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Duplicated Steiner triple systems with self-orthogonal near resolutions

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Abstract

A Steiner triple system, STS(v), is a family of 3-subsets (blocks) of a set of v elements such that any two elements occur together in precisely one block. A collection of blocks consisting of two copies of each block of an STS is called a duplicated Steiner triple system, DSTS. A resolvable (or near resolvable) DSTS is called self-orthogonal if every pair of distinct classes in the resolution has at most one block in common. We provide several methods to construct self-orthogonal near resolvable DSTS and settle the existence of such designs for all values of v with only four possible exceptions. This addresses a recent question of Bryant, Davies and Neubecker.

1 Introduction

A *balanced incomplete block design* (BIBD) is a pair $(V, \mathcal{B})$, where $\mathcal{B}$ is a collection of subsets (*blocks*) taken from a finite set V of v elements with the properties:

(1) Every block contains exactly k elements.
(2) Every pair of distinct elements of V is contained in precisely λ blocks of $\mathcal{B}$.

We denote such a design as a (v, k, λ)-BIBD. The necessary divisibility conditions for the existence of a (v, k, λ)-BIBD are

$$\lambda(v-1) \equiv 0 \pmod{k-1} \text{ and} \tag{1.1}$$
$$\lambda v(v-1) \equiv 0 \pmod{k(k-1)}. \tag{1.2}$$

A *Steiner triple system* of order v, denoted by STS(v), is a $(v, 3, 1)$-BIBD. It is well known that STS(v) exist for all $v \equiv 1$ or $3 \pmod 6$, [11].

A (v, k, λ)-BIBD $\mathcal{D} = (V, \mathcal{B})$ is called *resolvable* if its block collection $\mathcal{B}$ can be partitioned into classes $R_1, R_2, \ldots, R_r$ (resolution classes) where $r = \frac{\lambda(v-1)}{k-1}$ such that each element of V is contained in precisely one block of each class. The resolution

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classes $R_1, R_2, \ldots, R_r$, sometimes also called parallel classes, form a *resolution* of $\mathcal{D}$. Sometimes the notation (v, k, λ)-RBIBD is used for a resolvable BIBD. Necessary conditions for the existence of a (v, k, λ)-RBIBD are (1.1) and $v \equiv 0 \pmod{k}$. Together, these imply (1.2).

A $(v, 3, 1)$-RBIBD or resolvable STS(v) is called a *Kirkman triple system* and denoted KTS(v). Note that in this case, there are $|\mathcal{B}| = v(v-1)/6$ blocks and $r = (v-1)/2$ resolution classes where each class contains $v/3$ blocks. The necessary conditions amount to $v \equiv 3 \pmod{6}$. The existence question for KTS(v) was a celebrated open problem for over 100 years, and known as Kirkman's schoolgirl problem. In 1971 [16], Ray-Chaudhuri and Wilson proved that a KTS(v) exists if and only if $v \equiv 3 \pmod{6}$. (An earlier and independent proof by Lu [15] was discovered several years later.)

A (v, k, λ)-BIBD $\mathcal{D} = (V, \mathcal{B})$ is said to be *near resolvable* if $\mathcal{B}$ can be partitioned into classes (which we also call resolution classes) such that for each element $x \in V$, there is precisely one class which does not contain x in any of its blocks and each class contains exactly $v-1$ distinct elements. The number of resolution classes in the near resolvable setting equals $\lambda v/(k-1)$. It follows from the above that necessary conditions are $v \equiv 1 \pmod{k}$ and $\lambda = k-1$. For block size 3, Hanani [10] established the existence of near resolvable $(v, 3, 2)$-BIBDs for all $v \equiv 1 \pmod{3}$.

Suppose $(V, \mathcal{B})$ is an STS(v). We can easily construct a $(v, 3, 2)$-BIBD on V by taking two copies of each block in $\mathcal{B}$. The resulting block collection can be denoted as $2\mathcal{B}$. We call such a design a *duplicated* STS(v) and denote it by DSTS(v). In a recent paper [5], near resolvable DSTS(v) are constructed for all $v \equiv 1 \pmod{6}$, $v \geq 19$; the cases $v \in \{7, 13\}$ were shown to be definite exceptions. Most of the direct constructions in [5] produce designs with an additional interesting property, which we examine here in more detail.

Consider a resolvable or near resolvable DSTS(v), for $v \equiv 3 \pmod{6}$ or $1 \pmod{6}$, respectively. If $|R_i \cap R_j| \leq 1$ for all $i \neq j$, then the design is called a *self-orthogonal* DSTS(v). Note that, in this case, we do not distinguish between copies of the same block. We caution that there are other uses of the term 'orthogonal' in the context of Steiner triple systems and other designs, and these can have different meanings; see for example [6, 8]. The notion of self-orthogonality in [5] and in what follows here is a property of resolutions. As such, we adopt the standard abbreviations 'R' and 'NR' for resolvability and near resolvability, respectively, and we append $*$ to denote the property that the resolutions are self-orthogonal. In [5], the NR DSTS(v) constructed for $19 \leq v \leq 85$ are all self-orthogonal. The existence of NR*DSTS(v) for all $v \equiv 1 \pmod{6}$, $v \geq 19$, was posed as an open problem. We settle this question in the affirmative with only a small number of possible exceptions for v.

Theorem 1.1 *There exists an NR*DSTS(v) for all $v \equiv 1 \pmod{6}$, $v \geq 19$, except possibly for $v \in \{115, 133, 175, 259\}$.*

There is a close connection between NR*DSTSs and other designs with orthogonal resolutions.

