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Peaks and valleys of nonintersecting Dyck paths

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Abstract

Deutsch proved that the number of high peaks and the number of valleys are equidistributed over Dyck paths. The proof is by constructing a bijection on Dyck paths. In this paper, we consider the high peaks and valleys over k-tuples of nonintersecting Dyck paths for any positive integer k. We establish a bijection over k-tuples of nonintersecting Dyck paths, exchanging the number of high peaks and the number of valleys. This leads to the equidistribution of the number of high peaks and the number of valleys over k-tuples of nonintersecting Dyck paths. When $k = 1$, our result applies to the equidistribution over Dyck paths due to Deutsch. In this case, our bijection differs from the bijection of Deutsch.

1 Introduction

Dyck paths are classical objects in combinatorics, which are counted by the Catalan numbers. See, for example, the book of Stanley [25] for a thorough introduction. Dyck paths have received much attention in recent years; see for example [6,8,13,21].

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In this paper, we are concerned with high peaks and valleys of Dyck paths. By constructing a bijection $\Gamma : D_n \to D_n$, where D_n denotes a Dyck path of length n, Deutsch [7] proved that the number of high peaks and the number of valleys are equidistributed over Dyck paths. The main purpose of this paper is to generalize this result to k-tuples of nonintersecting Dyck paths for any positive integer k. This is achieved by establishing a bijection $\Omega : \mathrm{Path}(\lambda, k) \to \mathrm{Path}(\lambda, k)$, where $\mathrm{Path}(\lambda, k)$ denotes the set of k-tuples of nonintersecting paths in λ. By this operation, we find that the high peaks and valleys of k-tuples of nonintersecting Dyck paths have the same distribution. The construction of the bijection rests on an intermediate structure of certain fillings of Young diagrams. An example is shown in Figure 3.11. When restricted to Dyck paths (namely, the case $k = 1$), our bijection differs from the bijection of Deutsch [7].

The number of valleys over k-tuples of nonintersecting Dyck paths has recently been studied by Hwang, Kim, Yoo and Yun [15]. Let $N_{n,k}(q)$ denote the generating polynomial of the number of valleys over k-tuples of nonintersecting Dyck paths of semilength n. Hwang, Kim, Yoo and Yun [15] found a determinantal formula for $N_{n,k}(q)$ in order to prove two conjectures posed by Morales, Pak and Panova [22, 23] in their study of the hook length formulas for skew diagrams.

Using the above determinantal formula, we show that $N_{n,k}(q)$ is a symmetric polynomial. Note that in the case $k = 1$, this is known as the symmetry of the Narayana polynomials, while in the case $k = 2$, such a symmetry property has been proved by Courtiel, Fusy, Lepoutre and Mishna [5] based on a bijection between pairs of nonintersecting Dyck paths and Schnyder woods due to Bernardi and Bonichon [1]. As an application of the symmetry of $N_{n,k}(q)$, we obtain an alternative proof of a conjecture posed by Reiner, Tenner and Yong [24] for the staircase shape.

This paper is organized as follows. In Section 2, we state the main result of this paper, whose proof is given in Section 3. In Section 4, we show that the general Narayana polynomial $N_{n,k}(q)$ is a symmetric polynomial.

2 Main result

In this section, we state the main result of this paper, that is, that the number of high peaks and the number of valleys are equidistributed over k-tuples of nonintersecting Dyck paths. As will be seen, we shall establish a stronger result by considering nonintersecting lattice paths in a Young diagram.

Let us start with a review of the definition of a Dyck path. A *Dyck path* of semilength n is a diagonal lattice path from $(0, 0)$ to $(2n, 0)$, consisting of n up-steps (along the vector $(1, 1)$) and n down-steps (along the vector $(1, -1)$), such that the path never goes below the x-axis. We will refer to n as the semilength of the path. It is well known that the Dyck paths of semilength n are counted by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. For example, there are five Dyck paths of semilength 3, as illustrated in Figure 2.1.

