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Online 3-choosability of a planar graph without certain cycles

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Abstract

Online list coloring of a graph G is a dynamic version of list coloring in which lists are not predetermined. It has been proven that planar graphs without cycles of length $4, r, s, 9$, where $r < s$ and $r, s \in \{5, 6, 7, 8\}$, are 3-choosable. In this paper, we extend the above results by proving that planar graphs without cycles of length $4, r, s, 9$, where $r < s$ and $r, s \in \{5, 6, 7, 8\}$ (except $r = 5$ and $s = 7$), are online 3-choosable.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A function $\phi : V(G) \to \{1, 2, 3, \dots, k\}$ is called a proper k-coloring of a graph G if $\phi(u) \neq \phi(v)$ for all edges $uv \in E(G)$. The chromatic number $\chi(G)$ of a graph G is the smallest integer k such that G has a proper k-coloring.

Vizing [9] and Erdős et al. [3] independently introduced the concept of list coloring in graphs.

Definition 1.1. For a given function $f : V(G) \to \mathbb{N}$, each vertex $u \in V(G)$ is assigned a list of $f(u)$ available colors. If there exists a proper coloring for every such list assignment then G is said to be f-choosable. If $f(u) = k$ for all $u \in V(G)$, then G is said to be k-choosable. The choice number of G is the smallest integer k such that G is k-choosable, and is denoted by $\chi_\ell(G)$ or $Ch(G)$.

Schauz [4] and Zhu [14] independently introduced the concept of online list coloring (paintability) in the form of a game. Online list coloring is a Marker-Remover game played on a graph G. For a function $f : V(G) \to \mathbb{N}$, there are $f(u) - 1$ erasers

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allotted to each vertex $u \in V(G)$. In the i-th round, Marker marks a non-empty subset M_i of the uncolored vertices of $V(G)$ with color i. After that, Remover colors and removes a maximal independent subset R_i of M_i. For every vertex marked with color i, but not removed, Remover needs to use one of its allotted erasers to remove its mark. Marker wins if, at the end of some round, there is a vertex u for which no eraser is left, and Remover wins if all the vertices are removed giving a proper coloring.

Definition 1.2. For a function $f : V(G) \to \mathbb{N}$, if Remover wins whenever $f(u) - 1$ erasers are allotted to each vertex u, then G is called online f-choosable. A graph G is called online k-choosable if $f(u) = k$.

Definition 1.3. The online choice number of G is the minimum value of k such that G is online k-choosable. Online choice number is denoted by $\chi_p(G)$ or $ch^{OL}(G)$.

The recursive definition of online list coloring is given by Schauz [4].

Definition 1.4. Online f-choosability of a graph G for a given function $f : V(G) \to \mathbb{N}$ is defined as follows:

(1) Empty graph is online f-choosable.
(2) A non-empty graph G is online f-choosable if every non-empty subset $V_M \subseteq V(G)$ contains an independent subset V_R such that $(G - V_R)$ is online $f - \mathbf{1}_{(V_M \setminus V_R)}$ choosable, where characteristic function $\mathbf{1}_X$ of a set X is defined as $\mathbf{1}_X(v) = 1$ if $v \in X$ and $\mathbf{1}_X(v) = 0$ if $v \notin X$.

Suppose that Marker writes down all the colors used to mark the vertex u in a list $L(u)$. When the game is over, the list $L(u)$ has at most $f(u)$ entries, since Remover can erase the mark at u at most $f(u)-1$ times. The color assigned to a vertex u of a graph G belongs to the list $L(u)$. Thus online list coloring may be seen as a dynamic version of list coloring where the lists $L(u)$ for $u \in V(G)$ are not predetermined. Since we have the natural inequalities $\chi_p(G) \geq \chi_\ell(G) \geq \chi(G)$, extending the known results from choosability to online choosability strengthens those results. Thomassen [6] proved that a planar graph is 5-choosable. He also proved that a planar graph with girth at least 5 is 3-choosable [7], and gave a short list-color proof of Grötzsch's theorem [8]. Schauz [4] proved that a planar graph is online 5-choosable. Chang and Zhu [2] proved that a planar graph with no 3-cycle and no 4-cycle adjacent to a 4-cycle or 5-cycle, is online 3-choosable.

We begin by introducing the terminology and definitions used throughout this paper. A graph is called planar if it can be embedded in the plane so that its edges do not cross each other and intersect only at their endpoints. Let G be a planar graph with vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. For a positive integer k, a cycle of length k is called a k-cycle. The boundary of a face f is a closed walk around f. The number of edges on the boundary of f is called the degree of f. We call a face f a k-face if its degree is equal to k. A face f is called simple if its boundary forms a cycle. Let $V(f)$ and $E(f)$ denote the set of vertices and edges, respectively, on the boundary of the face f. If $v \in V(f)$ then a vertex v is said to be incident with a face f. In a plane graph, a simple face is specified by the sequence of its vertices traversed in either the clockwise or counterclockwise direction. We call a vertex v a k-vertex, k^--vertex, or k^+-vertex if its degree $d(v)$ is equal to k, at most k, or at least k, respectively. An edge uv is denoted by $(d(u), d(v))$. The minimum degree of a vertex in a graph G is denoted by $\delta(G)$. For $A \subset V(G)$, $G - A$ denotes the subgraph of G induced by the vertex set $V(G) \setminus A$. $G - v$ denotes the induced subgraph of G obtained by deleting a vertex $v \in V(G)$. Function $f|_A$ denotes the restriction of the function f to the set A. The set of neighbors of a vertex v in a graph G is denoted by $N_G(v)$, i.e., $N_G(v) = \{u \mid u \in V(G), uv \in E(G)\}$. A chord of a cycle C is an edge that connects two non-adjacent vertices of C. A chord xy is called an internal chord if it lies entirely within the region enclosed by C. Otherwise, xy is called an external chord of C. A directed graph is a graph where each edge is assigned a direction. In a directed edge xy, the vertex x is the tail, and the vertex y is the head. Vertex x is an in-neighbor of y, and y is an out-neighbor of x. A directed cycle $(v_1, v_2, v_3, \ldots, v_n, v_1)$ is a sequence of directed edges $v_1v_2,\ v_2v_3,\ v_3v_4, \ldots,\ v_nv_1$, forming a closed path in which each vertex appear exactly once, except for the first and last, which coincide.

