$ contains only of pairs of adjacent vertices, i.e., $\langle A \rangle$ is a complete graph.

Pick any pair of vertices $u, v \in \langle A \rangle$. Since G is a strongly regular graph, $|N(u, v)| = \lambda$. Also, $\langle A \rangle = N(u, v) \cup \{u, v\}$. Hence, $\langle A \rangle$ consists of $\lambda + 2$ vertices and $\langle A \rangle \cong K_{\lambda+2}$.

Case 2: $\langle A \rangle$ contains pairs of both adjacent and non-adjacent vertices.

If $u, v \in \langle A \rangle$ are adjacent, then $N(u, v) \in \langle A \rangle$ and $|N(u, v)| = \lambda$. If u, v are non-adjacent, then $N(u, v) \in \langle A \rangle$ and $|N(u, v)| = \mu$. Either way, $\langle A \rangle$ is a (λ, μ)-graph. So, $\langle A \rangle$ is either an irregular (λ, μ)-graph or a regular (λ, μ)-graph, i.e., a strongly regular graph. □

Corollary 4.10. *If G is an $srg(n, k, \lambda, \mu)$ and $A \subseteq V(G)$ with $|A| \geq 3$ is such that $\langle A \rangle$ is a maximal k-closure, then $\langle A \rangle$ under 2-percolation is a strongly regular graph with the same λ and μ as G.*

Proof. By Lemma 4.9, there are three possibilities for $\langle A\rangle$. By Lemma 3.4 (i) and the fact that $|A| \geq 3$, $\langle A\rangle$ is 2-connected, so $\langle A\rangle$ cannot be an irregular (λ, μ)-graph. By Lemma 3.4 (iv), each vertex in $\langle A\rangle$ has at most two neighbors in A, so $\langle A\rangle$ cannot be a complete graph. Thus, the only remaining possibility is that $\langle A\rangle$ is a strongly regular graph with the same λ and μ as G. □

We remark that Theorem 4.8 and Lemma 4.9 are unpublished results of Brouwer, but in the context of vertices fixed by an automorphism of G. These results were presented by Wilbrink in [51]. A special case of this result for diameter 2 Moore graphs was earlier shown by Aschbacher in [1].

Using Lemma 4.9, we can determine the 2-percolation number of three infinite families of graphs. We begin by providing an alternative proof for the following result:

Theorem 4.5 ([23])**.** *If G is a conference graph, then $m(G, 2) = 2$ unless $G = C_5$, in which case $m(G, 2) = 3$.*

Proof. Suppose G is a conference graph. Let $A = \{u, v\}$ be a pair of non-adjacent vertices of G. Because G is diameter 2, u, v have a common neighbor, so A is a proper subset of $\langle A\rangle$. By Lemma 4.9, the set $\langle A\rangle$ induces a $K_{\lambda+2}$, an irregular (λ, μ)-graph, or a strongly regular graph. The vertices of A are not adjacent, so $\langle A\rangle$ cannot be a copy of $K_{\lambda+2}$.

By Theorem 4.8, the set $\langle A\rangle$ can only induce an irregular (λ, μ)-graph when $\mu = 0, 1$. We are only concerned with connected graphs, so we can ignore the case when $\mu = 0$. If $\mu = 1$ and G is a conference graph, then by Lemma 4.1, $n = 5, \lambda = 0$, and $k = 2$. Hence G is a 2-regular, girth 5 graph on 5 vertices and so $G = C_5$.

If $G \neq C_5$, $\langle A\rangle$ cannot be an irregular (λ, μ)-graph and hence can only be a strongly regular graph with the same λ, μ as G. By Lemma 4.1, λ, μ determine n so that any two conference graphs with the same λ, μ have the same order and hence neither can be an induced subgraph of the other. In addition, A is a pair of non-adjacent vertices, so $\langle A\rangle$ cannot be a $K_{\lambda+2}$.

We have now ruled out all cases where $\langle A\rangle$ could be a proper subset of $V(G)$, so $\langle A\rangle$ must be $V(G)$. □

We now determine the 2-percolation number of complementary prisms of Paley graphs:

Theorem 4.6. *If $G = QR(q)\overline{QR(q)}$ and q is a prime power congruent to* 1 (mod 4) *such that $q \neq 5, 9$ then $m(G, 2) = 2$. If $q = 5, 9$, then $m(G, 2) = 3$.*

Before proving this theorem, we introduce three lemmas. The following lemma was proved by Cappelle et al.