Let $\mathcal{R}$ and $\mathcal{R}'$ be two resolutions of the blocks of a (v,k,λ)-BIBD, D. We say that $\mathcal{R}$ and $\mathcal{R}'$ are *orthogonal* if $|R_i \cap R'_j| \leq 1$ for all $R_i \in \mathcal{R}$ and $R'_j \in \mathcal{R}'$. It should be noted that the blocks of the design are considered as labeled so that if a subset of the elements occurs as a block more than once the blocks are treated as distinct. If $\mathcal{D}$ is a (v,k,λ)-RBIBD with a pair of orthogonal resolutions, it is called *doubly resolvable* and is denoted by DR(v,k,λ)-BIBD. If D is an NR(v,k,λ)-BIBD with a pair of orthogonal near resolutions, it is called *doubly near resolvable* and is denoted by DNR(v,k,λ)-BIBD.

A great deal of work has been done on the existence of doubly resolvable and doubly near resolvable designs with block size 3. Necessary and sufficient conditions are known for the existence of DR$(v,3,2)$-BIBDs, [4, 13] and DNR$(v,3,2)$-BIBDs, [4, 12]. DR$(v,3,1)$-BIBDs are equivalent to KS$_3(v;1,1)$ (Kirkman squares). KS$_3(v;1,1)$ are shown to exist in [9] with 23 possible exceptions; 11 of these exceptions are constructed in [4], $v=351$ is constructed in [2], and $v=249$ and $v=357$ are in [1]. (The last two designs are a result of special frames constructed directly using computer searches in [1].) So we have the following updated existence result.

Theorem 1.2 *Let v be a positive integer, $v \equiv 3 \pmod 6$, $v \neq 9, 15$. There exists a $KS_3(v;1,1)$ except possibly for $v \in \{21, 141, 153, 165, 177, 189, 231, 261, 285\}$. Furthermore, there do not exist $KS_3(v;1,1)$ for $v=9$ and $v=15$.*

It is easy to use DR$(v,3,1)$-BIBDs with $v \equiv 3 \pmod 6$ to construct R*DSTS(v). This construction illustrates the connection between doubly resolvable designs and self-orthogonal designs.

Lemma 1.3 *If there exists a DR$(v,3,1)$-BIBD, there exists an R*DSTS(v).*

Proof. Let $\mathcal{R}$ and $\mathcal{R}'$ be a pair of orthogonal resolutions for a DR$(v,3,1)$-BIBD. Then $\mathcal{R} \cup \mathcal{R}'$ is a set of $2r = v-1$ resolution classes for an R*DSTS(v). □

This immediately gives us the following existence result for R*DSTS(v).

Lemma 1.4 *There exists an R*DSTS(v) for all $v \equiv 3 \pmod 6$, $v \geq 21$ except possibly for $v \in \{21, 141, 153, 165, 177, 189, 231, 261, 285\}$.*

Our constructions for NR*DSTS(v) for $v \equiv 1 \pmod 6$ use many of the techniques and ideas that were used in [9]. In Section 2, we describe direct constructions for self-orthogonal DSTS based on finite fields. This is similar to the methods in [18], except that the need for duplicated STS requires extra conditions in selecting field elements. Our direct constructions simplify and unify many of the constructions in [5] into an algebraic framework. In Section 3, we provide definitions and existence results for $(1,1;3)$-frames and describe our main recursive construction using these frames to construct self-orthogonal near resolvable DSTS. In the following section, the existence of NR*DSTS(v) is established for $v \equiv 1 \pmod 6$ with at present four possible exceptions. Finally, in Section 5, we summarize our results and discuss generalizations for larger block sizes.

2 Direct constructions

2.1 Cyclotomic methods

For a prime power q, let $\mathbb{F}_q$ denote the finite field of order q and let $\mathbb{F}_q^\times$ denote its multiplicative group. Given a subgroup $G \le \mathbb{F}_q^\times$ of even order, a *halfset* of G is a subset $H \subset G$ such that $|H| = |G|/2$ and $G = \pm H := H \cup (-H)$.

Let $q \equiv 7 \pmod{12}$ be a prime power, and let $C_0, \ldots, C_5$ be the cosets of index 6 in $\mathbb{F}_q^\times$. If ω is a generator of $\mathbb{F}_q^\times$, we have $C_j = \{\omega^{6i+j} : 0 \le i < (q-1)/6\}$ for each $j = 0, 1, \ldots, 5$. Since $q \equiv 7 \pmod{12}$, these cosets have odd size, and it follows that $H = C_0$ is a halfset of the subgroup $G = C_0 \cup C_3$ of index 3 in $\mathbb{F}_q^\times$.

Suppose t, x, y, z are elements of $\mathbb{F}_q^\times$ having the following properties:

(1) the differences $x - y, y - z, z - x$ are all nonzero and belong to distinct cosets of G; and

(2) the elements $x, y, z, x + t, y + t, z + t$ are all nonzero and belong to distinct cosets of H.