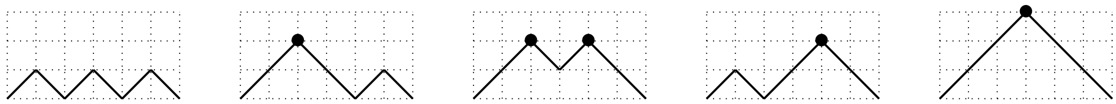


Figure 2.1: Dyck paths of semilength 3.

A joint node is called a *peak* if it is formed by an up-step followed by a down-step, and is called a *valley* if it is formed by a down-step followed by an up-step. By a *high peak* we mean a peak at a level strictly greater than 1. For example, the high peaks of the Dyck paths in Figure 2.1 are drawn with solid dots.

Two Dyck paths P_1 and P_2 are called *non-intersecting* if for every point (x_1, y_1) on P_1 and (x_1, y_2) on P_2, it holds that $y_1 \leq y_2$. A k-tuple of nonintersecting Dyck paths means there are k Dyck paths that form the nonintersecting Dyck paths. For example, both figures in Figure 2.2 are referred to as nonintersecting Dyck paths. The left one depicts a 3-tuple of nonintersecting Dyck paths of semilength 5 and the right one is called a 2-tuple of nonintersecting Dyck paths of semilength 5.

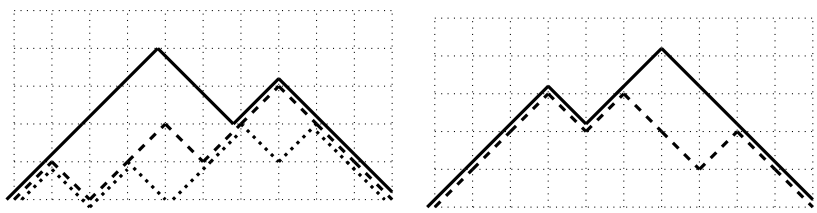


Figure 2.2: two nonintersecting Dyck paths.

Theorem 2.1 *Let n and k be positive integers. For any nonnegative integer i, the number of k-tuples of nonintersecting Dyck paths of semilength n with i high peaks is equal to the number of k-tuples of nonintersecting Dyck paths of semilength n with i valleys.*

Taking $k = 1$, Theorem 2.1 is specialized into a statement for ordinary Dyck paths, which was originally proved by Deutsch [7]. Our bijection for the $k = 1$ case is different from the bijection of Deutsch. We shall provide a stronger equidistribution by considering nonintersecting paths in a Young diagram, which specifies to Theorem 2.1 when the Young diagram is a staircase shape.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be an integer partition, that is, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$. The *Young diagram* of λ is a left-justified array of squares with λ_i squares in row i. If no confusion arises, we do not distinguish a partition and its Young diagram. A (lattice) path L in λ is a path from the bottom left point of λ to the top right point of λ with unit north steps or unit east steps. Note that the paths in λ are in one-to-one correspondence with subshapes of λ. More precisely, the southeast border of a subshape of λ determines a path in λ, and vice versa. In Figure 2.3, we illustrate a path in a Young diagram, where we use stars to indicate the corresponding subshape.

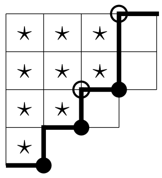


Figure 2.3: A path in $\lambda = (4, 4, 3, 1)$ and its left and right turns.

We proceed to introduce the notion of a left turn and a right turn of a path L in λ as introduced by Chan et al. [3]. A *left turn* of L is an east step followed immediately by a north step, while a *right turn* of L is a north step followed immediately by an east step with the additional condition that these two consecutive steps are entirely contained in λ. The path in Figure 2.3 has three left turns and two right turns, which are signified by solid dots and open dots, respectively.

A k-tuple $L = (L_1, L_2, \ldots, L_k)$ of paths in λ is *nonintersecting* if for $1 \leq i \leq k-1$, the subshape determined by the path L_i is contained in the subshape determined by the path L_{i+1}. Figure 2.4 illustrates a 3-tuple (L_1, L_2, L_3) of nonintersecting paths in $\lambda = (5, 4, 3, 1)$, where the paths L_1, L_2 and L_3 in λ are drawn with thick line, dashed line and dotted line respectively.