Lemma 4.11 ([18], Lemma 4)**.** *Suppose G is a 2-connected, diameter 2 graph, $A \subseteq V(G)$ is an initial set of infected vertices of G and $\langle A\rangle$ dominates $G - \langle A\rangle$. When $r = 2$, we can infect one vertex of $G - \langle A\rangle$ and the infection will percolate to the remaining vertices of G.*

Lemma 4.12 (P_3 trick)**.** *If G is a 2-connected, diameter 2 graph which can be decomposed into two connected subgraphs H_1, H_2 such that $V(H_1) \cup V(H_2) = V(G)$, and H_1 is joined to H_2 by a perfect matching, then $m(G, 2) = 2$ if either H_1 or H_2 contains an adjacent pair of vertices which percolates within H_1 or H_2, respectively.*

Proof. Suppose G, H_1, and H_2 are as described and without loss of generality, assume H_1 contains a pair of adjacent vertices, u_1, v_1 which percolate within H_1, i.e., $\langle\{u_1, v_1\}\rangle = V(H_1)$. Since H_1 and H_2 are joined by a perfect matching, there must be a vertex $v_2 \in H_2$ such that v_2 is adjacent to v_1. Infect v_2 and u_1. These two vertices infect v_1, then u_1, v_1 infect H_1. By Lemma 4.11, since v_2 is infected, H_2 also becomes infected. See Figure 6 for a diagram. □

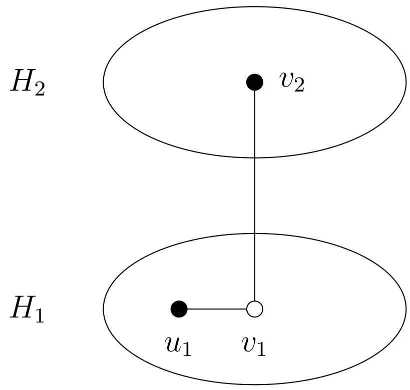

Figure 6: The P_3 trick

A *clique* in a graph, G, is a complete subgraph of G. The *clique number* of G, denoted $\omega(G)$, is the order of the largest clique of G. A *coclique* is another word for an independent set of a graph G, which forms a clique in the complement of G.

Lemma 4.13 (Clique-coclique bound, Corollary 4 of [16])**.** *If a graph G is vertex-transitive, then $\alpha(G)\omega(G) \leq |G|$*

We now prove Theorem 4.6.

Proof. We first show that $m(G, 2) = 3$ for the Petersen graph and $QR(9)\overline{QR(9)}$. The Petersen graph is girth 5 and hence any pair of vertices either has no mutual neighbors (if the two vertices are adjacent) or one mutual neighbor (if the two vertices are not adjacent). Hence, any girth 5 graph with more than 3 vertices must have 2-percolation number greater than 2. Any independent set of 3 vertices which are not all contained in the same neighborhood will percolate in the Petersen graph, so its 2-percolation number is 3.

The graph $QR(9)\overline{QR(9)}$ is the complementary prism of the 3 x 3 rook's graph (which is also $K_3 \square K_3$ and the line graph of $K_{3,3}$). We show that $m(QR(9)\overline{QR(9)}, 2) > 2$. We can choose a pair of vertices from the complementary prism in two ways. Either both vertices can be from the same complement or one vertex can be from each complement. Paley graphs are self-complementary, so it suffices to show that a pair of vertices in $QR(9)$ will not percolate. By Theorem 4.5, $m(QR(9), 2) = 2$ (in fact, any pair of non-adjacent vertices percolates). But even if we infect one of the $QR(9)$'s, the perfect matching implies that any vertex from the other is only adjacent to a single vertex in the first $QR(9)$ and so cannot become infected.

If we infect one vertex in each complement, we have two cases.

Case 1: The two vertices are adjacent.

Since each vertex is only adjacent to a single vertex in each complement, the infection must stop here.