Put $\mathcal{B} = \{\{a + hx, a + hy, a + hz\} : h \in H, a \in \mathbb{F}_q\}$. We claim that $(\mathbb{F}_q, 2\mathcal{B})$ is an NR*DSTS(q). From condition (1), it readily follows that $\mathcal{B}$ is the block collection of an STS(q) on $\mathbb{F}_q$. We are interested in the DSTS(q) with block collection $2\mathcal{B}$. From condition (2), the two blocks $\{x, y, z\}$, $\{x + t, y + t, z + t\}$ of $2\mathcal{B}$ develop under multiplication by H into a partition R of $\mathbb{F}_q^\times$. The set of q translates of R in $\mathbb{F}_q$ thus provide a near resolution of $2\mathcal{B}$. A block B belongs to two distinct translates of R only if $a_1 + B$ and $a_2 + B$ are both in R for distinct $a_1, a_2 \in \mathbb{F}_q$. Without loss of generality, we can take $a_1 = 0$, $B = \{hx, hy, hz\}$, and $a_2 = ht$ for some $h \in H$. This uniquely determines B and shows that the self-orthogonality property holds.

We illustrate the construction in detail for $q = 103$, which is a value that [5] settles for NRDSTS but not for NR*DSTS.

Example 2.1 Let $q = 103$. The element -3 has order 17 in $\mathbb{F}_q^\times$, and hence generates C_0. If we take $\{x, y, z\} = \{1, 4, 6\}$ and $t = 20$, conditions (1) and (2) above can be easily verified. So, this produces an NR*DSTS(103). Table 1 gives the base blocks, with each row having two blocks in the same cyclic orbit. The translates between these are indicated in the last column of the table.

The preceding construction can, in some cases, be modified to work when $q \equiv 1 \pmod{12}$. Let $q = 6 \times 2^e t + 1$, where t is odd and e is a nonnegative integer. Let G and H be the subgroups of $\mathbb{F}_q^\times$ of index 3×2^e and 6×2^e, respectively. We again have H being a halfset of G. This time, however, we need 2^e distinct quadruples (x_i, y_i, z_i, t_i), $i = 1, \ldots, 2^e$, such that

(1) the differences $\{x_i - y_i, y_i - z_i, z_i - x_i : i = 1, \ldots, 2^e\}$ belong to distinct cosets of G;

(2) the elements $\{x_i, y_i, z_i, x_i + t_i, y_i + t_i, z_i + t_i : i = 1, \ldots, 2^e\}$ belong to distinct cosets of H; and

(3) the translates $\{t_i : i = 1, \ldots, 2^e\}$ belong to distinct cosets of G.

Table 1: Base blocks and translates for an NR*DSTS(103).

base blocks		±translates
$\{1,4,6\}$	$\{21,24,26\}$	$20,83$
$\{100,91,85\}$	$\{40,31,25\}$	$43,60$
$\{9,36,54\}$	$\{86,10,28\}$	$77,26$
$\{76,98,44\}$	$\{51,73,19\}$	$78,25$
$\{81,15,74\}$	$\{53,90,46\}$	$75,28$
$\{66,58,87\}$	$\{47,39,68\}$	$84,19$
$\{8,32,48\}$	$\{65,89,2\}$	$57,46$
$\{79,7,62\}$	$\{11,42,97\}$	$35,68$
$\{72,82,20\}$	$\{70,80,18\}$	$101,2$
$\{93,63,43\}$	$\{99,69,49\}$	$6,97$
$\{30,17,77\}$	$\{12,102,59\}$	$85,18$
$\{13,52,78\}$	$\{67,3,29\}$	$54,49$
$\{64,50,75\}$	$\{5,94,16\}$	$44,59$
$\{14,56,84\}$	$\{88,27,55\}$	$74,29$
$\{61,38,57\}$	$\{45,22,41\}$	$87,16$
$\{23,92,35\}$	$\{71,37,83\}$	$48,55$
$\{34,33,101\}$	$\{96,95,60\}$	$62,41$

Similar to before, put $\mathcal{B} = \{\{a+hx_i, a+hy_i, a+hz_i\} : h \in H, a \in \mathbb{F}_q, i = 1,\dots,2^e\}$. By condition (3), the values $\pm ht_i$, $h \in H$, $i = 1,\dots,2^e$ are distinct and we again obtain an NR*DSTS(q).

The following result summarizes our application of the above cyclotomic methods. Table 2 gives more details, including generator elements, starter blocks, and translates we found for each value of $q = 6x + 1$. Our results are limited to what is necessary for the forthcoming proof of the main result, Theorem 1.1, but the methods work well for values smaller and even larger than the given range.

Lemma 2.2 *There exist NR*DSTS*$(6x+1)$ *for all* $x \in \{17, 18, 21, 23, 25, 26, 27, 30, 33, 37, 38, 46, 47, 51, 58\}$.

The preceding constructions are less likely to succeed when the largest odd divisor of $(q-1)/3$ is small. In fact, when $q = 6 \times 2^e + 1$ for some e, there does not appear to be a systematic way to choose a halfset of G to carry out the construction in general. In spite of this, we were able to find a construction in one of these cases by searching over different initial blocks and halfsets, getting lucky that one such choice admitted translates for self-orthogonality.