2 Some results in 3-choosbility

Borodin [1] proved the following theorem:

Theorem 2.1. *A planar graph without cycles of length k, for $4 \leq k \leq 9$, is 3-colorable.*

Forbidding cycles of certain lengths provides sufficient conditions for 3-choosability.

Theorem 2.2. *A planar graph is 3-choosable if it contains no cycles of length*

(1) [13] 4, 5, 6, 7, 8 *and* 9*; or*
(2) [13] 4, 5, 6 *and* 9*; or*
(3) [12] 4, 7, 8 *and* 9*; or*
(4) [5] 4, 6, 8 *and* 9*; or*
(5) [10] 4, 6, 7 *and* 9*; or*
(6) [11] 4, 5, 8 *and* 9.

If a certain structure appears in a minimal counterexample G and leads to a contradiction with the assumed property, then it is called a reducible structure. The reducible structure in the above results is an even cycle C in which all vertices have degree 3. Since an even cycle C is 2-choosable, the list coloring of $G - C$ can be extended to the list coloring of G. In this paper, we prove that the absence of cycles of certain lengths gives sufficient conditions for online 3-choosability.

3 Lemmas

Using the recursive definition of online f-choosability, Schauz [4] proved the following lemmas. For a graph G, let f, g, g_1, and g_2 be functions from $V(G)$ to the set of natural numbers $\mathbb{N}$.

Lemma 3.1. *If $g(v) \leq f(v)$ for all $v \in V(G)$ then online g-choosablility of G implies online f-choosablility of G.*

Lemma 3.2. *Let $H = \{v : deg(v) < f(v)\}$ be a subset of $V(G)$. G is online f-choosable if $G - H$ is online $f|_{G \setminus H}$ choosable.*

Lemma 3.3. *Let G be a graph with a function $g : V(G) \to \mathbb{N}$. Let A be an independent subset of $V(G)$ such that $g(v) = 1$ for all $v \in A$. Let $f : V(G) \setminus A \to \mathbb{N}$ be defined as $f(v) = g(v) - |A \cap N_G(v)|$ then G is online g-choosable if and only if $G - A$ is online f-choosable.*

Lemma 3.4. *Let $G = A \cup B$ such that A and B are online g-choosable and h-choosable respectively, where $h(v) = 1$ for all $v \in V(A) \cap V(B)$. Let a function $f : (V(A) \cup V(B)) \to \mathbb{N}$ be defined as $f(v) = g(v)\mathbf{1}_{V(A)}(v) + h(v)\mathbf{1}_{[V(B) \setminus V(A)]}(v)$. Then $A \cup B$ is online f-choosable.*

4 Main results

Theorem 4.1. *A planar graph G without k-cycles for $4 \leq k \leq 9$ is online 3-choosable.*

Proof. Suppose G is a graph of least order that does not contain k-cycles for $4 \leq k \leq 9$ and G is not online 3-choosable. For such a graph G, we have $\delta(G) \geq 3$. Otherwise, there exists a vertex $v \in V(G)$ such that $d(v) < 3$. As G is the smallest counterexample, $G - v$ is online 3-choosable. By Lemma 3.2, G is online 3-choosable. This is a contradiction.

Borodin [1] proved that a planar graph G with $\delta(G) \geq 3$ and no adjacent triangles contains either a cycle of length between 4 to 9, or a 10-face incident with ten 3-vertices and adjacent to five triangles. Since counterexample G does not contain k-cycles for $4 \leq k \leq 9$ and has minimum vertex degree $\delta(G) \geq 3$, the graph G must contain a 10-face F that is incident with ten 3-vertices and adjacent to five triangles. Let $V_1 = \{v_1, v_2, \ldots, v_{10}\}$ be the set of 3-vertices incident to the 10-face F. Let $A = \{u \mid u \in N_G(v),\ v \in V(F)\} \setminus V(F)$. Thus, $A = \{u_1, u_2, u_3, u_4, u_5\}$ is the set of vertices incident with 3-faces adjacent to the 10-face F, but do not lie on the boundary of F (see Figure 1).

Let $G_1 = G - V(F)$ be the subgraph of G induced by $V(G) \setminus V(F)$. Let $g_1 : V(G_1) \to \mathbb{N}$ be defined by $g_1(v) = 3$ for all $v \in V(G_1)$. Since G is the smallest counterexample, it follows that the subgraph G_1 is online g_1-choosable.

Let G_2 be the subgraph of G with $V(G_2) = V(F) \cup A$ and $E(G_2) = E(F) \cup \{vu \mid v \in V(F), u \in A, vu \in E(G)\}$ (see Figure 1). Observe that the set A forms an independent set in G_2. Let $g_2 : V(G_2) \to \mathbb{N}$ be defined by $g_2(v) = 1$ for all $v \in A$ and $g_2(v) = 3$ for all $v \in V(F)$. We prove that G_2 is online g_2-choosable.

