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Uniform shared neighborhood structures in edge-regular graphs

Jared DeLeo

Department of Mathematics and Statistics
Auburn University
Auburn, AL, 36849 U.S.A.
jmd0150@auburn.edu

Abstract

A shared neighborhood structure (SNS) in a graph is a subgraph induced by the intersection of the open neighbor sets of two adjacent vertices. If a SNS is the same for all adjacent vertices in an edge-regular graph, call the SNS a uniform shared neighborhood structure (USNS). USNS-forbidden graphs (graphs which cannot be a USNS of an edge-regular graph) and USNS in graph products of edge-regular graphs are examined.

1 Preliminaries

Let $G = (V, E)$ be a finite, simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $uv \in E(G)$ for vertices $u, v \in V(G)$, then their adjacency is denoted $u \sim v$. The *degree* of a vertex is the number of edges it is incident to. Because G is simple, the degree of $v \in V(G)$ is also the number of vertices it is adjacent to. A graph G is *regular* if the degrees of the vertices in $V(G)$ are all the same. The *open neighborhood* of a vertex u in G, denoted $N_G(u)$, is the set of vertices u is adjacent to. If G is understood, this open neighborhood will be denoted $N(u)$. A graph G is *edge-regular* if G is both regular and, for some λ, every pair of adjacent vertices in G have exactly λ common (or shared) neighbors. If G is edge-regular, we say $G \in ER(n, d, \lambda)$, where $|V(G)| = n$, G is regular of degree d, and $|N(u) \cap N(v)| = \lambda$ for all $uv \in E(G)$.

An *induced subgraph* of G is a graph H such that $V(H) \subseteq V(G)$, $E(H)$ contains all of the edges of G among the vertices of $V(H)$, and only those edges. The induced subgraph H of G is denoted as $G[V(H)]$. If $G[N_G(u) \cap N_G(v)] \cong H$ for all $u \sim v$; with $u, v \in V(G)$, where $\cong$ denotes a graph isomorphism, then G has a *uniform shared neighborhood structure*, abbreviated *USNS*. For instance, letting K_n denote the complete graph on n vertices, $G = K_3 \in ER(3, 2, 1)$ has USNS K_1.

For graphs G and H, define $G + H$ to be the graph formed from G and H where $V(G + H) = V(G) \cup V(H)$ (such that $V(G)$ and $V(H)$ are disjoint) and

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$E(G+H) = E(G) \cup E(H)$. Further, for a graph G and positive integer m, define mG to be the union, or sum, of m disjoint copies of G. That is, $mG = G+G+\cdots+G$.

Edge-regular graphs do not need to have a USNS. If G is the Cartesian product of K_4 and $K_6 \setminus \{\text{a perfect matching in } K_6\}$, $G \in ER(24, 7, 2)$ has two different shared neighborhood structures (SNS): K_2 and $2K_1$. Also, a SNS for one pair of adjacent vertices may also be the SNS for a different pair of adjacent vertices. Suppose G is $K_6 \setminus \{\text{a perfect matching in } K_6\}$ as in Fig. 1. Then $G \in ER(6, 4, 2)$ has a $2K_1$ as a USNS, and each of the three $2K_1$'s in G is the SNS of two disjoint pairs of adjacent vertices.

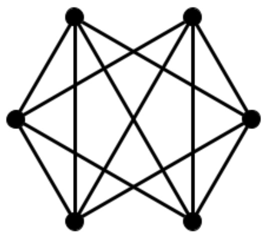

Figure 1: K_6 with a perfect matching removed

A number of studies of edge-regular graphs have focused on the parameter λ. These graphs with $\lambda = 1$ have been studied in [1] and [5], while those with $\lambda = 2$ have been studied in [4]. Additionally, in [1], [5], and [4], constructions are described for edge-regular graphs.

Outside of specific λ values, relations amongst the parameters of an edge-regular graph have also been studied, notably when $d = \lambda + k$ for $k \in \{1, 2, 3\}$ in [6]. The research in [7] also examines parameter relations, specifically as it pertains to n, λ, and the number of vertices missing from any shared neighborhood.

The research presented in this paper will pertain more to the structure of edge-regular graphs, akin to the research presented in [3], which constructs a specific type of edge-regular graph, a *Neumaier* graph. Within the body of this research, there is an emphasis on families of graphs that cannot be a USNS in any edge-regular graph, as well as corresponding constructions of graphs in these families.

2 Forbidden USNS

There are families of graphs that cannot be a USNS in any edge-regular graph; call these USNS-forbidden graphs. Our results about such graphs will be proved by contradiction. For a graph G and $u, v \in V(G)$, let $A(u, v)$ denote the set of vertices in G that are adjacent to u but not to v, and let $B(u, v)$ denote the set of vertices in G that are adjacent to v but not to u. Finally, let $X(u, v)$ denote the set of vertices in G that are adjacent neither to u nor v.

Let P_m be the path graph on m vertices.