Example 2.3 Let $q = 97$, and use the generator $\omega = 5$ of $\mathbb{F}_q$. With the help of a computer, we found a block $B = \{1, 13, 17\}$ and ordered halfset of G, namely

$$h = (1, 28, 8, 67, 33, 46, 70, 77, 22, 63, 18, 78, 50, 55, 12, 52).$$

Table 2: Base blocks and translates for an NR*DSTS(103).

x	q	2^e	ω	starter block(s)	translate(s)
17	103	1	5	$\{1,4,6\}$	20
18	109	2	6	$\{22,48,56\}$	104
				$\{27,62,86\}$	67
21	127	1	3	$\{1,9,12\}$	50
23	139	1	2	$\{1,2,4\}$	21
25	151	1	6	$\{1,5,10\}$	2
26	157	2	5	$\{29,127,151\}$	132
				$\{16,68,107\}$	61
27	163	1	2	$\{1,2,4\}$	7
30	181	2	2	$\{27,107,153\}$	125
				$\{69,143,161\}$	128
33	199	1	3	$\{1,3,9\}$	41
37	223	1	3	$\{1,9,12\}$	26
38	229	2	6	$\{116,170,173\}$	94
				$\{45,114,212\}$	15
46	277	2	5	$\{28,114,152\}$	185
				$\{161,182,250\}$	77
47	283	1	3	$\{1,7,23\}$	30
51	307	1	5	$\{1,15,21\}$	7
58	349	2	2	$\{58,105,124\}$	306
				$\{118,138,333\}$	170

The resulting cyclic STS(q) with base blocks $\{h_iB : i = 1,\dots,16\}$ admits translates

$$t = (41, 93, 79, 55, 48, 45, 74, 54, 34, 53, 20, 1, 25, 35, 70, 12)$$

such that $\cup_{i=1}^{16}\{h_iB, t_i + h_iB\}$ is a starter for an NR*DSTS(97).

2.2 Other constructions

Here, we provide two constructions of NR*DSTS(v) for non-prime v, namely for $v \in \{91, 121\}$. The latter value is a prime power, but in this case the cyclotomic method above has the extra complications that $e = 4$ and that the additive group is not cyclic. Instead, we give cyclic constructions with multiplier automorphisms.

Example 2.4 Let $v = 91$. The element $m = 9$ satisfies $m^3 = 1$ in $\mathbb{Z}_v$. Consider the blocks

$$\{1, 42, 74\}, \{5, 36, 39\}, \{7, 33, 43\}, \{48, 59, 73\}, \{53, 75, 82\}$$

and respective translates $79, 21, 11, 13, 2$ in $\mathbb{Z}_v$. If we develop these multiplicatively under $1, m, m^2$ and then additively, the resulting near resolvable DSTS(v) has the self-orthogonality property.

Example 2.5 Let $v = 121$. The element $m = 3$ satisfies $m^5 = 1$ in $\mathbb{Z}_v$. Consider the blocks

$$\{4, 10, 103\}, \{14, 78, 118\}, \{22, 96, 104\}, \{71, 85, 106\}$$

and respective translates $49, 27, 28, 98$ in $\mathbb{Z}_v$. If we develop these multiplicatively under $1, m, m^2, m^3$, m^4 and then additively, the resulting near resolvable DSTS(v) has the self-orthogonality property.

3 Frames

Let T denote an integer partition of v and let K be a set of positive integers. A *group divisible design* of *type* T with block sizes in K is a triple $(V, \mathcal{G}, \mathcal{B})$ which satisfies the following properties.

(1) V is a set of v points.
(2) $\mathcal{G} = \{G_1, G_2, \dots, G_m\}$ is a partition of V into groups so that $T = (|G_1|, \dots, |G_m|)$.
(3) $\mathcal{B} \subseteq \cup_{k \in K} \binom{V}{k}$ is a set of blocks meeting each group in at most one point.
(4) Every pair of elements from distinct groups occurs in precisely λ blocks. (λ is called the index.)

We denote such a design as a $GDD(v; K; G_1, G_2, \dots, G_m; 0, \lambda)$; see [17] for the more general definition of GDDs. It is convenient to use exponential notation for the type of a GDD; we say a GDD has type $t_1^{u_1} t_2^{u_2} \dots t_\ell^{u_\ell}$ if there are u_i G_j's of cardinality t_i, $1 \le i \le l$. A $GDD(v; K; G_1, \dots, G_m; 0, 1)$ is often denoted as a K-GDD or GDD(v, K) of *type* $t_1^{u_1} t_2^{u_2} \dots t_\ell^{u_\ell}$. A GDD with a single block size k is often denoted simply as a k-GDD or a GDD(v, k).

We will use two special types of GDDs with index $\lambda = 1$ in our recursive constructions. A *pairwise balanced block design* or PBD, denoted by PBD(v, K), is a GDD(v, K) of type 1^v. A *transversal design* TD(k, n) is a GDD which has block size k and precisely k groups of size n, a k-GDD of type n^k. In this case, the blocks are transversals of the partition. It is well known that a TD(k, n) is equivalent to a set of $k - 2$ mutually orthogonal latin squares (MOLS) of order n. We refer to [3] for results on the existence of transversal designs.

Let V be a set of v elements. Let $G_1, G_2, \dots, G_m$ be a partition of V into m sets. A $\{G_1, G_2, \dots, G_m\}$-*frame* F with block size k, index λ, and *latinicity* μ is a square array of side length $t = \lambda v/(\mu(k-1))$ which satisfies the properties listed below. Let $t_i = \lambda |G_i|/(\mu(k-1))$, let $g_k = \sum_{i=1}^{k} t_i$ and define $g_0 = 0$. We index the rows and columns of F with the elements $0, 1, \dots, \lambda v/(\mu(k-1)) - 1$.

(1) Each cell is either empty or contains a k-subset of V.
(2) Let F_i be the subsquare of F indexed by $g_{i-1}, g_{i-1}+1, \dots, g_i - 1$. F_i is empty for $i = 1, 2, \dots, m$. That is, the main diagonal of F consists of empty subsquares of sides $t_i \times t_i$ for $i = 1, 2, \dots, m$.