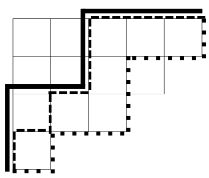


Figure 2.4: A 3-tuple of nonintersecting paths in $(5, 4, 3, 1)$.

Theorem 2.2 *Let λ be a partition, and k be a positive integer. For any nonnegative integer i, the number of k-tuples of nonintersecting paths in λ with i left turns is equal to the number of k-tuples of nonintersecting paths in λ with i right turns.*

When λ is the staircase partition $\delta_n = (n-1, n-2, \ldots, 1)$, each k-tuple of nonintersecting paths in λ naturally corresponds to a k-tuple of nonintersecting Dyck paths of semilength n. The correspondence is illustrated in Figure 2.5. Moreover,

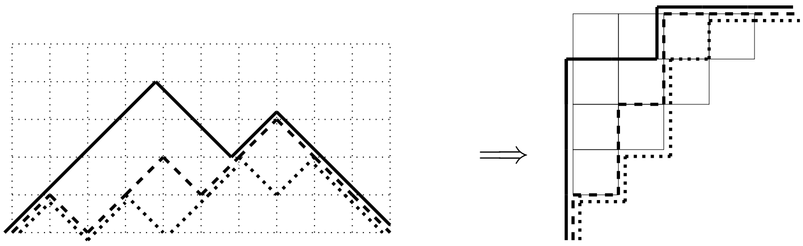


Figure 2.5: Nonintersecting lattice paths in δ_n.

by definition, it is easy to check that a left turn corresponds to a valley, and a right turn corresponds to a high peak. So, when λ is restricted to δ_n, Theorem 2.2 can be specialized into Theorem 2.1. The proof of Theorem 2.2 will be discussed in Section 3.4.

3 Proof of Theorem 2.2

To prove Theorem 2.2, we establish a bijection Ω on k-tuples of nonintersecting paths in λ, exchanging the number of left turns and the number of right turns. To do this, we construct two bijections between k-tuples of nonintersecting paths in λ and certain fillings of λ.

3.1 Fillings of Young diagrams and reverse plane partitions

A filling F of a *Young diagram* λ is an assignment of nonnegative integers into the squares of λ such that each square receives exactly one integer. Before defining the southeast chain, let us first provide a definition of weakly below and weakly to the right. We say "M" is *weakly below* to "N" if "M" is positioned at the same height or lower vertically compared to "N". We say "M" is *weakly to the right* of "N" if "M" is positioned at the same horizontal level or farther right relative to "N". A *southeast chain* (or, SE-chain) of F is a sequence of non-zero entries in the filling such that any entry in the sequence is weakly below and weakly to the right of the preceding entry in the sequence. The *length* of an SE-chain is defined as the sum of all the entries in the chain. Let $\mathrm{F}(\lambda, k)$ denote the set of fillings of λ such that the longest SE-chain has length at most k.

Figure 3.6 gives two fillings of $\lambda = (5, 3, 3, 2)$, where the longest SE-chain in the left filling has length 3, while the longest SE-chain in the right filling has length 4.

We say a sequence is *weakly increasing* if each element is greater than or equal to the one before it. A reverse plane partition of shape λ is a filling of λ such that the entries in each row and in each column are weakly increasing; see Stanley [26, Chap. 7]. Let $\mathrm{RPP}(\lambda, k)$ denote the set of reverse plane partitions of shape λ with entries not exceeding k. Let $\mathrm{Path}(\lambda, k)$ denote the set of k-tuples of nonintersecting

0	2	0	1	0
1	0	1		
1	1	0		
1	0			

0	0	0	2	0
0	3	1		
2	0	0		
1	0			

Figure 3.6: Two fillings of $\lambda = (5, 3, 3, 2)$.