Case 2: The two vertices are not adjacent. Since $QR(9)\overline{QR(9)}$ is diameter 2, our initial infected vertices must have a common neighbor. Without loss of generality, let us call the initially infected vertices u_1, v_2 and their counterparts u_2, v_1, where two vertices have the same subscript if they are in the same complement. If u_1v_1 is an edge in $QR(9)$, then u_2, v_2 is not an edge in $\overline{QR(9)}$. Hence, u_1 and v_2 can have at most one mutual neighbor. See Figure 7 for a diagram of this situation. In the second round, u_1 and v_2 infect v_1. In $QR(9)$, each pair of adjacent vertices is contained in a unique triangle, so the closure of any pair of adjacent vertices in $QR(9)$ is a triangle. Let w_1 be the third vertex of the triangle containing u_1, w_1 and w_2 be the counterpart of w_1 in the complement. Then, w_1 is infected in the third round. The infection cannot percolate further because u_1, v_1, w_1 is a triangle, so u_2 in the complement is not adjacent to either v_2 or w_2.

However, if we were to infect a pair of non-adjacent vertices in the $QR(9)$ and a single vertex in the complement, then by Lemma 4.11 $QR(9)\overline{(QR(9))}$ would become infected, so the 2-percolation number of this graph is 3.

Now, we show that $m(QR(q)\overline{QR(q)}, 2) = 2$ when q is a prime power congruent to 1 (mod 4) such that $q \neq 5, 9$. Let G be a complementary prism for some $QR(q)$, where $q > 9$. If $QR(q)$ contains a pair of adjacent vertices which percolate, then we can infect G by the P_3 trick (Lemma 4.12). Thus, we will show that every Paley graph with $q > 9$ contains a pair of adjacent vertices which percolate.

By Lemma 4.9, the only way that a pair of adjacent vertices can fail to percolate in a Paley graph is if they percolate to a $K_{\lambda+2}$. We will demonstrate that this cannot happen if $q > 9$. It is well-known that the clique number of the Paley graphs is bounded above by $\sqrt{q}$. This can be shown directly from properties of quadratic residues [52]. Alternatively, we can use Lemma 4.13. Since Paley graphs are self-complementary in addition to being vertex transitive, their independence and clique numbers are equal. Hence, $\omega(QR(q))^2 \leq q$, which implies $\omega(QR(q)) \leq \sqrt{q}$.

For a Paley graph, $\lambda + 2 = \frac{q-5}{4} + 2 = \frac{q+3}{4}$. We want to show that if $q > 9$, then $\frac{q+3}{4}$ exceeds $\sqrt{q}$. We first determine when $\frac{q+3}{4} = \sqrt{q}$, i.e., when $\frac{q^2+6q+9}{16} = q$. This equation can be transformed into $q^2 - 10q + 9 = 0$. The quadratic function on the left hand side has roots at 9 and 1 and is positive when $q < 1$ and $q > 9$. In other words, when q exceeds 9, $\lambda + 2$ exceeds the clique number of $QR(q)$. Hence, $QR(q)$ contains no clique of order $\lambda + 2$ and thus every pair of adjacent vertices must percolate. □

Before determining the 2-percolation number of the McKay–Miller–Širáň graphs,

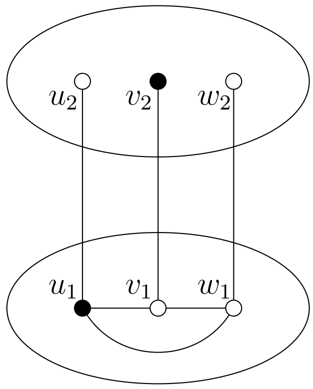

Figure 7: Attempting to infect $QR(9)\overline{QR(9)}$ with 2 vertices, where black vertices are initially infected

we note the following useful facts about these graphs:

1. Each vertex of a $QR(q)$ is adjacent to exactly one vertex in each complement and each vertex in a complement is adjacent to exactly one vertex in each copy of $QR(q)$.

2. If u, v are a pair of vertices in distinct $QR(q)$'s or distinct $\overline{QR(q)}$'s, then u, v are not adjacent.

3. If u, v are a pair of vertices where each is in a distinct copy of $QR(q)$, then u, v have exactly one common neighbor, contained in a $\overline{QR(q)}$. In fact, if v is a vertex in a fixed $QR(q)$ and S is in a different $QR(q)$, then v has a distinct common neighbor with each vertex of S, each one in a different complement. Likewise, if each vertex of the pair is in a different $\overline{QR(q)}$, then they have exactly one common neighbor, contained in a $QR(q)$. See Figure 5 for a diagram illustrating these features of these graphs.