Theorem 2.1. *If $G \in ER(n,d,3)$ with a USNS, then the USNS $\ncong P_3$.*

Proof. By way of contradiction, let $u \sim v$, and let $N(u) \cap N(v) = \{w_1, w_2, w_3\}$, where $G[N(u) \cap N(v)] \cong P_3$. Without loss of generality, let $w_1 \sim w_2 \sim w_3$ and $w_1 \nsim w_3$. Then as $w_1 \sim w_2$, $G[N(w_1) \cap N(w_2)] \cong P_3$. As two of w_1 and w_2's common neighbors are u and v, there must exist a third vertex, say z, such that $N(w_1) \cap N(w_2) = \{u, v, z\}$ and $G[N(w_1) \cap N(w_2)] \cong P_3$.

Without loss of generality, suppose $z \sim u$. Then $\{w_1, v, w_3, z\} \subseteq N(u) \cap N(w_2)$, contradicting $\lambda = 3$. Thus, $G[N(w_1) \cap N(w_2)] \ncong P_3$. □

It should be noted that Theorem 2.1 is a special case of Theorem 2.5, found later in the paper.

Naturally, there are a variety of graphs to sum with P_3 to see if it is a possible USNS for some edge-regular graph. There is a partial result for $P_3 + H$ where H is an arbitrary graph.

Theorem 2.2. *Suppose that G is edge-regular with USNS $P_3 + H$ for some graph H. Then H has at least one edge. Further: if, for some $u, v \in V(G)$, $u \sim v$, and $G[N(u) \cap N(v)]$ contains a P_3 component with vertices $w_1 \sim w_2 \sim w_3$, and if the edge uv is an edge of a P_3 component of $G[N(w_1) \cap N(w_2)]$, then H has a P_4 subgraph and a K_2 component.*

Proof. Suppose $u, v \in V(G)$, $u \sim v$, and $w_1w_2w_3$ is a P_3 component of $G[N(u) \cap N(v)]$. Then uv is an edge of $G[N(w_1) \cap N(w_2)] \cong P_3 + H$. If uv is not an edge of a P_3 component of $P_3 + H$ then $uv \in E(H)$. Therefore, the theorem will be proven if we prove that H contains a P_4 and a K_2 component, under the assumption that uv is an edge of a P_3 component of $P_3 + H = G[N(w_1) \cap N(w_2)]$.

The third vertex of P_3 in $G[N(w_1) \cap N(w_2)]$ must be an element of $A(u,v)$ or $B(u,v)$. Without loss of generality, suppose the remaining vertex is $a_1 \in A(u,v)$ (that is, a_1 is adjacent to u but not to v). Then every vertex of H in $G[N(w_1) \cap N(w_2)]$ must be in $X(u,v)$, as any vertex in $A(u,v)$ or $B(u,v)$ would have an adjacency to u or v, respectively. Thus, $N(w_1) \cap N(w_2) = \{a_1, u, v, x_1, \ldots, x_{|H|}\}$.

Consider the adjacent vertices u and w_2. Notice that $\{a_1, w_1, v, w_3\} \subseteq N(u) \cap N(w_2)$, and $G[a_1, w_1, v, w_3]$ is connected. As these four vertices are part of the same component in $N(w_2) \cap N(u)$, then they cannot contain the P_3 component and thus are contained in the H component so H must contain a P_4.

Now consider the adjacent vertices v and w_1. As $\{w_2, u\} \subseteq N(v) \cap N(w_1)$ and $w_2 \sim u$, then w_2 and u are in the same component of $G[N(v) \cap N(w_1)]$. The only other vertices in $N(v) \cap N(w_1)$ are in $B(u,v)$, and none of these can be adjacent to u, nor to w_2, since $N(w_1) \cap N(w_2)$ has no elements in $B(u,v)$ and no vertices in $B(u,v)$ are adjacent to u.

Consequently, the single edge uw_2 is a component of $G[N(v) \cap N(w_1)] \cong P_3 + H$, and is obviously not a P_3. Therefore H has a K_2 component. □

A natural corollary follows from the above theorem to forbid a union of isolated vertices with P_3.

Corollary 2.1. *If $G \in ER(n, d, 3 + \ell)$, $l \geq 1$, with a USNS, then the USNS $\ncong P_3 + \ell K_1$.*

Corollary 2.2. *Suppose m is a positive integer. Then mP_3 is USNS-forbidden.*

Proof. For $m = 1$, see Theorem 2.1. Assume that $m > 1$. If G is edge-regular with USNS mP_3, then because every $uv \in E(G)$ is in a component of $G[N(x) \cap N(y)]$ for any $x \sim y$ in $N(u) \cap N(v)$, every $uv \in E(G)$ is in a P_3 component of two adjacent vertices in a P_3 component of $G[N(u)\cap N(v)]$. Therefore, by Theorem 2.2, $G[N(u)\cap N(v)] \cong P_3 + (m-1)P_3$ contains a P_4 subgraph. Obviously, this is impossible. □

Since P_3 is a forbidden USNS, it is natural to ask if longer paths are also forbidden. The theorem below asserts that P_4, like P_3, is USNS-forbidden.

Theorem 2.3. *If $G \in ER(n, d, 4)$ with a USNS, then the USNS $\ncong P_4$.*

Proof. Suppose for contradiction there exists $G \in ER(n, d, 4)$ with USNS $\cong P_4$. Let $u \sim v$, and let $N(u) \cap N(v) = \{w_1, w_2, w_3, w_4\}$, where $G[N(u) \cap N(v)] \cong P_4$ with endpoints w_1 and w_4 and $w_1 \sim w_2$. $G[N(w_1) \cap N(w_2)] \cong P_4$, as G has a P_4 USNS.