paths in λ. There is a simple bijection between the set $\mathrm{Path}(\lambda, k)$ and the set $\mathrm{RPP}(\lambda, k)$. We denote this bijection by Φ, which will be sketched below, see also Kamioka [16] or Krattenthaler [18]. Given $L = (L_1, L_2, \dots, L_k) \in \mathrm{Path}(\lambda, k)$, define $P = \Phi(L)$ as follows. Let L_0 and L_{k+1} respectively denote the northwest border of λ and the southeast border of λ. Taking Figure 3.7 as an example, if $L = (L_1, L_2)$, the red line represents L_0, the black thick line represents L_1, the dashed line represents L_2 and the blue line represents L_3.

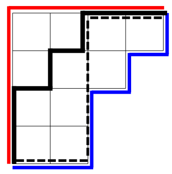
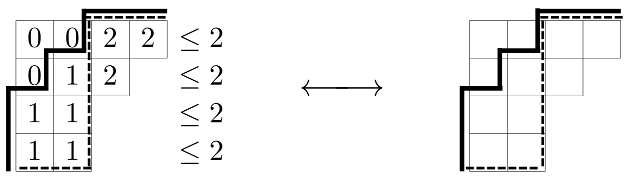


Figure 3.7: An example for $L = (L_1, L_2)$

For $0 \le i \le k$, fill each of the squares between L_i and L_{i+1} with the entry i. It is not hard to check that P is a reverse plane partition. The reverse procedure can also be easily recovered. Figure 3.8 is an illustration of the bijection Φ, where L_1 is the thick line and L_2 is the dashed line.



P $\qquad$ $L = (L_1, L_2)$

Figure 3.8: An illustration of the bijection Φ.

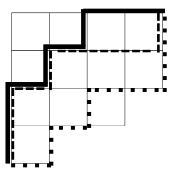
3.2 The first bijection from $\mathrm{Path}(\lambda, k)$ to $\mathrm{F}(\lambda, k)$

Let us describe the first bijection Ω_1 from $\mathrm{Path}(\lambda, k)$ to $\mathrm{F}(\lambda, k)$. Let L be a k-tuple of nonintersecting paths in λ. Let $P = \Phi(L)$ be the reverse plane partition corresponding to L. For a square (i, j) of λ, let $P(i, j)$ denote the entry of P filled in (i, j). Define a filling F of λ by letting the entry in the square (i, j) be

$$F(i,j) = \min\{P(i+1,j), P(i,j+1)\} - P(i,j), \tag{3.1}$$

where we set $P(i, j) = k$ if the square (i, j) is not contained in λ. Set $\Omega_1(L) = F$.

For example, we give in Figure 3.9 an illustration of the map Ω_1.



$\longleftrightarrow$

0	0	1	1
0	2	2	2
2	2	3	
2			

$\longleftrightarrow$

0	1	0	1
2	0	0	1
0	1	0	
1			

L $\qquad$ P $\qquad$ F

Figure 3.9: An illustration of the bijection Ω_1.

Theorem 3.1 *The map Ω_1 is a bijection from the set $\mathrm{Path}(\lambda, k)$ to the set $\mathrm{F}(\lambda, k)$. Moreover, for any $L \in \mathrm{Path}(\lambda, k)$, the number of left turns of L equals the total sum of entries in $\Omega_1(L)$.*

Proof. We first show that for $L \in \mathrm{Path}(\lambda, k)$, the filling $F = \Omega_1(L)$ of λ belongs to $\mathrm{F}(\lambda, k)$. Let $\mathrm{m}(F, (i, j))$ denote the maximal length of any SE-chain such that each entry in the SE-chain lies weakly below and weakly to the right of the square (i, j).