During the following proof, we will refer to these properties by their numbers.

Theorem 4.7. *If G is a McKay–Miller–Širáň of order $2q^2$ where q is a prime power congruent to* $1 \pmod 4$ *and G is not the Hoffman-Singleton graph, then $m(G, 2) = 3$.*

Proof. Let H_q be a McKay–Miller–Širáň graph with $q \equiv 1 \pmod 4$ and $q \neq 5$ (i.e., H_q is not the Hoffman-Singleton graph). We will first show that $m(H_q, 2) > 2$. Let u, v be a pair of infected vertices in H_q. We have three cases. Since Paley graphs are self-complementary, it suffices to consider these cases from the perspective of $QR(q)$.

Case 1: Both u, v are contained in the same $QR(q)$.

Let S be the $QR(q)$ containing u and v. In this case, u, v could potentially infect $QR(q)$. However, each vertex outside S is adjacent to one or zero vertices in S (by 1. and 2.), so the infection must stop there.

Case 2: One vertex is in a $QR(q)$ and one vertex is in a $\overline{QR(q)}$.

Let S be the $QR(q)$ and T be the $\overline{QR(q)}$. By Theorem 4.6, it is possible that these two vertices might infect the entire complementary prism $S \cup T$. But then, each vertex in some $QR(q) \neq S$ is adjacent to no vertices in S and one vertex of T

and each vertex in a $\overline{QR(q)} \neq T$ is adjacent to no vertices of T and one vertex of S (by 1. and 2.). Either way, no vertex outside $S \cup T$ is adjacent to more than one vertex inside $S \cup T$, so the infection cannot spread further.

Case 3: One vertex is in one $QR(q)$ and one vertex is in another $QR(q)$

Let S_1 be the first $QR(q)$ and S_2 be the second $QR(q)$. There are no common neighbors of u, v amongst the $QR(q)$'s and u, v have only one common neighbor in a single $\overline{QR(q)}$ (by 3.), so while u, v infect one more vertex, the infection must stop after that.

Now, we will show that if H_q is not the Hoffman-Singleton graph, then we can always find a percolating set with 3 vertices. Select u, v, w such that u, v are a percolating set of one $QR(q)$ and w is in a different $QR(q)$. Let S_1 be the $QR(q)$ containing u, v, S_2 be the $QR(q)$ containing w and T_1 be some $\overline{QR(q)}$. w has a common neighbor, y, with a vertex, x, of S_1 in T_1, so together w and x infect y. S_1 is joined to each complement by a perfect matching, so S_1 dominates each complement. By Lemma 3.1, x together with S_1 infects all of T_1. T_1 dominates S_2, so by Lemma 3.1, T_1 and w infect all of S_2.

Recall that w has a distinct common neighbor with each vertex of S_1, each in a different one of the complements. Hence, by the aforementioned lemma, these common neighbors and S_1 infect the rest of the $\overline{QR(q)}$'s. Likewise, if T_2 is some other complement, then x along with its common neighbors with each vertex of T_2 infect the rest of the $QR(q)$'s.

Since $m(H_q, 2) > 2$ and $m(H_q, 2) \leq 3$, we conclude that $m(H_q, 2) = 3$. □

4.3 Other results

In this subsection, we give some other applications of Lemma 4.9. Cappelle et. al found that the 2-percolation number of the Hoffman-Singleton graph is 4 using computation. Lemma 4.9 allows us to show $m(H_5, 2) = 4$ by hand, as well as determine the 2-percolation number of the Games graph. In addition, the lemma also allows us to find an upper bound on the 2-percolation number of hypothetical graphs, such as the missing Moore graph. We also can find an upper bound on the number of induced strongly regular subgraphs of strongly regular graphs with the same λ and μ.