(3) Let $x \in \{g_{i-1}, g_{i-1}+1, \ldots, g_i - 1\}$. Row x of F contains each element of $V - G_i$ μ times and column x of F contains each element of $V - G_i$ μ times.

(4) The blocks obtained from the nonempty cells of F form a GDD$(v; k; G_1, G_2, \ldots, G_m; 0, \lambda)$.

The type of a $\{G_1, G_2, \ldots, G_m\}$-frame is the type of the underlying GDD, $\{|G_1|, |G_2|, \ldots, |G_m|\}$. So, we also use exponential notation to describe the type of a frame. A$\{G_1, G_2, \ldots, G_m\}$-frame with block size k, index λ, and latinicity μ is denoted as a $(\mu, \lambda; k, \{G_1, \ldots, G_m\})$-frame or a $(\mu, \lambda; k)$-frame of type $t_1^{u_1} t_2^{u_2} \ldots t_\ell^{u_\ell}$ if there are u_i G_j's of cardinality t_i, $1 \leq i \leq \ell$. If $|G_i| = h$ for all i, the frame F is often called a $(\mu, \lambda; k, m, h)$-frame.

Our main recursive construction uses $(1,1;3)$-frames to construct self-orthogonal near resolvable DSTSs.

Theorem 3.1 *Suppose there exists a $(1,1;3)$-frame with group partition $\{G_1, G_2, \ldots, G_m\}$. If there exists an NR*DSTS$(|G_i|+1)$ for all $i = 1, \ldots, m$ then there exists an NR*DSTS$(\sum_{i=1}^{m} |G_i| + 1)$.*

PROOF. Let $V = \cup_{i=1}^{m} G_i$. We construct an NR*DSTS$(v+1)$ on $V \cup \{\infty\}$ where $|V| = v$. Let D_i denote an NR*DSTS$(|G_i|+1)$ defined on $G_i \cup \{\infty\}$ with $|G_i| = t_i$. D_i has $t_i + 1$ resolution classes, $P_{i,\infty}, P_{i,1}, P_{i,2}, \ldots, P_{i,t_i}$ where $P_{i,\infty}$ denotes the resolution class missing the element ∞.

Suppose F is a $(1,1;3,\{G_1, G_2, \ldots, G_m\})$-frame. F has $\frac{t_i}{2}$ rows, $R_{i,j}$ for $j = 1, 2, \ldots, \frac{t_i}{2}$, which are missing the elements of G_i; that is, every element of $V \setminus G_i$ occurs once in each of these rows. F also has $\frac{t_i}{2}$ columns, $C_{i,j}$ for $j = 1, 2, \ldots, \frac{t_i}{2}$, which are missing the elements of G_i. Adjoin a resolution class from D_i to each of these rows and to each of these columns. The resulting (near) resolution classes, $\mathcal{P}_{i,j}$, $j = 1, 2, \ldots, t_i$, are each missing precisely one element of G_i.

$$\begin{array}{lcll} \mathcal{P}_{i,j} & = & R_{i,j} \cup P_{i,j} & \text{for} \quad 1 \leq j \leq \frac{t_i}{2}, \\ \mathcal{P}_{i,\frac{t_i}{2}+j} & = & C_{i,j} \cup P_{i.\frac{t_i}{2}+j} & \text{for} \quad 1 \leq j \leq \frac{t_i}{2}. \end{array}$$

Let $\mathcal{P}_\infty = \cup_{i=1}^{m} P_{i,\infty}$; every element of V occurs once in this class. Then $\mathcal{P}_\infty$ together with $\mathcal{P}_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, t_i$ is a set of $1 + \sum_{i=1}^{m} t_i$ (near) resolution classes.

First, note that by taking the rows and the columns of the frame F, we have used two copies of the underlying GDD with block size 3 and $\lambda = 1$. Since we are filling in the groups, G_i, of the GDD with DSTS defined on $G_i \cup \{\infty\}$, the resulting design is a DSTS$(v+1)$.

Next we check that the $v+1$ resolution classes are self-orthogonal. Observe that $C_{i,j} \cap R_{i,k} = \emptyset$ for $1 \leq j, k \leq t_i$. (The rows and columns in F which are missing the group G_i are disjoint.) Since D_i is an NR*DSTS, $|\mathcal{P}_{i,j} \cap \mathcal{P}_{i,k}| \leq 1$ for $j \neq k$ and $|\mathcal{P}_{i,j} \cap \mathcal{P}_\infty| \leq 1$. Since F is a $(1,1;3)$-frame, $|\mathcal{P}_{i,j} \cap \mathcal{P}_{\ell,k}| \leq 1$ for $i \neq \ell$ and $j = 1, \ldots, t_i$, $k = 1, \ldots, t_\ell$. Therefore, we have constructed an NR*DSTS$(v+1)$ on $V \cup \{\infty\}$. □

In order to apply this result, we need some existence results for $(1,1;3)$-frames. The following two standard recursive constructions for frames, the singular direct product and the Fundamental Construction, will be useful.