To verify $F \in \mathrm{F}(\lambda, k)$, we need to show that

$$\mathrm{m}(F, (1, 1)) \leq k. \tag{3.2}$$

To this end, we claim that for any square (i, j) in λ,

$$P(i, j) = k - \mathrm{m}(F, (i, j)). \tag{3.3}$$

The proof is by induction on the hook length of the square (i, j). Let $h_{i,j}$ denote the hook length of the square (i, j), namely, the number of squares of λ directly to the right or directly below (i, j), counting (i, j) itself once. We first consider the case when $h_{i,j} = 1$. In this case, (i, j) is a corner of λ. This means that neither the square $(i+1, j)$ nor the square $(i, j+1)$ is in λ. Moreover, we see that $\mathrm{m}(F, (i, j)) = F(i, j)$. Hence, we have

$$P(i,j) \quad = \quad \min\{P(i+1,j), P(i,j+1)\} - F(i,j)$$

$$\begin{aligned} &= k - F(i,j) \\ &= k - \mathrm{m}(F,(i,j)). \end{aligned}$$

Assume that (3.3) is true for the square (i,j) with $h_{i,j} = t \geq 1$. We now consider the case for the square (i,j) with $h_{i,j} = t+1$. Note that $h_{i+1,j} \leq t$ and $h_{i,j+1} \leq t$. By the induction hypothesis,

$$\begin{aligned} P(i,j) &= \min\{P(i+1,j), P(i,j+1)\} - F(i,j) \\ &= \min\{k - \mathrm{m}(F,(i+1,j)), k - \mathrm{m}(F,(i,j+1))\} - F(i,j) \\ &= k - \max\{\mathrm{m}(F,(i+1,j)), \mathrm{m}(F,(i,j+1))\} - F(i,j) \\ &= k - \mathrm{m}(F,(i,j)), \end{aligned}$$

which verifies (3.3).

In particular, restricting $(i,j) = (1,1)$ in (3.3) yields that

$$P(1,1) = k - \mathrm{m}(F,(1,1)),$$

and so we have

$$\mathrm{m}(F,(1,1)) = k - P(1,1) \leq k.$$

This concludes (3.2). Thus F is a filling of λ belonging to $\mathrm{F}(\lambda,k)$.

On the other hand, given a filling F of λ in $\mathrm{F}(\lambda,k)$, we can define a k-tuple L of nonintersecting paths in λ as follows. Let P be a tableau defined by setting

$$P(i,j) = k - \mathrm{m}(F,(i,j)).$$

It is obvious that

$$0 \leq P(i,j) \leq k.$$

Moreover,

$$P(i+1,j) = k - \mathrm{m}(F,(i+1,j)) \geq k - \mathrm{m}(F,(i,j)) = P(i,j)$$

and

$$P(i,j+1) = k - \mathrm{m}(F,(i,j+1)) \geq k - \mathrm{m}(F,(i,j)) = P(i,j).$$

Therefore, P is a reverse plane partition in $\mathrm{RPP}(\lambda,k)$. Let $L = \Psi(P)$ be the k-tuple of nonintersecting paths in λ. It is easy to check that the above procedures are reversible, and thus the map Ω_1 is a bijection.

By the construction of the bijection Ψ, it is easily checked that for each square (i,j) of λ, the number of left turns of $L = \Psi(P)$ that border the square (i,j) is exactly equal to

$$\min\{P(i,j+1), P(i+1,j)\} - P(i,j) = F(i,j).$$

Hence the number of left turns of L equals the total sum of integers in $\Omega_1(L)$. This completes the proof. ∎

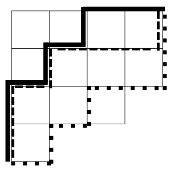
3.3 The second bijection from $\mathrm{Path}(\lambda, k)$ to $\mathrm{F}(\lambda, k)$

We next describe the second bijection Ω_2 from $\mathrm{Path}(\lambda, k)$ to $\mathrm{F}(\lambda, k)$. Let L be a k-tuple of nonintersecting paths in λ. Again, we let $P = \Phi(L)$ be the reverse plane partition corresponding to L. Define a filling F' of λ by setting

$$F'(i,j) = P(i,j) - \max\{P(i-1,j), P(i,j-1)\}, \tag{3.4}$$

where we set $P(0,j) = P(i,0) = 0$. Set $\Omega_2(L) = F'$.