Lemma 4.14. *If the order of G is more than 3 and G has girth 5 or greater, then $m(G, 2) > 2$.*

Proof. Suppose G has girth at least 5 and G has order at least 4. A pair of adjacent vertices forms a triangle with each of their common neighbors, while a pair of non-adjacent vertices forms a C_4 with every two common neighbors. Since G contains no cycles on less than 5 vertices, each pair of adjacent vertices of G cannot have any common neighbors, while each pair of non-adjacent vertices of G can have at most one common neighbor. Hence, any pair of initially infected verties of G can infect at most a P_3, but the order of G is at least 4, so G needs at least three vertices to become infected. □

Proposition 4.15. *The 2-percolation number of the Hoffman-Singleton graph is 4.*

Before beginning the proof, we present a definition. If I is an independent set of a graph G such that every vertex of I is contained in the neighborhood of some vertex, v, then I is a type I independent set. If I is an independent set of a graph where every vertex of I is not all contained in the same neighborhood, then I is a type II independent set.

Proof. First, we show that $m(H_5, 2) > 3$. The Hoffman-Singleton graph has girth 5, so by Lemma 4.14, $m(H_5, 2) > 2$. Triples of vertices can have 0, 1, or 2 edges (3 edges is impossible because the graph is girth 5). Let x, y, z be a triple of vertices of H_5. If x, y, z has 2 edges, then the vertices induce a P_3. Since no pair of adjacent vertices has a common neighbor, no more vertices can be infected.

If x, y, z have one edge, then the vertices induce a disjoint union of K_2 and K_1. Within the Hoffman-Singleton graph the closure of K_2+K_1 is C_5, but by Lemma 4.9, once we percolate to a girth 5 Moore graph within another girth 5 Moore graph, the percolation process halts. If x, y, z have 0 edges, but all are contained in the same neighborhood, then x, y, z infect one more vertex. But the infection must stop there because each pair of non-adjacent vertices has only one common neighbor. Hence, $\langle\{x, y, z\}\rangle = K_{1,3}$.

The other case when x, y, z have 0 edges is when x, y, z are not all in the same neighborhood. We claim that all such triples of vertices percolate to a Petersen graph. First, let $A = \{x, y, z\}$ be a type II independent set of H_5. Because G is diameter 2 and girth 5, each pair of vertices in A has exactly one common neighbor, so after one round, A percolates to a 6-cycle. Each of x, y, z, then is distance 3 from the vertex directly opposite to it in the C_6, so the infection will continue to proceed. It is not hard to verify that because H_5 is girth 5, each of these three pairs of vertices must have a distinct common neighbor, so we infect 3 more vertices. This scenario is shown on the left in Figure 8. In order to show that this percolation process stops at a Petersen graph, we must show that the 3 last infected vertices are all contained in the same neighborhood of another vertex. Girth 5 and diameter 2 alone are not enough to show this, so we make use of a result of James [41].

James showed that the Hoffman-Singleton graph is unique, i.e., any girth 5, diameter 2, 7-regular graph must be isomorphic to the Hoffman-Singleton graph. In particular, James proved that any 5-cycle in the Hoffman-Singleton graph divides the graph into 5 pentagons and 5 pentagrams, which in turn are related by Robertson's equations. The 5 pentagrams are formed from the vertices adjacent to the C_5 which has been picked out and the four remaining pentagons are formed from the vertices at distance 2 from the C_5.

Let H be the 9 vertex graph formed in the percolation process from a type II independent set described above. Pick out some 5-cycle in H and call it F. Denote the vertices of F by 1,2,3,4,5. Each of the four remaining vertices of H is adjacent to exactly one vertex of F, so label these vertices from the set $\{a, b, c, d, e\}$ where a would be adjacent to 1, b to 2, etc. An example of such a labelling of H is shown in Figure 8. The black vertices in the diagram are the initially infected vertices, here labelled with $1, 3$ and d. By James's result the 4 vertices of H not contained in F must be contained in a pentagram in the Hoffman-Singleton graph. Furthermore, since no two pentagrams have any adjacent vertices and these 4 vertices induce a P_4, they must be contained in the same pentagram. This is shown on the right of Figure 8. This implies that H consists of 9 vertices all contained in the same Petersen graph. Hence, the last 3 vertices infected ($5, b, c$ in Figure 8) must have a common neighbor within the same Petersen graph (e in Figure 8) and hence the percolation process halts there.