Theorem 3.2 **([9])** *If there exists a $(1,1;3,\{G_1,G_2,\ldots,G_m\})$-frame and a set of three mutually orthogonal Latin squares of side n, then there exists a $(1,1;3)$-frame of type $\{n|G_i| \mid i=1,2,\ldots,m\}$.*

Theorem 3.3 **([9])** *Let G be a GDD$(v;K;G_1,G_2,\ldots,G_m;0,1)$. Suppose there exists a function $w : V \rightarrow \mathbb{Z}^+ \cup \{0\}$ (a weight function) which has the property that for each block $b=\{x_1,x_2,\ldots,x_k\} \in \mathcal{B}$, there exists a $(1,1;3)$-frame of type $\{w(x_1),w(x_2),\ldots,w(x_k)\}$. Then there exists a $(1,1;3)$-frame of type*

$$\left\{\sum_{x\in G_1} w(x), \sum_{x\in G_2} w(x), \ldots, \sum_{x\in G_m} w(x)\right\}.$$

The connection between Kirkman squares, $\mathrm{KS}_3(v+1;1,1)$, and $(1,1;3)$-frames of type $2^{\frac{v}{2}}$ is described in [9, Lemma 4.4]. The first result for $(1,1;3)$-frames is a consequence of Theorem 1.2 on the existence of Kirkman squares (DR$(v,3,1)$-BIBDs).

Theorem 3.4 *There exist $(1,1;3)$-frames of type 2^u for $u \equiv 1 \pmod 3$ and $u \geq 10$ except possibly for $u \in \{10,70,76,82,88,94,115,130,142\}$.*

Theorem 3.5 *There exist $(1,1;3)$-frames of type 6^u for $7 \leq u \leq 19$, $u \in \{31,49,50,56,57,58\}$, and $u \geq 63$.*

Proof. We update the existence result in [2]. $(1,1;3)$-frames of type 6^u have been constructed for $u=15,16,17,18$ in [1]. □

Several small $(1,1;3)$-frames will also be used in Section 4.

Lemma 3.6 *There exist $(1,1;3)$-frames of the following types:*

(1) [7] 4^7; *and*
(2) [1] 18^8, $24^7 18^1$, $24^7 36^1$, $30^7 36^1$.

4 Existence

In this section, we construct self-orthogonal near resolvable DSTS(v) for $v=6x+1$ for $x \geq 3$ with at present four possible exceptions for x. Our main recursive construction uses group divisible designs.

Theorem 4.1 *Let G be a GDD$(v; K; G_1, G_2, \ldots, G_m; 0, 1)$. Suppose there exist $(1, 1; 3)$-frames of type 6^k for each $k \in K$ and an NR*DSTS$(6|G_i| + 1)$ for each i, $i = 1, 2, \ldots, m$. Then there exists an NR*DSTS$(6v + 1)$.*

Proof. We use the Fundamental Construction for frames, Theorem 3.3 with $w(x) = 6$ for all x and then apply the Basic Frame Construction, Theorem 3.1. □

We first use PBDs to construct group divisible designs and then apply Theorem 4.1. This will be our main construction for v large.

Theorem 4.2 *If there exists a PBD$(x + 1; \{7, 8, 9\})$, then there exists an NR*DSTS$(6x + 1)$.*

Proof. We delete one element of a PBD$(x+1; \{7, 8, 9\})$ to construct a $\{7, 8, 9\}$-GDD with $|G_i| \in \{6, 7, 8\}$ for all i. By Theorem 3.5, there exist $(1, 1; 3)$-frames of types 6^k for $k \in \{7, 8, 9\}$, and NR*DSTS$(6m + 1)$ are constructed in [5] for $m = 6, 7, 8$. So, we can apply Theorem 4.1 to construct NR*DSTS$(6x + 1)$. □

Truncated transversal designs can also be used to construct group divisible designs. The next result describes several types of group divisible designs that are useful for the existence of NR*DSTSs. This is [9, Lemma 5.3].

Lemma 4.3 ([9], Truncation of Transversal Designs)

(1) *If there exists a TD$(8, n)$, then there exists a $\{7, 8\}$-GDD of type $n^7 w^1$ where w is an integer, $0 \le w \le n$.*
(2) *If there exists a TD$(9, n)$, then there exists a $\{7, 8, 9\}$-GDD of type $n^7 w^1 y^1$ where w and y are integers, $0 \le w, y \le n$.*
(3) *If there exists a TD$(10, n)$, then there exists a $\{7, 8, 9, 10\}$-GDD of type $(n - 1)^8 w^1 y^1$ where w and y are integers, $0 \le w, y \le n - 1$.*
(4) *There exists a $\{7, 8, 9, w, 19\}$-GDD of type $7^{19-w} 8^w y^1$ whenever $0 \le w \le 19$ and $0 \le y \le 18$.*

In order to apply Theorem 4.2, we need the existence of PBD$(v; \{7, 8, 9\})$s. Let $E_{789} = [10, 48] \cup [51, 55] \cup [59, 62]$ and $X_{789} = [93, 111] \cup [116, 118] \cup \{132\} \cup [138, 168] \cup [170, 174] \cup [180, 216] \cup [219, 223] \cup [228, 230] \cup [242, 258] \cup [261, 279] \cup [283, 286] \cup [298, 300] \cup [303, 307] \cup [311, 335] \cup [339, 342]$ where the notation $[x, y]$ is used to denote the set of integers no smaller than x and no larger than y. We use the existence result, [9, Theorem 5.6]; it updates the existence result in [14]

Theorem 4.4 ([9]) *For any integer $v \ge 10$, there is a PBD$(v; \{7, 8, 9\})$ except possibly when v is in X_{789}, and definitely when v is in E_{789}.*

One other recursive construction is useful for several small cases. It follows immediately from Theorem 3.2 and Theorem 3.1.