Figure 3.10 is an illustration of the bijection Ω_2.



$\longleftrightarrow$

0	0	1	1
0	2	2	2
2	2	3	
2			

$\longleftrightarrow$

0	0	1	0
0	2	0	0
2	0	1	
0			

L P F'

Figure 3.10: An illustration of the bijection Ω_2.

Theorem 3.2 *The map Ω_2 is a bijection from the set $\mathrm{Path}(\lambda, k)$ to the set $\mathrm{F}(\lambda, k)$. Moreover, for any $L \in \mathrm{Path}(\lambda, k)$, the number of right turns of L equals the total sum of entries in $\Omega_2(L)$.*

Proof. For each square (i,j) of λ, let $\overline{\mathrm{m}}(F', (i,j))$ denote the maximal length of any SE-chain of F' such that each entry in the SE-chain lies weakly above and weakly to the left of (i,j).

We claim that for any square (i,j) of λ,

$$P(i,j) = \overline{\mathrm{m}}(F', (i,j)). \tag{3.5}$$

The proof is by induction on the reverse hook length of (i,j). Here, the reverse hook length of a square is defined to be 1 plus the number of squares above it, plus the number of squares to its left, see Dukes and Reifegerste [9]. Let $\overline{h}_{i,j}$ denote the reverse hook length of the square (i,j).

If $\overline{h}_{i,j} = 1$, then (i,j) must be the square $(1,1)$. According to (3.4), it is clear that

$$P(1,1) = F'(1,1) = \overline{\mathrm{m}}(F', (1,1)).$$

Assume that (3.5) is true for any square (i,j) with $\overline{h}_{i,j} = t \geq 1$. We now consider the case for the squares with $\overline{h}_{i,j} = t+1$. Notice that $\overline{h}_{i-1,j} \leq t$ and $\overline{h}_{i,j-1} \leq t$. Hence, by the induction hypothesis,

$$P(i,j) \quad = \quad F'(i,j) + \max\{P(i-1,j), P(i,j-1)\}$$

$$\begin{aligned} &= F'(i,j) + \max\{\overline{\mathrm{m}}(F',(i-1,j)), \overline{\mathrm{m}}(F',(i,j-1))\} \\ &= \overline{\mathrm{m}}(F',(i,j)). \end{aligned}$$

This verifies the claim in (3.5). So F' is a filling of λ in $\mathrm{F}(\lambda,k)$.

The reverse procedure of Ω_2 can be described as follows. Given a filling F' of λ in $\mathrm{F}(\lambda,k)$, we first define a filling P by setting

$$P(i,j) \;=\; \overline{\mathrm{m}}(F',(i,j)).$$

Obviously, $0 \leq P(i,j) \leq k$. Moreover,

$$P(i,j) = \overline{\mathrm{m}}(F',(i,j)) \geq \overline{\mathrm{m}}(F',(i,j-1)) = P(i,j-1)$$

and

$$P(i,j) = \overline{\mathrm{m}}(F',(i,j)) \geq \overline{\mathrm{m}}(F',(i-1,j)) = P(i-1,j).$$

So we see that P is a reverse plane partition in $\mathrm{RPP}(\lambda,k)$. Let $L = \Psi(P)$ be the k-tuple of nonintersecting paths in λ.

By the construction of Ψ, for every square (i,j) of λ, the number of right turns of $L = \Psi(P)$ bordering the square (i,j) is equal to

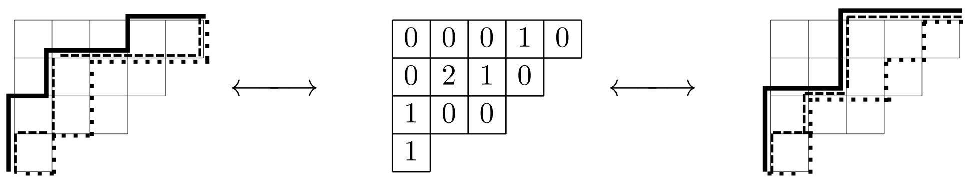
$$P(i,j) - \max\{P(i,j-1), P(i-1,j)\} = F'(i,j).$$