We now show that $m(H_5, 2) \leq 4$. Let a, b, c be an independent set of vertices of H_5 which are not all contained in a single neighborhood. We have shown that the closure of a, b, c is a Petersen graph. Let d be a vertex not contained in the closure of a, b, c. If we infect d, there is no girth 5 Moore graph other than the Hoffman-Singleton graph where the infection can stop, so a, b, c, d form a percolating set. □

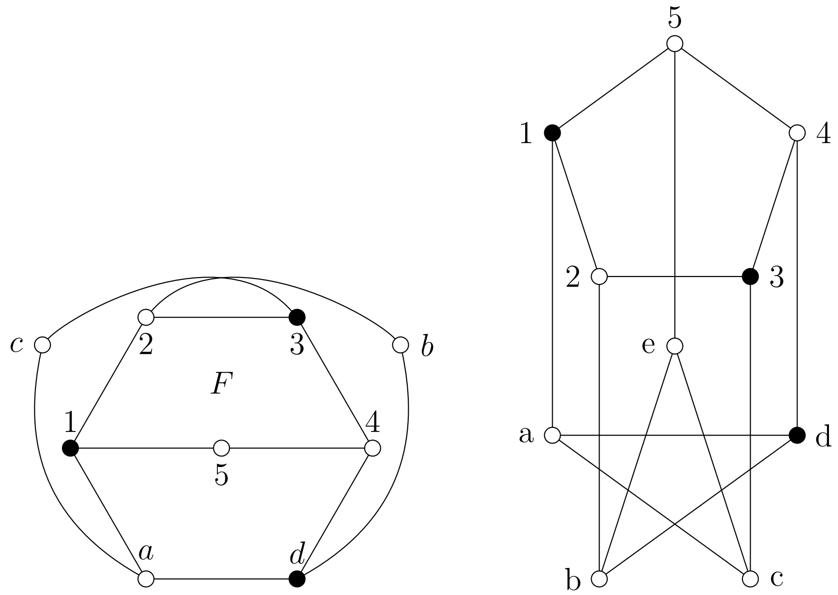

Figure 8: 3 vertices percolating to a Petersen graph in the Hoffman-Singleton graph

The Games graph is the unique strongly regular graph with parameters $(729, 112, 1, 20)$ [11]. This was the only graph in Cappelle et al.'s table whose 2-percolation number was not determined. Using Lemma 4.9, we can determine the 2-percolation number of the Games graph by hand. We perform eigenvalue calculations of the same kind as those used by Hoffman and Singleton [38] but with the parameters $\lambda = 1$ and $\mu = 20$, rather than $\lambda = 0$ and $\mu = 1$. We do not reproduce the calculations here, but it is straightforward to show that the Games graph has the smallest possible order among graphs with $\lambda = 1$ and $\mu = 20$. Hence $m(Games, 2) = 2$. We corroborated this result by computation and list it in Table 6.

Next, we show that the missing Moore graph has 2-percolation number either 3 or 4.

Proposition 4.16. *Let MMG be the missing Moore graph.* $3 \leq m(MMG, 2) \leq 4$.

Proof. The MMG is girth 5, so $m(MMG, 2) \neq 2$. To show that $m(MMG, 2) \leq 4$, recall that by Lemma 4.9, a closure of a diameter 2 Moore graph can only be a star or another diameter 2 Moore graph. Let a, b, c be an independent set of vertices of type II. Because these vertices form an independent set of type II, they cannot percolate to a star. The independence number of C_5 is 2, so this set cannot percolate to a C_5. It is possible that $\langle\{a, b, c\}\rangle$ is a Petersen graph. If $\langle\{a, b, c\}\rangle$ is not a Petersen graph, then a, b, c must be a percolating set of MMG. This is because every cardinality 3 independent set not contained in the same neighborhood percolates to a Petersen graph in the Hoffman-Singleton graph. So, a, b, c are contained in neither a Petersen graph nor a Hoffman-Singleton graph.