Case 1. $N(w_1)\cap N(w_2) = \{a_1, u, v, b_1\}$, such that $G[N(w_1)\cap N(w_2)] \cong P_4$ having endpoints a_1 and b_1, with $a_1 \in A(u, v)$ and $b_1 \in B(u, v)$. See Fig. 2 for reference.

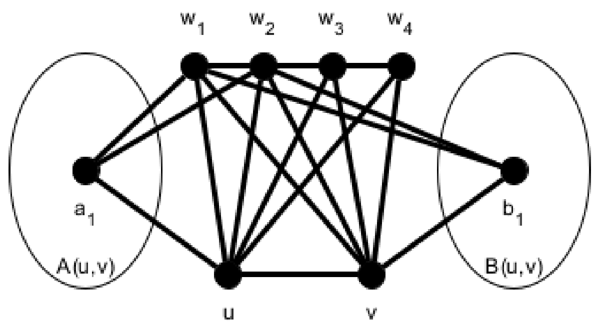

Figure 2: Beginning of case 1 in the proof of Theorem 2.3

Consider the vertices u and w_1, which are adjacent by assumption. As vertices u and w_1 have common neighbors a_1, w_2, and v, then there must exist another vertex in their shared neighborhood adjacent either to a_1 or to v. As w_1 is not adjacent to w_3 and w_4, and u is only adjacent to v, the w_i vertices, and vertices in $A(u, v)$, then the 4th vertex in this common neighborhood must be some $a_2 \in A(u, v)$. So $a_2 \sim a_1$, and $G[N(u) \cap N(w_1)] \cong P_4$ with endpoints a_2 and v.

Now consider adjacent vertices u and w_2. $N(u) \cap N(w_2) = \{a_1, w_1, v, w_3\}$ is completely determined from previous assumptions. As $G[N(u) \cap N(w_2)] \cong P_4$, then this must have endpoints w_3 and a_1, so $w_3 \nsim a_1$.

Now consider the adjacent vertices v and w_1. Then $N(v)\cap N(w_1) = \{b_2, b_1, w_2, u\}$, where $b_1, b_2 \in B(u,v)$. Using similar logic to how $N(u) \cap N(w_1)$ was constructed, then we conclude that $G[N(v)\cap N(w_1)] \cong P_4$ having endpoints b_2 and u, with $b_2 \nsim w_2$ and $b_2 \sim b_1$.

Now consider the adjacent vertices v and w_2. $N(v) \cap N(w_2) = \{b_1, w_1, u, w_3\}$ is completely determined from previous assumptions. As $G[N(v) \cap N(w_2)] \cong P_4$, then this must have endpoints w_3 and b_1, so $w_3 \nsim b_1$.

Lastly, consider the adjacent vertices w_2 and w_3. As $\{u, v\} \in N(w_2) \cap N(w_3)$, there exists $z \in \{N(w_2) \cap N(w_3)\} \setminus \{u, v\}$ such that $z \in A(u, v)$ or $z \in B(u, v)$. As $w_2 \nsim a_2$ and $w_2 \nsim b_2$ (from $N(u) \cap N(w_2)$ and $N(v) \cap N(w_2)$, respectively), then $z \neq a_2$ and $z \neq b_2$. As $w_3 \nsim a_1$ and $w_3 \nsim b_1$ (implied from $N(u) \cap N(w_2)$ and $N(v) \cap N(w_2)$, respectively), then $z \neq a_1$ and $z \neq b_1$. Without loss of generality, say $z \in A(u, v)$. Then $N(u) \cap N(w_2)$ contains $z \in A(u, v) \setminus \{a_1\}$, a contradiction. Thus, $N(w_1) \cap N(w_2) \neq \{a_1, u, v, b_1\}$.

Case 2. $N(w_1) \cap N(w_2) = \{v, u, a_1, x_1\}$, where $a_1 \in A(u, v)$ and $x_1 \in X(u, v)$. By assumption, $u \sim a_1$, $u \nsim x_1$, and $v \nsim x_1$, so v and x_1 are endpoints of $G[N(w_1) \cap N(w_2)]$.

Consider adjacent vertices w_1 and v. Then $N(w_1) \cap N(v) = \{u, w_2, b_2, b_3\}$ for some $b_2, b_3 \in B(u, v)$. This follows from the facts that v has no neighbors in $A(u, v) \cup X(u, v)$ and w_1 is adjacent to no w_j; $j > 2$. Therefore, the two vertices in $N(w_1) \cap N(v)$ other than u and w_2 must be in $B(u, v)$. By assumption, u is not adjacent to any vertex in $B(u, v)$, so w_2 must be adjacent to one of $\{b_2, b_3\}$. Without loss of generality, $w_2 \sim b_2$. However, this implies $N(w_1)\cap N(w_2)$ contains b_2, a contradiction. So $N(w_1) \cap N(w_2) \neq \{v, u, a_1, x_1\}$.

Case 3. $N(w_1) \cap N(w_2) = \{a_2, a_1, u, v\}$, where $a_1, a_2 \in A(u, v)$. By assumption, $u \sim a_2$ and $u \sim a_1$, so u is not an endpoint of $G[N(w_1) \cap N(w_2)]$. v must be an endpoint, as v is only adjacent to u. Without loss of generality, say a_2 is an endpoint and a_1 is not an endpoint in $G[N(w_1)\cap N(w_2)]$. As $a_2 \sim u$, then $G[N(w_1 \cap N(w_2)] \ncong P_4$, a contradiction. So $N(w_1) \cap N(w_2) \neq \{a_2, a_1, u, v\}$.