Hence the number of right turns of L equals the total sum of numbers in F'. This completes the proof. ∎

3.4 Proof of Theorem 2.2

By Theorems 3.1 and 3.2, we obtain a one-to-one correspondence $\Omega = \Omega_1^{-1}\Omega_2$ on $\mathrm{Path}(\lambda,k)$ such that for each $L \in \mathrm{Path}(\lambda,k)$, the number of right turns of L is equal to the number of left turns of $\Omega(L)$. This finishes the proof of Theorem 2.2.

Figure 3.11 gives an example to illustrate the bijection Ω.

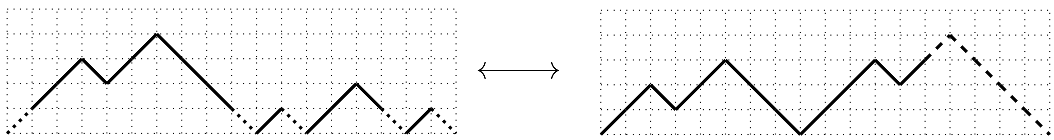


L $\qquad\qquad$ $\Omega_2(L)$ $\qquad\qquad$ $\Omega(L) = \Omega_1^{-1}\Omega_2(L)$

Figure 3.11: An illustration of the bijection Ω.

When λ is the staircase shape δ_n and $k = 1$, the bijection Ω can be specialized into a bijection on Dyck paths. In this case, given a Dyck path D, the number of high peaks of D equals the number of valleys of $\Omega(D)$. As mentioned in the Introduction, Deutsch [7] also established a bijection on Dyck paths, which switches the number of high peaks to the number of valleys. We next recall this bijection, and give an example to illustrate that when restricted to Dyck paths, Ω differs from the bijection of Deutsch.

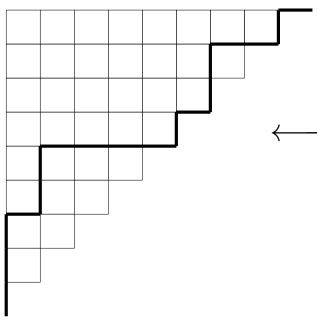
The bijection in [7] can be described as follows. The return of a Dyck path D is defined as the down-step landing on the horizontal axis. Assume that D has k returns. Delete the very first step of D and all the k return steps, see the dotted steps in Figure 3.12(a). Draw the remaining steps continuously, and then add an up step followed by k down steps, see the dashed steps in Figure 3.12(b). The resulting path is denoted by $\Gamma(D)$. It is not hard to check that Γ is a bijection. Moreover, the number of high peaks of D equals the numbers of valleys of $\Gamma(D)$.



(a) (b)

Figure 3.12: Deutsch's bijection Γ on Dyck paths.

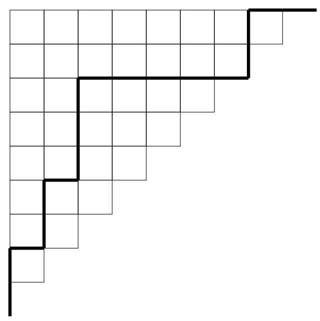
For comparison, for the Dyck path in Figure 3.12(a), the resulting Dyck path by applying the bijection Ω is given in Figure 3.13, which is different from the Dyck path in 3.12(b).



$\longleftrightarrow$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	
0	0	0	0	0	0		
0	0	0	0	0			
0	1	0	0				
0	0	0					
1	0						
0							

$\longleftrightarrow$



D $\Omega_2(D)$ $\Omega(D) = \Omega_1^{-1}\Omega_2(D)$

Figure 3.13: Our bijection Ω on Dyck path.