It is also possible that the missing Moore graph is not unique, but since all such graphs have the same number of vertices none can be a proper subset of another, so the percolation cannot stop at any Moore graph and must infect all vertices. If $H = \langle\{a, b, c\}\rangle$ is a Petersen graph, then let d be a vertex at distance 2 from every vertex of H. Because H is a closure, every vertex of MMG adjacent to H is adjacent to only a single vertex of H. Since d is at distance 2 from every vertex of H, d must have a common neighbor with every vertex of H. The degree of d in $\langle\{a, b, c, d\}\rangle$ is at least 10, so our percolation cannot stop at a copy of the 7-regular Hoffman-Singleton and a, b, c, d is a percolating set of the missing Moore graph. □

It is also possible to prove Proposition 4.16 by using a fact about the percolating sets of the Hoffman-Singleton graph, along with Lemma 3.4, and Corollary 4.10. If $A \subseteq V(MMG)$ with $|A| = 4$ and $\langle A\rangle$ is a maximal 4-closure, then by Corollary 4.10 the subgraph induced by $\langle A\rangle$ is isomorphic to the Hoffman-Singleton graph. However, it is a fact that any edge of the Hoffman-Singleton graph can be extended to a percolating 4-set. Hence, there is a 4-set A' whose closure is equal to that of A and where two vertices in A' are adjacent. Using the adjacent vertices in A' and the P_3-trick, we can then show that $\langle A\rangle$ was in fact not a maximal 4-closure.

It is unknown whether the missing Moore graph contains even a single copy of the Petersen graph [24]. However, it is known that the missing Moore graph can contain at most 266,266,000 induced Petersen graphs. This upper bound was found by Kováčiková using an algorithmic approach [42]. Using percolation, we can provide an alternative proof of this result.

First, we prove a lemma:

Lemma 4.17. *Suppose* $J, K \subseteq V(G)$ *are both closures under* 2*-percolation, i.e.,* $J = \langle A\rangle$ *and* $K = \langle B\rangle$*. Then, if* $J \cap K \supseteq A$*,* $J \subseteq K$*.*

Proof. When A percolates, the process stops at J. Since K is a closure and K contains A, K must also contain the closure of A. Hence, $J \subseteq K$. □

Theorem 4.18. *The missing Moore graph has between* 0 *and* $266,266,000$ *induced copies of the Petersen graph. Furthermore, the following hold:*

1. *The missing Moore graph has* $266,266,000$ *Petersen graphs if and only if its* 2*-percolation number is* 4.
2. *The missing Moore graph has* 0 *copies of the Petersen graph if and only if every type II independent set of size* 3 *is a percolating set of the Missing Moore graph.*

Proof. Claim: Each type II independent set of cardinality 3 is contained in at most one induced Petersen graph.

Each induced Petersen graph of the MMG is a closure in the MMG by Lemma 4.9. Suppose that P and Q are induced Petersen subgraphs of the missing Moore graph and both P and Q contain the same type II independent set with 3 vertices. Then, by Lemma 4.17 either $V(P) \subseteq V(Q)$ or $V(Q) \subseteq V(P)$, but since both graphs are induced Petersen subgraphs, this is only possible if $V(P) = V(Q)$.

The claim implies that the number of type II independent sets of cardinality 3 in the missing Moore graph provides an upper bound on the number of induced Petersen subgraphs. The missing Moore graph contains $\binom{3250}{3}$ subsets of cardinality 3. We will count the number of type II independent sets by subtracting all other subsets of 3 vertices from this number. Since the missing Moore graph is girth 5, all sets of 3 vertices contain 2,1, or 0 edges. The number of such sets with 2 edges is the same as the number of induced P_3's, which is the same as the number of pairs of non-adjacent vertices, which is 5,187,000. The missing Moore graph contains 92625 edges, each of which has 3136 vertices at distance 2 from it, which counts the number of sets of 3 vertices with 1 edge: 290,472,000. The sets with 0 edges are either type II independent sets or type I independent sets. Each type I independent set corresponds to a claw, of which there are $3250\binom{57}{3} = 95{,}095{,}000$. Altogether, we have 5,325,320,000 type II independent sets of size 3 in the missing Moore graph. The Petersen graph contains 20 such independent sets, so the missing Moore graph must contain at most 266,266,000 induced Petersen graphs.

We showed in Theorem 4.16 that if a type II independent set of cardinality 3 does not percolate to the Petersen graph, then it is a percolating set of the missing Moore graph. Hence, $m(MMG, 2) = 4$ only when every type II independent set of cardinality 3 in the missing Moore graph is contained in a Petersen graph. This occurs when the MMG contains the maximum number of Petersen graphs. This proves 1.