Theorem 4.5 *If there exists a $(1,1;3)$-frame of type t^m, three mutually orthogonal Latin squares of order n, and an NR* DSTS$(tn+1)$, then there exists an NR* DSTS$(tmn+1)$.*

We are now in a position to prove our main result on the existence of NR*DSTS(v) for $v \equiv 1 \pmod 6$. We start by constructing designs for small v; recall that these do not exist [5] for $v \in \{7, 13\}$.

Lemma 4.6 *There exist NR* DSTS(v) for all $v = 6x+1$, $3 \leq x \leq 52$, except possibly for $x \in \{19, 22, 29, 43\}$.*

PROOF. NR*DSTS$(6x+1)$ for $3 \leq x \leq 14$ are given in [5]. We use Lemma 2.2 for several cases with $15 \leq x \leq 52$. Constructions for the remaining cases are described in Table 3 in the Appendix. □

PROOF OF THEOREM 1.1. Lemma 4.6 establishes the existence of NR*DSTS$(6x+1)$ for $3 \leq x \leq 52$ and $x \notin \{19, 22, 29, 43\}$. For $53 \leq x \leq 369$, all but one of these values is done using the group divisible design construction, Theorem 4.1. These constructions are described in detail in Table 4 in the Appendix. In each case, we make use of the existence of the small designs constructed in Lemma 4.6 and the existence of the necessary frames of type 6^k, Theorem 3.5.

By Theorems 4.2 and 4.4, there exist PBD$(x+1; \{7, 8, 9\})$ and hence NR*DSTS$(6x+1)$ for all $x \geq 342$. □

5 Concluding Remarks

We have shown that there exist self-orthogonal near resolvable DSTS(v) for all $v \equiv 1 \pmod 6$, $v \geq 19$, with four possible exceptions. The largest exception could be constructed using a $(1,1;3)$-frame of type $18^{13}24^1$ if it were found. The remaining values are non-prime-powers divisible by either 5 or 7, and may require new ideas to obtain a construction.

One could consider a generalization of this work to constructing near resolvable $(v, k, k-1)$-BIBDs via $k-1$ copies of a $(v, k, 1)$-BIBD with $v \equiv 1 \pmod{k(k-1)}$. Let $(V, \mathcal{B})$ be a $(v, k, 1)$-BIBD with $v \equiv 1 \pmod{k(k-1)}$. We denote a self-orthogonal NR$(v, k, k-1)$-BIBD defined on V with block collection $(k-1)\mathcal{B}$ by NR* $(v, k, k-1)$-BIBD. The direct cyclotomic construction of Section 2 generalizes to larger block size and can be used to find examples. Here is an example of a NR* $(v, 4, 3)$-BIBD obtained as three disjoint translates of a cyclic $(v, 4, 1)$-BIBD.

Example 5.1 Let $q = 37$ and consider the cosets $H = C_0, \ldots, C_{11}$ of index 12 in $\mathbb{F}_q^\times$. The element $\omega = 2$ is a generator, so we have $H = \{2^0, 2^{12}, 2^{24}\} = \{1, 26, 10\}$. The starter block $B = \{1, 4, 6, 15\}$ has its 12 ordered differences in distinct cosets, and moreover $B \cup (10+B) \cup (18+B)$ intersects each coset C_j exactly once. Letting

$\mathcal{B} = \{a + hB : h \in H, a \in \mathbb{F}_q\}$, we obtain blocks of a $(37, 4, 1)$-BIBD. The blocks $\{hB, h(B + 10), h(B + 18) : h \in H\}$ provide a starter for a near resolvable $(37, 4, 3)$-BIBD. Since ± 10, ± 18 (the two translates), and ± 8 (their difference) are in distinct cosets, the tripled block collection $3\mathcal{B}$ is an NR* $(37, 4, 3)$-BIBD.

Computer searches found instances of Example 5.1 for $q = 109$ and $q = 157$ but, curiously, not for $q = 61$.

NR* $(v, k, k-1)$-BIBDs are PBD-closed, [19]. To see this, we construct a $v \times v$ array from an NR* $(v, k, k-1)$-BIBD as follows. Index the rows and columns of the array with the (near) resolution classes $R_1, R_2, \ldots, R_v$ ordered so that R_i is missing element i of V. For $i \neq j$, place the block from $R_i \cap R_j$ in cell (i, j). If $R_i \cap R_j = \emptyset$, the cell is left empty. The diagonal is also left empty. The resulting square array displays the resolution classes in the columns and in the rows.

Lemma 5.2 *If there exists a PBD(v, M) and an NR* $(m, k, k-1)$-BIBD for each $m \in M$, then there exists an NR* $(v, k, k-1)$-BIBD.*

Proof. Construct a $v \times v$ array by replacing each block of size m with the $m \times m$ array obtained from an NR* $(m, k, k-1)$-BIBD. It is straightforward to check that the resulting array displays a set of self-orthogonal resolutions for a NR* $(v, k, k-1)$-BIBD. □

Although PBDs are not useful for explicit constructions here due to the large block sizes required, Lemma 5.2 can be used to establish asymptotic existence results. For example, let $M = \{37, 157\}$. There exist PBD(v, M) for all sufficiently large $v \equiv 1 \pmod{12}$, [19]. Since NR* $(m, 4, 3)$-BIBDs exist for each $m \in M$, there exist NR* $(v, 4, 3)$-BIBDs for all sufficiently large $v \equiv 1 \pmod{12}$.

For general values of k, the existence of NR* $(v, k, k-1)$-BIBDs is an interesting topic left for future work.