This exhausts all possibilities for $N(w_1) \cap N(w_2)$, so G cannot have P_4 as a USNS. □

Theorem 2.4. *Let $G \in ER(n, d, \lambda)$ with a P_λ USNS for $\lambda \geq 5$, and let $u \sim v$ in G with $N(u)\cap N(v) = \{w_1, w_2, \ldots, w_\lambda\}$, where w_1 is an endpoint of $G[N(u)\cap N(v)]$. If $w_1 \sim w_2$, then $N(w_1)\cap N(w_2)$ contains exactly one vertex from $N(u)\setminus(N(u)\cap N(v))$ and exactly one vertex from $N(v) \setminus (N(u) \cap N(v))$.*

Proof. **Case 1.** We first assume that $N(w_1)\cap N(w_2)$ contains no vertex from $A(u, v)$. So $N(w_1) \cap N(w_2)$ contains u, v, a vertex in $B(u, v)$, and $\lambda - 3$ vertices in $X(u, v)$.

Consider adjacent vertices u and w_1. Then $N(u)\cap N(w_1)$ contains v and w_2. But as u is not adjacent to any vertex in the set $B(u, v)$ nor $X(u, v)$, the remainder of the vertices in this common neighborhood must be elements of $A(u, v)$. Yet there is no adjacency from these vertices in $A(u, v)$ to v. If any of these vertices in $A(u, v)$ were to be adjacent to w_2, then $N(w_1)\cap N(w_2)$ would contain a vertex from $A(u, v)$, contradicting our case assumption. As $\lambda \geq 5$, then $G[N(u) \cap N(w_1)] \not\cong P_\lambda$, a contradiction.

Case 2. We assume that $N(w_1) \cap N(w_2)$ contains more than one vertex from $A(u, v)$, say m vertices from $A(u, v)$. Then u in $G[N(w_1) \cap N(w_2)]$ has degree $m+1$. As $m \geq 2$, then $G[N(w_1) \cap N(w_2)] \not\cong P_\lambda$, a contradiction.

Thus, $N(w_1) \cap N(w_2)$ must contain exactly one vertex from $A(u, v)$ and exactly one vertex from $B(u, v)$. □

While paths are far from completely decided upon as a family of USNS-forbidden graphs, there are other families of graphs that are. The following theorems tackle a few of these families, namely the family of complete bipartite graphs of different partition sizes, star graphs, and wheel graphs.

Theorem 2.5. *If $G \in ER(n, d, m_1 + m_2)$ with a USNS, then for all $m_1 \neq m_2$, the USNS $\not\cong K_{m_1,m_2}$.*

Proof. Let $u \sim v$, and $N(u) \cap N(v) = \{w_1, w_2, \ldots, w_{m_1}, z_1, z_2, \ldots, z_{m_2}\}$, where $G[w_1, w_2, \ldots, w_{m_1}, z_1, z_2, \ldots, z_{m_2}] \cong K_{m_1,m_2}$ with $w_1, \ldots, w_{m_1}$ in one part and $z_1, \ldots, z_{m_2}$ in the other part.

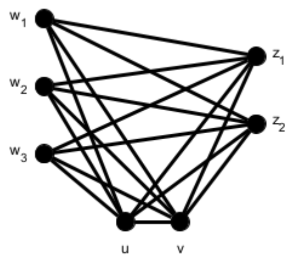

Figure 3: A $K_{3,2}$ shared neighborhood of vertices u and v.

Consider the adjacent vertices w_1 and z_1. Then without loss of generality,

$$N(w_1) \cap N(z_1) = \{u, v, a_1, \ldots, a_{m_2-1}, b_1, \ldots, b_{m_1-1}\},$$

where $a_1, \ldots, a_{m_2-1} \in A(u, v)$ and $b_1, \ldots, b_{m_1-1} \in B(u, v)$. So $G[N(w_1) \cap N(z_1)] \cong K_{m_1,m_2}$, where $v, a_1, \ldots, a_{m_2-1}$ are in one part and $u, b_1, \ldots, b_{m_1-1}$ are in the other part.

Now consider the adjacent vertices u and w_1. Then by previous assumptions, $N(u) \cap N(w_1)$ contains $\{z_1, \ldots, z_{m_2}, a_1, \ldots, a_{m_2-1}, v\}$. Further, as $\lambda = m_1 + m_2$ by assumption and $|N(u) \cap N(w_1)| \geq 2m_2$, then $m_2 \leq m_1$. By symmetry, $m_1 \leq m_2$, so $m_1 = m_2$. Thus, K_{m_1,m_2} is only possible as a USNS when $m_1 = m_2$. □

A graph such as the one in Fig. 3 is a forbidden USNS, where $m_1 = 3$ and $m_2 = 2$. What immediately follows from Theorem 2.5 is a fact about the *star graph* S_l, which is a graph with one central vertex and $l - 1$ vertices adjacent to it, but not to each other.