On the other hand, if every type II independent set of cardinality 3 is a percolating set of the missing Moore graph, then none is contained in a Petersen graph. Since each Petersen graph contains 20 such sets, in this scenario, the missing Moore graph would contain no induced Petersen graphs. This proves 2. □

In general, Lemma 4.9 can be used to bound the number of strongly regular graphs contained within other strongly regular graphs with the same λ and μ. Suppose G is an $\mathrm{srg}(n, k, \lambda, \mu)$ and H is an $\mathrm{srg}(n', k', \lambda, \mu)$. If we know the structure of a minimum 2-percolating set of H, then Lemma 4.17 implies that each such minimum 2-percolating set is contained in a unique copy of H. Hence, the total number of such sets in G divided by the number of such sets in H bounds above the number of induced copies of H in G. We provide another example:

It is unknown whether there is a strongly regular graph with parameters $(99, 14, 1, 2)$ [51]. Analogous to the problem of whether the missing Moore graph contains a Petersen graph is the question of whether a (99,14,1,2) graph contains a copy of $K_3 \square K_3$, the unique (9,4,1,2) graph [19, 32]. The minimum 2-percolating sets of $K_3 \square K_3$ are pairs of non-adjacent vertices, of which there are 18. Hence, by finding the total number of pairs of non-adjacent vertices in a hypothetical (99,14,1,2) graph and dividing by 18, we can bound above the number of induced copies of $K_3 \square K_3$. By adapting the methods of Theorem 4.18, we have the following result.

Theorem 4.19. *Let NNG be a hypothetical strongly regular graph with parameters* $(99, 14, 1, 2)$. *The following hold:*

1. $2 \leq m(NNG, 2) \leq 3$.
2. *The NNG contains at most* 231 *copies of* $K_3 \square K_3$ *and contains the maximum number of copies of* $K_3 \square K_3$ *if and only if* $m(NNG, 2) = 3$.
3. *The NNG contains* 0 *copies of* $K_3 \square K_3$ *if and only if every pair of non-adjacent vertices is a percolating set of the NNG.*

5 Open Problems

In this article, we have investigated the effects of diameter and connectivity on the cardinality of minimum 2-percolating sets of graphs. There are still many unanswered questions about 2-percolation in general and even about 2-percolation specifically in diameter 2, 2-connected graphs. We present a few problems here:

1. Does there exist a constant c such that if G is a diameter 2, 2-connected graph, then $m(G, 2) \leq c$?

   The Hoffman-Singleton graph is the only diameter 2, 2-connected graph which we are aware of with 2-percolation number equal to 4, so this leads to the problem:

2. Find another diameter 2, 2-connected graph with 2-percolation number equal to 4 or more.

   Ibrahim [40] showed that $m(G, r)$ is unbounded in general, but the construction depends on having a large number of vertex disjoint r-forbidden subgraphs.

3. If G is a graph such that every two r-forbidden subgraphs of G intersect, i.e., G does not contain even 2 vertex disjoint r-forbidden subgraphs, then is $m(G, r)$ bounded above by a constant?

4. It would also be interesting to investigate 2-percolation in diameter 3 graphs. It is challenging to find natural examples of 2-connected, diameter 3 graphs with high 2-percolation numbers.

   We determined the 2-percolation numbers of the unique $(6, 5)$-cage (which can be formed by deleting a Petersen graph from the Hoffman-Singleton graph) and the four $(5, 5)$-cages by computation. These graphs are all diameter 3 and their 2-percolation numbers are shown in Table 6. Grippo et al. [31] determined that the 2-percolation number of the diameter 3 Kneser graph $K(7, 3)$ is 5.

   The McKay–Miller–Siran graphs provide an infinite family of diameter 2, 2-connected graphs with $m(G, 2) = 3$.

5. Find infinite families of 2-connected, diameter 3 graphs with 2-percolation numbers above 2.

6. Is it possible to find constant upper or lower bounds on r-percolation number for families containing graphs with multiple diameters?

6 Data

Here is a table of the 2-percolation numbers for some cages.

Graph	$m(G, 2)$	URL
$(6, 5)$-cage	4	Link
$(5, 5)$-cage (Wong Graph)	3	Link
$(5, 5)$-cage (Foster Cage)	3	Link
$(5, 5)$-cage (Meringer graph)	4	Link
$(5, 5)$-cage (Robertson-Wegner graph)	4	Link
srg$(729, 112, 1, 20)$ (Games Graph)	2	Link

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