Let $H = G_2 - A$ be the subgraph of G_2 induced by $V(G_2) \setminus A$, with $h : V(H) \to \mathbb{N}$ defined by $h(v) = g_2(v) - |A \cap N_{G_2}(v)|$. Therefore, $h(v) = 2$ for all $v \in V(H)$. The subgraph H is an even cycle C_{10}. Zhu [14] proved that the even cycle C_{2n} is online 2-choosable. Hence, H is online 2-choosable. It follows from Lemma 3.3 that G_2 is online g_2-choosable.

Note that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = A$. Let $f : V(G) \to \mathbb{N}$ be defined by $f(v) = g_1(v)\mathbf{1}_{V(G_1)}(v) + g_2(v)\mathbf{1}_{[V(G_2)\setminus V(G_1)]}(v)$. Thus, $f(v) = 3$ for all $v \in V(G)$. By Lemma 3.4, $G = G_1 \cup G_2$ is online f-choosable. Hence, we obtain a contradiction. □

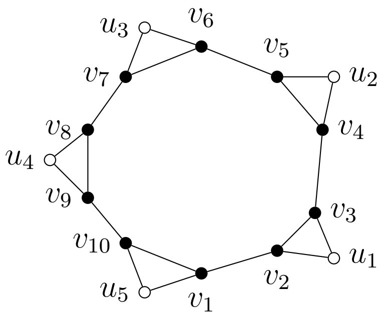


Figure 1: G_2:10-face with adjacent 3-faces

Zhang and Wu [13] proved that the absence of cycles of length 4, 5, 6 and 9 in a graph G is sufficient to have a 10-face incident with ten vertices of degree three and adjacent to five triangles. Proceeding as in Theorem 4.1, we obtain the following result.

Theorem 4.2. *A planar graph G without 4, 5, 6 and 9-cycles is online 3-choosable.*

Theorem 4.3. *A planar graph G without 4, 7, 8 and 9-cycles is online 3-choosable.*

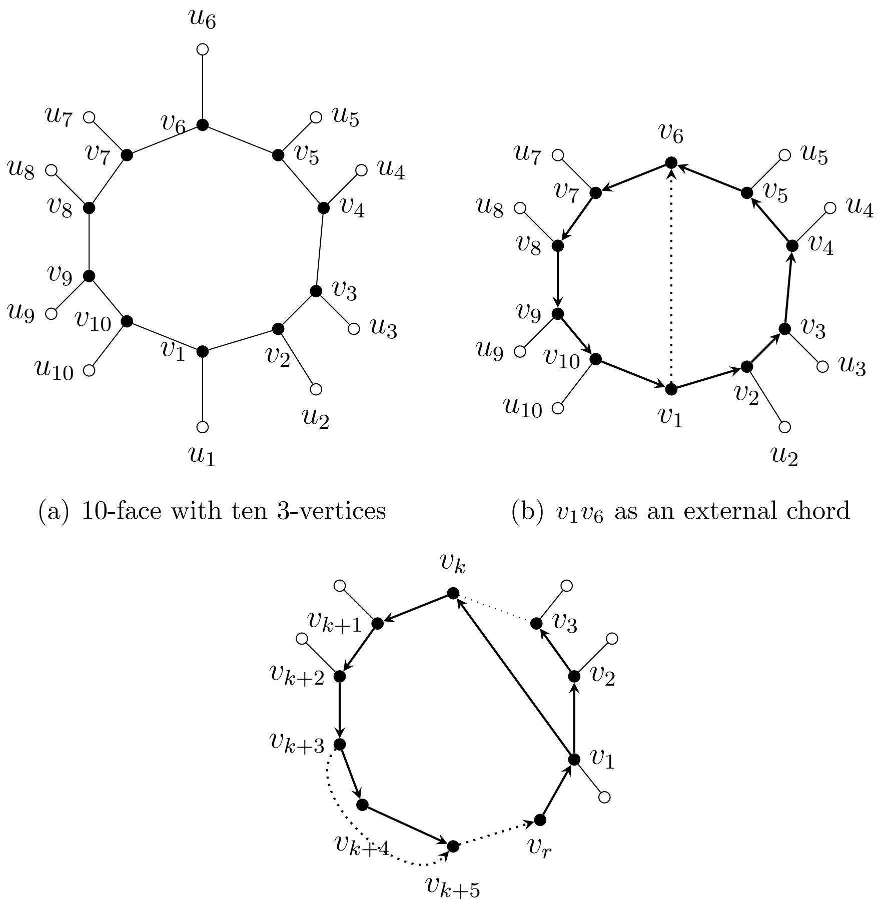
Proof. Suppose G is a graph of least order that does not contain 4, 7, 8 and 9-cycles, and G is not online 3-choosable. If $\delta(G) < 3$, then there exists a vertex $v \in V(G)$ such that $d(v) < 3$. Since G is smallest counterexample, it follows that $G - v$ is online 3-choosable. By Lemma 3.2, G is online 3-choosable. Hence, we obtain a contradiction. It follows that $\delta(G) \geq 3$. A θ-graph consists of two distinct vertices connected by three internally pairwise disjoint paths. It was shown in [12] that G contains either a 10-face incident with ten 3-vertices, or a special θ-like induced subgraph S with the following properties (see Figure 2):

(1) $\delta(S) = 2$;

(2) S contains a cycle C that spans all the vertices of S;

(3) the removal of external chords of C, if any, leaves C with only one internal chord, which is the edge $(3, 4^-)$ in G;

(4) all vertices of the subgraph S have degree 3 in G, with the possible exception of one endpoint of the internal chord $(3, 4^-)$.