Corollary 2.3. *If $G \in ER(n, d, \ell)$ with a USNS, then for all $l \geq 3$, the USNS $\ncong S_\ell$.*

Proof. Let $m_1 = 1$ and $m_2 = \ell - 1$. Then $K_{m_1,m_2} \cong S_\ell$. So S_ℓ cannot be a USNS by Theorem 2.5. □

As noted earlier, Theorem 2.5 generalizes Theorem 2.1, as $P_3 \cong K_{1,2}$.

This is not to suggest that complete bipartite graphs with equal part sizes are also USNS-forbidden. On the contrary, consider K_4, which has a $K_2 \cong K_{1,1}$ USNS.

In the following result, define the *wheel graph* W_m to be a connected graph on $m + 1$ vertices, such that m vertices induce a cycle, and the $(m + 1)^{st}$ vertex is adjacent to all vertices in the cycle.

Theorem 2.6. *If $G \in ER(n, d, m + 1)$, $m \geq 4$, has a USNS, then the USNS $\ncong W_m$.*

Proof. Suppose for contradiction $u \sim v$ such that $G[N(u) \cap N(v)] \cong W_m$ consisting of vertices $w_1, \ldots, w_{m+1}$ such that $w_2, \ldots, w_{m+1}$ are the vertices in the cycle and w_1 is adjacent to the vertices in the cycle.

Consider adjacent vertices u and w_1. $N(u) \cap N(w_1) = \{w_2, w_3, \ldots, w_{m+1}, v\}$. So w_1 is not adjacent to any vertex in $A(u, v)$.

Similarly, w_1 is not adjacent to any vertex in $B(u, v)$.

As $G[u, v, w_1] \cong K_3$ and $N(w_2) \cap N(w_3)$ contain u, v, w_1, then this K_3 is an induced subgraph of $G[N(w_2) \cap N(w_3)]$. As $m \geq 4$, one of u, v, w_1 must be the center of this wheel.

If u is the center, the other $m-2$ vertices in $N(w_2) \cap N(w_3)$ besides u, v, w_1 must be in $A(u, v)$, so $w_1 \sim a_i$ for some $a_i \in A(u, v)$, a contradiction.

If v is the center, the other $m-2$ vertices in $N(w_2) \cap N(w_3)$ besides u, v, w_1 must be in $B(u, v)$, so $w_1 \sim b_i$ for some $b_i \in B(u, v)$, a contradiction.

If w_1 is the center, then as $m-2 > 0$, u and v are adjacent vertices on a cycle C_m in $G[N(w_2) \cap N(w_3)]$ of length $m \geq 4$ which cannot contain any w_j, $j > 3$ (because $w_2 \not\sim w_j$). Then there is a P_4 $auvb$ on C_m with $a \in A(u, v)$, $b \in B(u, v)$. But then w_1, as the center of the wheel, is adjacent in G to both a and b, whereas either adjacency contradicts a previous inference.

Thus, W_m is not a possible USNS when $m \geq 4$. □

A *component*-regular graph is a graph such that each component is regular. Every known USNS graph is component-regular, and every aforementioned USNS-forbidden graph is not component-regular. Is it true that every USNS graph is component-regular?

3 Constructions of $ER(n, d, \lambda)$ with USNS

Given graphs G_1 and G_2, the Cartesian product of G_1 and G_2 is denoted $G_1 \square G_2$. The vertex set is defined by $V(G_1 \square G_2) = V(G_1) \times V(G_2)$. The edge set is defined by, given two vertices (u, u') and $(v, v') \in V(G_1 \square G_2)$, $(u, u') \sim (v, v')$ if and only if either $u = v$ and $u' \sim v'$ (in G_2) or $u \sim v$ (in G_1) and $u' = v'$.

It was shown in [4] that if $G_1 \in ER(n_1, d_1, \lambda)$ and $G_2 \in ER(n_2, d_2, \lambda)$, then $G_1 \square G_2 \in ER(n_1 n_2, d_1 + d_2, \lambda)$. However, it is rare that the Cartesian product of two edge-regular graphs that each have a USNS will have a USNS.

Theorem 3.1. *Let $n_1, n_2 \geq 1$ and $d_1, d_2 \geq 0$. If $G_1 \in ER(n_1, d_1, \lambda)$ with a USNS $\cong X$ and $G_2 \in ER(n_2, d_2, \lambda)$ with a USNS $\cong Y$, then $G_1 \square G_2 \in ER(n_1 n_2, d_1 + d_2, \lambda)$ has a USNS if and only if $X \cong Y$, in which case the USNS of $G_1 \square G_2$ is $X (\cong Y)$.*

Proof. Let $G_1 \in ER(n_1, d_1, \lambda)$ with USNS $\cong X$ and $G_2 \in ER(n_2, d_2, \lambda)$ with USNS $\cong Y$.

We assume that $G_1 \square G_2 \in ER(n_1 n_2, d_1 + d_2, \lambda)$ has a USNS. Suppose $(u, v) \sim (x, y)$ in $G_1 \square G_2$. Then by the definition of the Cartesian product, either $u = x$ in G_1 and $v \sim y$ in G_2 or $u \sim x$ in G_1 and $v = y$ in G_2.