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Appendix

Two tables are given for the construction of small designs needed for our proof.

Table 3: Constructions for NR*DSTS($6x+1$) for $3 \leq x \leq 52$

x	$6x+1$	Construction	Parameters
$3, 4, \dots, 14$	$19, 25, \dots, 85$	[5]	Direct
15	91	2.4	Direct
16	97	2.3	Direct
17	103	2.2	Direct
18	109	2.2	Direct
19	115	??	
20	121	2.5	Direct
21	127	2.2	Direct
22	133	??	
23	139	2.2	Direct
24	145	3.1	18^8-frame
25	151	2.2	Direct
26	157	2.2	Direct
27	163	2.2	Direct
28	169	4.5	6^7-frame, $n=4$
29	175	??	
30	181	2.2	Direct
31	187	3.1	$24^7 18^1$-frame
32	193	4.5	6^8-frame, $n=4$
33	199	2.2	Direct
34	205	3.1	$24^7 36^1$-frame
35	211	4.5	6^7-frame, $n=5$
36	217	4.5	6^9-frame, $n=4$
37	223	2.2	Direct
38	229	2.2	Direct
39	235	4.5	2^{13}-frame, $n=9$
40	241	4.5	6^{10}-frame, $n=4$
41	247	3.1	$30^7 36^1$-frame
42	253	4.5	4^7-frame, $n=9$
43	259	??	
44	265	4.5	6^{11}-frame, $n=4$
45	271	4.5	6^9-frame, $n=5$
46	277	2.2	Direct
47	283	2.2	Direct
48	289	4.5	6^{12}-frame, $n=4$
49	295	4.5	6^7-frame, $n=7$
50	301	4.5	6^{10}-frame, $n=5$
51	307	2.2	Direct
52	313	4.5	6^{13}-frame, $n=4$

Table 4: Constructions for NR*DSTS($6x+1$) for $53 \le x \le 369$

x	Construction	Parameters	Ingredient design
$53 \le x \le 56$	4.1, 4.3(1)	$n=7,\ 4 \le w \le 7$	$\{7,8\}$-GDD of type 7^7w^1
57	4.2, 4.4		PBD($x+1;\{7,8,9\}$)
58	2.2	Direct	
$59 \le x \le 61$	4.1, 4.3(2)	$n=8,\ w=0,\ 3 \le y \le 5$	$\{7,8,9\}$-GDD of type $8^7w^1y^1$
$62 \le x \le 91$	4.2, 4.4		PBD($x+1;\{7,8,9\}$)
$83 \le x \le 100$	4.1, 4.3(3)	$n=11,\ 3 \le w,y \le 10$	$\{7,8,9,10\}$-GDD of type $10^8w^1y^1$
$99 \le x \le 120$	4.1, 4.3(3)	$n=13,\ 3 \le w,y \le 12$	$\{7,8,9,10\}$-GDD of type $12^8w^1y^1$
$118 \le x \le 130$	4.2, 4.4		PBD($x+1;\{7,8,9\}$)
131	4.1	[9, Lemma 5.4]	$\{7,8,9\}$-GDD of type $8^{14}12^17^1$
$132 \le x \le 136$	4.2, 4.4		PBD($x+1;\{7,8,9\}$)
$137 \le x \le 148$	4.1, 4.3(4)	$w=0,\ 4 \le y \le 15$	$\{7,8,9,19\}$-GDD of type $7^{19}y^1$
$144 \le x \le 155$	4.1, 4.3(4)	$w=7,\ 4 \le y \le 15$	$\{7,8,9,19\}$-GDD of type $7^{12}8^7y^1$
$150 \le x \le 161$	4.1, 4.3(4)	$w=13,\ 4 \le y \le 15$	$\{7,8,9,13,19\}$-GDD of type $7^68^{13}y^1$
162	4.1, 4.3(4)	$w=15,\ y=14$	$\{7,8,9,15,19\}$-GDD of type $7^48^{15}y^1$
163	4.1, 4.3(4)	$w=15,\ y=15$	$\{7,8,9,15,19\}$-GDD of type $7^48^{15}y^1$
$164 \le x \le 205$	4.1, 4.3(2)	$n=23,\ 3 \le w+y \le 44$ $w,y \in \{0,3,\ldots,23\} \setminus \{19,22\}$	$\{7,8,9\}$-GDD of type $23^7w^1y^1$
$178 \le x \le 225$	4.1, 4.3(2)	$n=25,\ 3 \le w+y \le 50$ $w,y \in \{0,3,\ldots 25\} \setminus \{19,22\}$	$\{7,8,9\}$-GDD of type $25^7w^1y^1$
$220 \le x \le 279$	4.1, 4.3(2)	$n=31,\ 3 \le w+y \le 62$ $w,y \in \{0,3,\ldots,31\} \setminus \{19,22,29\}$	$\{7,8,9\}$-GDD of type $31^7w^1y^1$
$262 \le x \le 333$	4.1, 4.3(2)	$n=37,\ 3 \le w+y \le 74$ $w,y \in \{0,3,\ldots,37\} \setminus \{19,22,29\}$	$\{7,8,9\}$-GDD of type $37^7w^1y^1$
$290 \le x \le 369$	4.1, 4.3(2)	$n=41,\ 3 \le w+y \le 82$ $w,y \in \{0,3,\ldots,41\} \setminus \{19,22,29\}$	$\{7,8,9\}$-GDD of type $41^7w^1y^1$

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