(a) 10-face with ten 3-vertices

(b) v_1v_6 as an external chord

(c) S

Figure 2: Subgraphs of G

Case 4.3.1. Suppose G contains a 10-face F incident with ten 3-vertices. Let $V_1 = \{v_1, v_2, \ldots, v_{10}\}$ be the set of 3-vertices incident with the face F. Let A be the set of neighbors in G of vertices in V_1, excluding those in V_1 itself, i.e., $A = \{u \mid u \in N_G(v),\ v \in V_1\} \setminus V_1$.

Let $G_1 = G - V_1$ be the subgraph of G induced by $V(G) \setminus V_1$. We define a function $g_1 : V(G_1) \to \mathbb{N}$ by $g_1(v) = 3$ for all $v \in V(G_1)$. Since G is the smallest counterexample, it follows that the subgraph G_1 is online g_1-choosable.

Let G_2 be the subgraph of G with $V(G_2) = V_1 \cup A$ and $E(G_2) = E(F) \cup \{vu \mid v \in V_1,\ u \in A,\ vu \in E(G)\}$. Note that the set A forms an independent set in G_2. Let $g_2 : V(G_2) \to \mathbb{N}$ be a function defined by $g_2(v) = 1$ for all $v \in A$ and $g_2(v) = 3$ for all $v \in V(G_2) \setminus A$. We prove that G_2 is online g_2-choosable.

Let $H = G_2 - A$ be the subgraph of G_2 induced by $V(G_2) \setminus A$. Let $h : V(H) \to \mathbb{N}$ be defined by $h(v) = g_2(v) - |A \cap N_{G_2}(v)|$. Therefore, $h(v) = 2$ for all $v \in V(H)$. Since $H = G - A$ is an even cycle C_{10}, it follows that H is online 2-choosable. By Lemma 3.3, G_2 is online g_2-choosable.

Note that $G = G_1 \cup G_2$, with $V(G_1) \cap V(G_2) = A$. Let $f : V(G) \to \mathbb{N}$ be defined by $f(v) = g_1(v)\mathbf{1}_{V(G_1)}(v) + g_2(v)\mathbf{1}_{[V(G_2)\setminus V(G_1)]}(v)$. Thus, $f(v) = 3$ for all $v \in V(G)$. By Lemma 3.4, $G = G_1 \cup G_2$ is online f-choosable. Thus, we arrive at a contradiction.

Case 4.3.2. Since there are no cycles of length 4, 7, 8 and 9, G may contain a 10-face F with exactly one external chord that divides F into two equal parts. Let $V_1 = \{v_1, v_2, \ldots, v_{10}\}$ be the set of 3-vertices incident to the face F. Without loss of generality, we may relabel the vertices of F so that they appear in the cyclic order $(v_1, v_2, \ldots, v_{10})$, with the external chord being v_1v_6. Let A be the set of neighbors in G of vertices in V_1, excluding those in V_1 itself, i.e., $A = \{u \mid u \in N_G(v),\ v \in V_1\} \setminus V_1$.

Let $G_1 = G - V_1$ be the subgraph of G induced by $V(G) \setminus V_1$. Let $g_1 : V(G_1) \to \mathbb{N} =$ be defined by $g_1(v) = 3$ for all $v \in V(G_1)$. Since G is the smallest counterexample, it follows that the subgraph G_1 is online g_1-choosable.

Let G_2 be the subgraph of G with $V(G_2) = V_1 \cup A$ and $E(G_2) = E(F) \cup \{vu \mid v \in V_1, u \in A, vu \in E(G)\} \cup \{v_1v_6\}$. Let $g_2 : V(G_2) \to \mathbb{N}$ be a function defined by $g_2(v) = 1$ for all $v \in A$, and $g_2(v) = 3$ for all $v \in V(G_2) \setminus A$. We prove that G_2 is online g_2-choosable.

Let $H = G_2 - A$ be the subgraph of G_2 induced by $V(G_2) \setminus A$. Let $h : V(H) \to \mathbb{N}$ defined by $h(v) = g_2(v) - |A \cap N_{G_2}(v)|$. It follows that $h(v) = 2$ for all $v \in V(H)$, except for the endpoints of the external chord v_1v_6, which evenly divides the face F. For these endpoints, $h(v_1) = 3$ and $h(v_6) = 3$. The subgraph $H = G_2 - A$ consists of an even cycle $C_{10} = (v_1, v_2, v_3, \ldots, v_9, v_{10}, v_1)$ and an external chord v_1v_6. We orient the cycle C_{10} to form a directed cycle (see Figure 2(b)). The external chord v_1v_6 is oriented from v_1 to v_6. In the i-th round, Marker marks a non-empty subset M_i of the uncolored vertices in $V(H)$ with color i. Remover colors and removes the maximal independent subset R_i of M_i, selected greedily with respect to the given orientation. Thus, a vertex $v \in M_i$ is in R_i if and only if it has no in-neighbor in M_i under a given orientation. Every vertex $v \in V(H)$ has exactly one in-neighbor, except for v_6, which has v_1 and v_5 as its two in-neighbors. Remover wins by applying this strategy. A vertex $v \in V(H) \setminus \{v_6\}$ is marked without being removed at most once for its in-neighbor. The endpoint v_6 of the external chord v_1v_6, is marked at most twice but not removed, once for each of its two in-neighbors. Hence, H is online h-choosable. By Lemma 3.3, G_2 is online g_2-choosable.