If $u = x$ in G_1 and $v \sim y$ in G_2, then $N_{G_1 \square G_2}(u, v) \cap N_{G_1 \square G_2}(x, y) = \{(u, z) | z \in N_{G_2}(v) \cap N_{G_2}(y)\}$ which induces, in $G_1 \square G_2$, a graph isomorphic to Y.

Similarly, if $u \sim x$ in G_1 and $v = y$ in G_2, then $N_{G_1 \square G_2}((u, v), (x, y))$ induces, in $G_1 \square G_2$, a graph isomorphic to X. However, $G_1 \square G_2$ has a USNS, by assumption. Thus, $X \cong Y$.

In the other direction, we assume that $X \simeq Y$. Then the argument above about SNS's in $G_1 \square G_2$ shows that $G_1 \square G_2$ has $X \simeq Y$ as USNS. □

The *tensor product* of G_1 and G_2 is denoted $G_1 \otimes G_2$. The vertex set is $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$. The edge set is defined by, given two vertices (u, u') and $(v, v') \in V(G_1 \otimes G_2)$, $(u, u') \sim (v, v')$ if and only if $u \sim v$ in G_1 and $u' \sim v'$ in G_2. By previous work in [4], if $G_1 \in ER(n_1, d_1, \lambda_1)$ and $G_2 \in ER(n_2, d_2, \lambda_2)$, then $G_1 \otimes G_2 \in ER(n_1 n_2, d_1 d_2, \lambda_1 \lambda_2)$. The following theorem extends the work in [4] to include the preservation and structure of the USNS in $G_1 \otimes G_2$.

Theorem 3.2. *If $G_1 \in ER(n_1, d_1, \lambda_1)$ with a USNS $\cong H_1$ and $G_2 \in ER(n_2, d_2, \lambda_2)$ with a USNS $\cong H_2$, then $G_1 \otimes G_2 \in ER(n_1 n_2, d_1 d_2, \lambda_1 \lambda_2)$ with a USNS $\cong H_1 \otimes H_2$.*

Proof. Suppose that $(u, v) \sim (x, y)$ in $G_1 \otimes G_2$. Then $(s, t) \in N_{G_1 \otimes G_2}(u, v) \cap N_{G_1 \otimes G_2}(x, y)$ if and only if $u \sim s$, $x \sim s$ in G_1 and $v \sim t$, $y \sim t$ in G_2. Thus, $N_{G_1 \otimes G_2}(u, v) \cap N_{G_1 \otimes G_2}(x, y) = (N_{G_1}(u) \cap N_{G_1}(x)) \times (N_{G_1}(v) \cap N_{G_1}(y)) = V(H_1) \times V(H_2)$, and this set induces $H_1 \otimes H_2$ in $G_1 \otimes G_2$. □

For example, $K_n \otimes K_m \cong K_{m,m,\ldots,m} \setminus \{n-1)$-factor edges$\}$, an n-partite graph with uniform part size m where the edges of a $(n-1)$-factor are the column edges when the vertices are arranged in a $n \times m$ matrix. Therefore, the USNS of $(K_n \otimes K_m) \cong K_{n-2} \otimes K_{m-2} \cong K_{m-2,m-2,\ldots,m-2} \setminus \{(n-3)$-factor edges$\}$, an $(n-2)$-partite graph with uniform part size $m-2$, where the edges of a $(n-3)$-factor are column edges when the vertices are arranged in a $(n-2) \times (m-2)$ matrix.

From this, it follows that given $G \in ER(n, d, \lambda)$ with USNS $\cong H$ where $|H| = \lambda$, $K_3 \otimes G$ has a USNS of $|H|K_1$. In other words, the tensor product of an edge-regular graph G with some USNS and a K_3 removes all of the edges of the USNS of G as a new USNS.

Another example: $G_1 \otimes G_2$, where $G_1 \in ER(n, d, \lambda)$ and G_2 is a triangle-free regular graph, has an empty graph USNS. That is, $G_1 \otimes G_2$ is also triangle-free.

Another useful graph construction for edge-regular graphs is the *shadow* of a graph. For any positive integer n, let $[n] = \{1, \dots, n\}$. Enlarging the definition in [8], given a graph G, define $D_m(G)$ to be the m^{th} *shadow graph* of G, by $V(D_m(G)) = \{v_j^i | i \in [m]; j \in [n]\}$, given that $V(G) = \{v_1, \dots, v_n\}$; for $j, l \in [n]$ and $i, k \in [m]$, the vertices v_j^i and v_l^k are adjacent in $D_m(G)$ if $v_j \sim v_l$ in G. See Fig. 4 for an example.

Theorem 3.3. *If $G \in ER(n, d, \lambda)$ with a USNS $\cong H$, then $D_m(G) \in ER(mn, md, m\lambda)$ with a USNS $\cong D_m(H)$.*

Proof. Let $G \in ER(n, d, \lambda)$. Then by construction the m^{th} shadow of G contains m copies of every vertex of G, so $|D_m(G)| = mn$.

Now suppose $N_G(v_i) = \{u_1, \dots, u_d\}$. Then v_i^k is adjacent to each of $\{u_1^1, \dots, u_d^1, u_1^2, \dots, u_d^2, \dots, u_1^m, \dots, u_d^m\}$ for $k \in [m]$. So $D_m(G)$ is regular of degree md.