Note that $G = G_1 \cup G_2$, with $V(G_1) \cap V(G_2) = A$. Let $f : V(G) \to \mathbb{N}$ be defined by $f(v) = g_1(v)\mathbf{1}_{V(G_1)}(v) + g_2(v)\mathbf{1}_{[V(G_2)\setminus V(G_1)]}(v)$. Thus, $f(v) = 3$ for all $v \in V(G)$. By Lemma 3.4, $G = G_1 \cup G_2$ is online f-choosable. Hence, we obtain a contradiction.

Case 4.3.3. Suppose G contains the special subgraph S. Let V_1 be the vertex set of the subgraph S. Let $C = (v_1, v_2, v_3, \ldots, v_k, v_{k+1}, v_{k+2}, v_{k+3}, \ldots, v_r, v_1)$ be a spanning cycle of S. After deleting all external chords of the cycle C, we obtain a cycle with exactly one internal chord $(3, 4^-)$. Without loss of generality, let v_1v_k be the internal chord with v_1 as 4^--vertex and v_k as 3-vertex in G. Let $A = \{u \mid u \in N_G(v),\ v \in V_1\} \backslash V_1$.

Let $G_1 = G - V_1$ be the subgraph of G induced by $V(G) \setminus V_1$. We define a function $g_1 : V(G_1) \to \mathbb{N}$ by $g_1(v) = 3$ for all $v \in V(G_1)$. Since G is the smallest counterexample, it follows that the subgraph G_1 is online g_1-choosable.

Let G_2 be the subgraph of G with $V(G_2) = V_1 \cup A$ and $E(G_2) = E(S) \cup \{vu \mid v \in V_1,\ u \in A,\ vu \in E(G)\}$. Let $g_2 : V(G_2) \to \mathbb{N}$ be defined by $g_2(v) = 1$ for all $v \in A$ and $g_2(v) = 3$ for all $v \in V(G_2) \setminus A$. We prove that G_2 is online g_2-choosable.

Let $H = G_2 - A$ be the subgraph of G_2 induced by $V(G_2) \setminus A$. Let $h : V(H) \to \mathbb{N}$ be defined by $h(v) = g_2(v) - |A \cap N_{G_2}(v)|$. Therefore, $h(v) = 2$ for all $v \in V(H)$, except for the endpoint v_k of the internal chord v_1v_k, for which $h(v_k) = 3$. We orient the spanning cycle C in S to form a directed cycle. The internal chord v_1v_k is oriented from v_1 to v_k. In the i-th round, Marker marks a non-empty subset M_i of the uncolored vertices in $V(H)$ with color i. Remover colors and removes the maximal independent subset R_i of M_i, selected greedily with respect to the given orientation. Thus, a vertex $v \in M_i$ is in R_i if and only if it has no in-neighbor in M_i under a given orientation. Every vertex $v \in V(H)$ has exactly one in-neighbor, except for the endpoint v_k of the internal chord v_1v_k, which has v_1 and v_{k-1} as its two in-neighbors. Remover wins applying this strategy, since $v \in V(H) \setminus \{v_k\}$ is marked without being removed at most once for its in-neighbor. The endpoint v_k of the internal chord v_1v_k, is marked at most twice but not removed, once for each of its two in-neighbors. If there is an external chord uv then $h(u) = 3$ and $h(v) = 3$. For a directed arc uv, the possibility of vertex v being marked twice but not removed, does not pose a problem, since $h(v) = 3$. For example, in case of the directed external chord $v_{k+3}v_{k+5}$, the vertex v_{k+5} may be marked twice without being removed as it has two in-neighbors v_{k+3} and v_{k+4}. Remover still wins, since $h(v_{k+5}) = 3$ (see Figure 2(c)). Hence, $H = G_2 - A$ is online h-choosable. By applying Lemma 3.3, we conclude that G_2 is online g_2-choosable.

Note that $G = G_1 \cup G_2$, with $V(G_1) \cap V(G_2) = A$. Let $f : V(G) \to \mathbb{N}$ be defined by $f(v) = g_1(v)\mathbf{1}_{V(G_1)}(v) + g_2(v)\mathbf{1}_{[V(G_2)\setminus V(G_1)]}(v)$. Thus, $f(v) = 3$ for all $v \in V(G)$. By Lemma 3.4, $G = G_1 \cup G_2$ is online f-choosable. Hence, we arrive at a contradiction.

Hence, a planar graph G without 4, 7, 8 and 9-cycles is online 3-choosable. □

Theorem 4.4. *A planar graph G without 4, 6, 8 and 9-cycles is online 3-choosable.*

Proof. Suppose G is a graph of least order that does not contain 4, 6, 8 and 9-cycles, and G is not online 3-choosable. For such a graph G, we have $\delta(G) \geq 3$. Otherwise, there exists a vertex $v \in V(G)$ such that $d(v) < 3$. Since G is smallest counterexample, $G - v$ is online 3-choosable. By Lemma 3.2, G is online 3-choosable. Hence, we obtain a contradiction.

As proved by Shen and Wang [5], G contains a 10-face incident with ten 3-vertices. Then following a similar strategy as in Case 4.3.1 of Theorem 4.3, we prove that a planar graph G with no cycles of length 4, 6, 8 and 9 is online 3-choosable. □

Theorem 4.5. *A planar graph G without 4, 6, 7 and 9-cycles is online 3-choosable.*

Proof. Suppose G is a graph of least order that does not contain 4, 6, 7 and 9-cycles, and G is not online 3-choosable. As proved in Theorem 4.4, $\delta(G) \geq 3$. Wang, Lu and Chen [10] proved that a planar graph G without 4, 6, 7 and 9-cycles contains either an 8-face incident with eight 3-vertices, or an 8-face incident with eight 3-vertices and containing exactly one external chord that divides an 8-face into two equal parts, or a 10-face with ten 3-vertices.

Case 4.5.1. Suppose G contains an 8-face F incident with eight 3-vertices. Let the vertex set of F be $V_1 = \{v_1, v_2, v_3, \ldots, v_8\}$. Since the face F is an even cycle, we proceed as in Case 4.3.1 of Theorem 4.3.

Case 4.5.2. Since there are no cycles of length 4, 6, 7 and 9, G may contain an 8-face F that is incident with eight 3-vertices and has exactly one external chord, which divides F into two equal parts. Let $V_1 = \{v_1, v_2, \ldots, v_8\}$ be the vertex set of the face F. Without loss of generality, we may relabel the vertices of F so that they appear in the cyclic order $(v_1, v_2, \ldots, v_8)$, with the external chord being v_1v_5. Let $A = \{u \mid u \in N_G(v),\ v \in V_1\} \setminus V_1$.

Let $G_1 = G - V_1$ be the subgraph of G induced by $V(G) \setminus V_1$. Let $g_1 : V(G_1) \to \mathbb{N}$ be defined by $g_1(v) = 3$ for all $v \in V(G_1)$. Since G is the smallest counterexample, it follows that the subgraph G_1 is online g_1-choosable.

Let G_2 be the subgraph of G with $V(G_2) = V_1 \cup A$ and $E(G_2) = E(F) \cup \{vu \mid v \in V_1, u \in A, vu \in E(G)\} \cup \{v_1v_5\}$. We define a function $g_2 : V(G_2) \to \mathbb{N}$ by $g_2(v) = 1$ for all $v \in A$ and $g_2(v) = 3$ for all $v \in V(G_2) \setminus A$. We prove that G_2 is online g_2-choosable.

Let $H = G_2 - A$ be the subgraph of G_2 induced by $V(G_2) \setminus A$, with $h : V(H) \to \mathbb{N}$ defined by $h(v) = g_2(v) - |A \cap N_{G_2}(v)|$. Therefore, $h(v) = 2$ for all $v \in V(H)$, except for the endpoints of the external chord v_1v_5 which evenly divides 8-face. For these endpoints, $h(v_1) = 3$ and $h(v_5) = 3$. The subgraph induced by H consists of an even cycle C_8 and external chord v_1v_5. We orient the cycle C_8 such that it forms a directed cycle. The external chord v_1v_5 is oriented from v_1 to v_5. In the i-th round, Marker marks a non-empty subset M_i of the uncolored vertices in $V(H)$ with color i. Remover colors and removes the maximal independent subset R_i of M_i, selected greedily with respect to the given orientation. Thus, a vertex $v \in M_i$ is in R_i if and only if it has no in-neighbor in M_i under a given orientation. Every vertex $v \in V(H)$ has exactly one in-neighbor, except for the endpoint v_5 of the external chord v_1v_5, which has v_1 and v_4 as its two in-neighbors. Remover wins by applying this strategy because $v \in V(H) \setminus \{v_5\}$ is marked but not removed, at most once for its in-neighbor. The endpoint v_5 of the external chord v_1v_5, is marked at most twice, but is not removed, once for each of its two in-neighbors. Hence, H is online h-choosable. Lemma 3.3 implies that G_2 is online g_2-choosable.

Note that $G = G_1 \cup G_2$, with $V(G_1) \cap V(G_2) = A$. Let $f : V(G) \to \mathbb{N}$ be defined by $f(v) = g_1(v)\mathbf{1}_{V(G_1)}(v) + g_2(v)\mathbf{1}_{[V(G_2)\setminus V(G_1)]}(v)$. Thus, $f(v) = 3$ for all $v \in V(G)$. By Lemma 3.4, $G = G_1 \cup G_2$ is online f-choosable. Thus, we arrive at a contradiction.

Case 4.5.3. Suppose G contains a 10-face F incident with ten 3-vertices. We proceed as in case 4.3.1 of Theorem 4.3. □

Theorem 4.6. *A planar graph G without 4, 5, 8 and 9-cycles is online 3-choosable.*

Proof. Suppose G is a graph of least order that does not contain 4, 5, 8 and 9-cycles, and G is not online 3-choosable.

A θ-graph consists of two distinct vertices joined by three internally disjoint paths. An induced subgraph S_θ of G is a special θ-graph, isomorphic to an r-cycle with one internal chord. All vertices of S_θ are of degree 3 in G, except for one endpoint of its internal chord, which is a 4^--vertex (see Figure 3(a)).

An induced subgraph T_θ is an altered version of S_θ in which two endpoints of the internal chord are replaced by two 3-faces. All vertices of T_θ are of degree 3, except for two vertices lying on the outer cycle, which are incident to one of the two 3-faces and can possibly have degree 4 in G (v_1 and v_r in Figure 3(b)).

Wang, Lu, and Chen [11] proved the following structural property. If a connected graph G satisfies the following properties:

(1) $3 \leq \delta(G)$;
(2) G does not contain 4, 5, 8 and 9-cycles;
(3) every simple even face contains at least one 4^+-vertex,

then G contains either S_θ or T_θ.