Using similar logic, say $v_i \sim v_j$ in G such that $N(v_i) \cap N(v_j) = \{u_1, \dots, u_\lambda\}$. Then $N(v_i^k) \cap N(v_j^l) = \{u_\beta^\alpha | \alpha \in [m]; \beta \in [\lambda]\}$ for $k, l \in [m]$. Thus, every pair of adjacent vertices in $D_m(G)$ share exactly $m\lambda$ vertices.

Further, as $G[\{v_1, \dots, v_\lambda\}] \cong H$, then $N(v_i^k) \cap N(v_j^l)$ contains exactly m copies of H, one in each shadow. The edge set among these m copies of H are as defined in the m^{th} shadow graph. Thus, $D_m(G)$ has a USNS $\cong D_m(H)$. □

Iteration of a USNS with the shadow graph function allows for additional infinite families of USNS.

Theorem 3.4. *$D_q(D_m(G)) \cong D_{qm}(G)$ for integers $q, m \geq 2$.*

Proof. Suppose $V(G) = \{v_1, \dots, v_n\}$ and $V(D_m(G)) = \{v_j^i \mid i = 1, \dots, m; j = 1, \dots, n\}$. Then $V(D_q(D_m(G))) = \{v_j^{i,k} \mid i = 1, \dots, m; j = 1, \dots, n; k = 1, \dots, q\}$; for all $1 \leq i \leq m, 1 \leq k \leq q, 1 \leq j \leq n$, $v_j^{i,k}$ is adjacent in $D_q(D_m(G))$ to every $v_r^{s,t}$ such that $v_j \sim v_r$ in G.

Arrange the qm copies of G in an $n \times m \times q$ array and label the vertices so that, with reference to a fixed list $v_1, \dots, v_n$ of the vertices of G, for $(s, t) \in [m] \times [q]$, the appearance of v_i in the line of the array consisting of places with coordinates $(-, s, t)$ is $v_i^{s,t}$. Now it is clear that adjacency in this incarnation of $D_{qm}(G)$ is the same as in $D_q(D_m(G))$.

Alternatively, in $V(D_{qm}(G)) = \{v_1, \dots, v_{qm}\}$, relabel the vertices such that the i^{th} vertex in the q^{th} copy of the m^{th} copy of the vertices is denoted $v_i^{m,q}$ for $i = 1, \dots, n$. So $\{v_{qm}\} = V(D_q(D_m(G)))$.

In both cases, $v_i^{m_1,q_1} \sim v_j^{m_2,q_2}$ if $v_i \sim v_j$ in G for $i \neq j; 1 \leq i, j \leq n; 1 \leq m_1, m_2 \leq m; 1 \leq q_1, q_2 \leq q$. Then $E(D_q(D_m(G))) = E(D_{qm}(G))$. So $D_q(D_m(G)) \cong D_{qm}(G)$ for all $q, m \geq 2$. □

For example, $D_m(K_n) \cong K_{m,m,\ldots,m} \cong T_{mn,n} \in ER(mn, m(n-1), m(n-2))$, a complete n-partite graph with uniform partition size m, commonly known as a (regular) *Turán graph*. $D_3(K_3) \cong T_{9,3}$ is shown in Fig. 4. So the USNS of $D_m(K_n)$ is $D_m(K_{n-2}) \cong K_{m,m,\ldots,m} \cong T_{m(n-2),n-2}$, the Turán graph on $m(n-2)$ vertices with partition size m and $n-2$ parts.

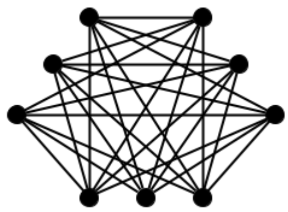

Figure 4: $D_3(K_3) \cong T_{9,3}$ with USNS of $D_3(K_1) \cong T_{3,1} = \overline{K_3}$

As stated in the preliminaries of the paper, not all edge-regular graphs have a connected USNS.

Consider $\mathcal{P}$, the Petersen graph; $\mathcal{P} \in ER(10, 3, 0)$. The complement of the Petersen graph, however, is the interesting case. This graph is also already known to be edge-regular, as discussed in the $d = \lambda + 3$ case of [6]: $\overline{\mathcal{P}} \in ER(10, 6, 3)$ with a USNS of $K_2 + K_1$.

4 Conway's 99-graph Problem

A strongly regular graph in $SR(n, d, \lambda, \mu)$ is a graph in $ER(n, d, \lambda)$ such that every pair of non-adjacent vertices share exactly μ common neighbors. Conway's 99-*graph problem* is an open problem that asks about the existence of a strongly regular graph in $SR(99, 14, 1, 2)$ [2]. Here we will show the non-existence of the 99-graph among Cartesian or tensor products of two edge-regular graphs.

A *regular clique assembly* is a regular graph in which each maximal clique (a complete subgraph) is maximum. The set of (isomorphism types of) regular clique assemblies on n vertices, of degree $d > 0$, with clique number = maximum order of a complete subgraph = $\omega(G) = k$ will be denoted $RCA(n, d, k)$. Observe that if $G \in RCA(n, d, k)$, then $k \geq 2$ and each edge of G is in exactly one maximum clique in G [5].

Next, we list results from [5] to use in the following theorem as lemmas.