We prove that $\delta(G) \geq 3$. If $\delta(G) < 3$, then there exists a vertex $v \in V(G)$ such that $d(v) < 3$. Since G is smallest counterexample, $G - v$ is online 3-choosable. By Lemma 3.2, G is online 3-choosable. Thus, we obtain a contradiction.

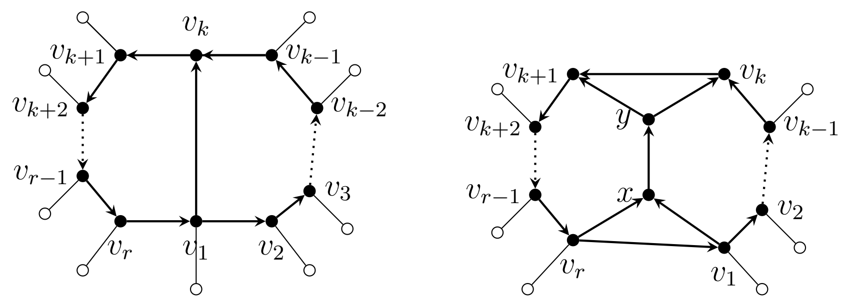
Every simple even face F of G contains at least one 4^+ vertex. Otherwise, boundary of a face F is an even cycle C with all vertices of degree 3. Let V_1 be the set of vertices incident to the face F. Let $A = \{u \mid u \in N_G(v),\ v \in V_1\} \backslash V_1$.

Let $G_1 = G - V_1$ be the subgraph of G induced by $V(G) \setminus V_1$. Let $g_1 : V(G_1) \to \mathbb{N}$ be defined by $g_1(v) = 3$ for all $v \in V(G_1)$. Since G is the smallest counterexample, it follows that the subgraph G_1 is online g_1-choosable.

Let G_2 be the subgraph of G with $V(G_2) = V(F) \cup A$ and $E(G_2) = E(F) \cup \{vu \mid v \in V_1, u \in A, vu \in E(G)\}$. Let $g_2 : V(G_2) \to \mathbb{N}$ be defined by $g_2(v) = 1$ for all $v \in A$ and $g_2(v) = 3$ for all $v \in V(G_2) \setminus A$. We prove that G_2 is online g_2-choosable.

Let $H = G_2 - A$ be the subgraph of G_2 induced by $V(G_2) \setminus A$. Let $h : V(H) \to \mathbb{N}$ defined by $h(v) = g_2(v) - |A \cap N_{G_2}(v)|$. Therefore, $h(v) = 2$ for all $v \in V(H)$. Since $H = G - A$ is an even cycle C, H is online 2-choosable, i.e., online h-choosable. It follows from Lemma 3.3, G_2 is online g_2-choosable.

Note that $G = G_1 \cup G_2$, with $V(G_1) \cap V(G_2) = A$. Let $f : V(G) \to \mathbb{N}$ be defined by $f(v) = g_1(v)\mathbf{1}_{V(G_1)}(v) + g_2(v)\mathbf{1}_{[V(G_2)\setminus V(G_1)]}(v)$. Thus, $f(v) = 3$ for all $v \in V(G)$. By Lemma 3.4, $G = G_1 \cup G_2$ is online f-choosable. Hence, we obtain a contradiction.



(a) S_θ (b) T_θ

Figure 3: Reducible structures

Case 4.6.1. Suppose G contains an induced subgraph S_θ. Let V_1 be the vertex set of S_θ. Let $A = \{u \mid u \in N_G(v),\ v \in V_1\} \setminus V_1$. Without loss of generality, we may relabel the vertices of S_θ so that they appear in the cyclic order $(v_1, v_2, \ldots, v_k, v_{k+1}, v_{k+2}, \ldots, v_r)$, with the external chord being v_1v_k, where v_1 is a 4^--vertex.

Let $G_1 = G - V_1$ be the subgraph of G induced by $V(G) \setminus V_1$. Define a function $g_1 : V(G_1) \to \mathbb{N}$ such that $g_1(v) = 3$ for all $v \in V(G_1)$. Since G is the smallest counterexample, it follows that the subgraph G_1 is online g_1-choosable.

Let G_2 be the subgraph of G with $V(G_2) = V(S_\theta) \cup A$ and $E(G_2) = E(S_\theta) \cup \{vu \mid v \in V_1, u \in A, vu \in E(G)\}$. Let $g_2 : V(G_2) \to \mathbb{N}$ be defined by $g_2(v) = 1$ for all $v \in A$ and $g_2(v) = 3$ for all $v \in V(G_2) \setminus A$. We show that G_2 is online g_2-choosable.

Let $H = G_2 - A$ be the subgraph of G_2 induced by $V(G_2) \setminus A$, with $h : V(H) \to \mathbb{N}$ defined by $h(v) = g_2(v) - |A \cap N_{G_2}(v)|$. Therefore, $h(v) = 2$ for all $v \in V(H)$, except for the endpoint v_k of the internal chord v_1v_k. For this endpoint, $h(v_k) = 3$. We orient the edges of H, i.e., S_θ as $v_1v_2,\ v_2v_3,\ v_3v_4, \ldots,\ v_kv_{k+1},\ v_{k+1}v_{k+2}, \ldots,\ v_{r-1}v_r,\ v_rv_1$ to form a directed cycle. The internal chord v_1v_k is oriented from the 4^--vertex v_1 to v_k (see Figure 3(a)). The winning strategy for Remover is as follows. In the i-th round, Marker marks a non-empty subset M_i of the uncolored vertices in $V(H)$ with color i. Remover colors and removes the maximal independent subset R_i of M_i, selected greedily with respect to the given orientation. Thus, a vertex $v \in M_i$ is in R_i if and only if it has no in-neighbor in M_i under a given orientation. Every vertex $v \in V(H)$ has exactly one in-neighbor, except for v_k, which has v_1 and v_{k-1} as its two in-neighbors. Remover wins by applying this strategy because $v \in V(H) \setminus \{v_k\}$ is marked, but not removed, at most once for its in-neighbor. The endpoint v_k of the internal chord v_1v_k, is marked at most twice but not removed, once for each of its two in-neighbors. Hence, H is online h-choosable. By applying Lemma 3.3, we conclude that G_2 is online g_2-choosable.

Note that $G = G_1 \cup G_2$, with $V(G_1) \cap V(G_2) = A$. Let $f : V(G) \to \mathbb{N}$ be defined by $f(v) = g_1(v)\mathbf{1}_{V(G_1)}(v) + g_2(v)\mathbf{1}_{[V(G_2)\setminus V(G_1)]}(v)$. Thus, $f(v) = 3$ for all $v \in V(G)$. By Lemma 3.4, $G = G_1 \cup G_2$ is online f-choosable. Hence, we obtain a contradiction.

Case 4.6.2. Suppose G contains an induced subgraph T_θ. Let V_1 be the vertex set of a subgraph T_θ. Let $A = \{u \mid u \in N_G(v),\ v \in V_1\} \setminus V_1$.

Let $G_1 = G - V_1$ be the subgraph of G induced by $V(G) \setminus V_1$. Define a function $g_1 : V(G_1) \to \mathbb{N}$ such that $g_1(v) = 3$ for all $v \in V(G_1)$. Since G is the smallest counterexample, it follows that the subgraph G_1 is online g_1-choosable.

Let G_2 be the subgraph of G with $V(G_2) = V(T_\theta) \cup A$ and $E(G_2) = E(T_\theta) \cup \{vu \mid v \in V_1, u \in A, vu \in E(G)\}$. Define a function $g_2 : V(G_2) \to \mathbb{N}$ by $g_2(v) = 1$ for all $v \in A$ and $g_2(v) = 3$ for all $v \in V(G_2) \setminus A$. We prove that G_2 is online g_2-choosable.

Let $H = G_2 - A$ be the subgraph of G_2 induced by $V(G_2) \setminus A$. Let $h : V(H) \to \mathbb{N}$ be defined by $h(v) = g_2(v) - |A \cap N_{G_2}(v)|$. Thus, $h(v) = 2$ for all $v \in V(H)$, except for vertices x, y, v_k, and v_{k+1}. Note that $h(x) = 3$, $h(y) = 3$, $h(v_k) = 3$ and $h(v_{k+1}) = 3$.

We orient the edges of H, i.e., T_θ such that each vertex with $h(v) = 2$ has exactly one in-neighbor, and each vertex with $h(v) = 3$ has at most two in-neighbors. We orient an outer cycle of T_θ as directed cycle $(v_1, v_2, v_3, \ldots, v_k, v_{k+1}, v_{k+2}, \ldots, v_{r-1}, v_r, v_1)$. Then, we orient the remaining edges inside the outer cycle of T_θ as $v_1x,\ v_rx,\ xy,\ yv_k$, and yv_{k+1} (see Figure 3(b)).

This provides the following winning strategy for Remover. In the i-th round, Marker marks a non-empty subset M_i of the uncolored vertices in $V(H)$ with color i. Remover colors and removes the maximal independent subset R_i of M_i, selected greedily with respect to the given orientation. Thus, a vertex $v \in M_i$ is in R_i if and only if it has no in-neighbor in M_i under a given orientation. Every vertex $v \in V(T_\theta)$ has only one preceding neighbor, except for x, v_k, and v_{k+1}, each of which has two in-neighbors. Remover wins by applying this strategy because $v \in V(T_\theta) \setminus \{x, v_k, v_{k+1}\}$ is marked but not removed, at most once for its in-neighbor. The vertices x, v_k, and v_{k+1} are marked at most twice, without being removed, once for each of its two in-neighbors. Hence, $H = G_2 - A$ is online h-choosable. Lemma 3.3 implies that G_2 is online g_2-choosable.

Note that $G = G_1 \cup G_2$, with $V(G_1) \cap V(G_2) = A$. Let $f : V(G) \to \mathbb{N}$ be defined by $f(v) = g_1(v)\mathbf{1}_{V(G_1)}(v) + g_2(v)\mathbf{1}_{[V(G_2)\setminus V(G_1)]}(v)$. Thus, $f(v) = 3$ for all $v \in V(G)$. By Lemma 3.4, $G = G_1 \cup G_2$ is online f-choosable. Thus, we obtain a contradiction. Therefore a planar graph G without 4, 5, 8 and 9-cycles is online 3-choosable. □

5 Conclusion

Using properties of reducible structure and greedy vertex coloring after orienting the edges of reducible structures as a winning strategy for Remover, we proved that planar graphs without cycles of length 4 to 9 or cycles of length $4, r, s$, and 9, where $r < s$ and $r, s \in \{5, 6, 7, 8\}$ (except $r = 5$ and $s = 7$), are online 3-choosable. Future work may be determining whether every planar graph without cycles of length 4, i, j, and k for $i < j < k$ and $i, j, k \in \{5, 6, 7, 8, 9\}$, is online 3-choosable.

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