Lemma 4.1. *$RCA(n, d, k) \subseteq ER(n, d, k-2)$, with equality when $k \in \{2, 3\}$.*

Lemma 4.2. *Suppose $ER(n, d, 1) \neq \emptyset$. Then*

1. *d is even;*
2. *$3|nd$;*
3. *for each $G \in ER(n,d,1)$ and $v \in V(G)$, $N_G[v]$ induces in G a friendship graph, $\{v\} \vee \frac{d}{2}K_2$;*
4. *if $d > 2$, each $G \in ER(n,d,1)$ is the clique graph of its clique graph, $CL(G) \in RCA(\frac{nd}{6}, \frac{3}{2}(d-2), \frac{d}{2})$.*

Lemma 4.3. *$ER(3(d-1), d, 1) \neq \emptyset$ if and only if $d \in \{2, 4, 6, 10\}$.*

If the 99-graph G exists, then it is necessarily an edge-regular graph in $ER(99, 14, 1)$. This is equivalent to a regular clique assembly on the parameters $RCA(99, 14, 3)$ by Lemma 4.3. The idea here is to try to construct $RCA(99, 14, 3)$ by a product of two graphs G_1 and G_2, and to show that there is no such combination if the product is either the Cartesian or the tensor product.

Theorem 4.1. *If Conway's 99-graph exists, then it cannot be constructed as the Cartesian product of two RCA graphs.*

Proof. Suppose $G_1 \in RCA(n_1, d_1, 3)$ and $G_2 \in RCA(n_2, d_2, 3)$. Then $G_1 \square G_2 \in RCA(n_1 n_2, d_1 + d_2, 3)$ by Theorem 3.1. Since G_1 and G_2 are regular graphs of odd order, $2 \mid d_i$, $i = 1, 2$. There are only two options for n_1 and n_2, namely the pairs $\{33, 3\}$ and $\{11, 9\}$.

Let $n_1 = 3$ and $n_2 = 33$. As with all regular graphs, $n > d$, so G_1 must have degree 2. So, $G_1 \in RCA(3, 2, 3) = ER(3, 2, 1) \cong K_3$. Then $G_2 \in RCA(33, 12, 3) = ER(33, 12, 1)$. By Lemma 4.3, $ER(33, 12, 1) = \emptyset$. So $\{33, 3\}$ is not a possible pair of orders of G_1 and G_2.

Let $n_1 = 9$ and $n_2 = 11$. Then for G_1 the only possible d_1 are $\{2, 4, 6, 8\}$ since $n_1 = 9 > d_1$.

If $G_1 \in RCA(9, 2, 3)$, then $G_2 \in RCA(11, 12, 3)$, impossible as $n_2 < d_2$. If $G_1 \in RCA(9, 4, 3)$, then $G_2 \in RCA(11, 10, 3) = ER(11, 10, 1)$. Given that $n_2 = d_2 + 1$, then G_2 would need to be K_{11}, of which $\lambda = 9 \neq 1$, so $RCA(11, 10, 3) = ER(11, 10, 1) = \emptyset$. If $G_1 \in RCA(9, 6, 3)$, then $G_2 \in RCA(11, 8, 3) = ER(11, 8, 1)$. By Lemma 4.2, since $3 \nmid nd = 88$, it follows that $ER(11, 8, 1) = \emptyset$.

Finally, if $G_1 \in RCA(9, 8, 3) = ER(9, 8, 1)$, then as $n_1 = d_1 + 1$, G_1 is K_9. Yet $K_9 = ER(9, 8, 7)$, so $ER(9, 8, 1) = \emptyset$.

Thus, the 99-graph cannot be the Cartesian product of two RCA graphs. □

Using similar logic, it is straightforward to show that the tensor product of two edge-regular graphs cannot yield Conway's 99-graph.

Theorem 4.2. *If Conway's 99-graph exists, then it cannot be constructed with the tensor product of edge-regular graphs.*

Proof. Suppose $G_1 \in ER(n_1, d_1, \lambda_1)$ and $G_2 \in ER(n_2, d_2, \lambda_2)$ such that $G_1 \otimes G_2 \in ER(99, 14, 1)$. It is straightforward to see that if G_1 or G_2 is disconnected, then

$G_1 \otimes G_2$ is disconnected, so we may assume that both G_1 and G_2 are connected graphs. By Theorem 3.2, $n_1 n_2 = 99$, $d_1 d_2 = 14$, and $\lambda_1 \lambda_2 = 1$. Thus, $\lambda_1 = \lambda_2 = 1$. Further, $d_1 d_2 = 1 \cdot 14$ or $d_1 d_2 = 2 \cdot 7$.

Suppose $d_1 d_2 = 1 \cdot 14$ and without loss of generality, $d_1 = 1$. Then $\lambda_1 = 1 = d_1$, a contradiction as $d > \lambda$ for all edge-regular graphs. Thus, $\{d_1, d_2\} \neq \{1, 14\}$.

Suppose $\{d_1, d_2\} = \{2, 7\}$ and without loss of generality, $d_1 = 2$. Then $\lambda_1 = 1$ and $d_1 = 2$ imply $n_1 = 3$. So $n_2 = 33$, $d_2 = 7$, and $\lambda_2 = 1$. An edge-regular graph $ER(33, 7, 1) = RCA(33, 7, 3)$ by Lemma 4.1. Yet $RCA(33, 7, 3)$ would be a regular graph of odd order and odd degree, an impossibility. □

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