3.5 Another proof of Theorem 3.2 and Theorem 2.2

In fact the bijection between reverse plane partitions of given shape λ with maximum entry at most k and fillings of λ with longest SE-chain at most k is a special case of Stanley's transfer map between the order polytope and chain polytope; see [27].

Let $\mathcal{P}$ be a finite partially ordered set (poset). As defined in [27], *the order polytope* of $\mathcal{P}$ is the set $\mathcal{O}(\mathcal{P})$ of labelings $f : \mathcal{P} \to [0,1]$ that are order-preserving (i.e. if $a \leq b$ in $\mathcal{P}$, then $f(a) \leq f(b)$). The *chain polytope* of $\mathcal{P}$ is the set $\mathcal{C}(\mathcal{P})$ of labelings $f : \mathcal{P} \to [0,1]$ such that the sum of the labels across every chain is at most 1.

The *transfer map* $\phi : \mathcal{O}(\mathcal{P}) \to \mathcal{C}(\mathcal{P})$, illustrated in [27], can be defined as follows: if $x = (x_1, x_2, \ldots, x_p) \in \mathcal{O}(P)$ and $t_i \in \mathcal{P}$, then $\phi(x) = (y_1, y_2, \ldots, y_p)$, where

$$y_i = \min\{x_i - x_j : t_i \text{ covers } t_j \text{in } \mathcal{P}\}.$$

Put another way,

$$y_i = x_i - \max\{x_j : t_i \text{ covers } t_j \text{in } \mathcal{P}\},$$

which is consistent with the relation (3.4).

As shown in Figure 3.14, the bijection between the reverse plane partition P and the filling F', first illustrated in Figure 3.10, can also be obtained by the transfer map ϕ. This provides the second proof method for Theorem 3.2.

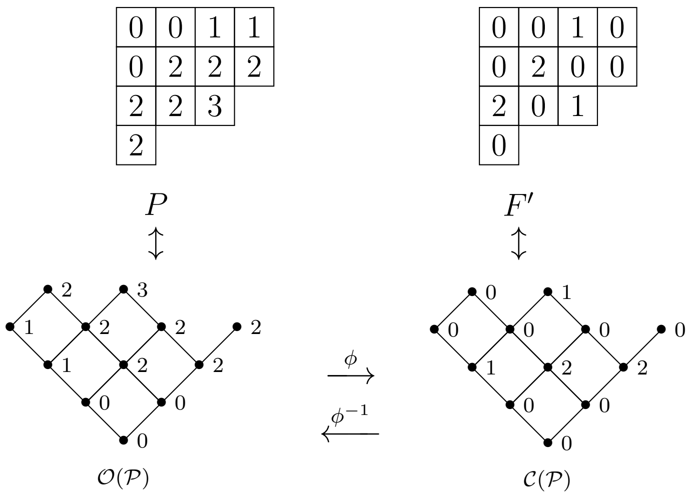


Figure 3.14: The transfer map ϕ.

Therefore, the bijection between reverse plane partitions of given shape λ with maximum entry at most k and fillings of λ with longest SE-chain at most k is a special case of Stanley's transfer map between the chain polytope and order polytope.

Next, we will introduce the piecewise-linear rowmotion defined by Einstein and Propp in [10]. The toggle $t_a : \mathcal{O}(\mathcal{P}) \to \mathcal{O}(\mathcal{P})$ is

$$t_a(x)_b = \begin{cases} x_b & \text{if } a \neq b \\ \min\limits_{a \lessdot c} x_c + \max\limits_{c \lessdot a} x_c - x_a & \text{if } a = b \end{cases}$$

for all $x \in \mathcal{O}(\mathcal{P})$ and $a \in P$. Then the piecewise-linear rowmotion is the map given by applying toggles along a linear extension from top to bottom, see Figure 3.16.

We extract the key steps for mapping Ω from Figure 3.11 and place them in Figure 3.15. Besides, the tableaux P and P' correspond to two order polytopes $\mathcal{O}(\mathcal{P})$ and $\mathcal{O}(\mathcal{P}')$, respectively. Obviously, our bijection Ω discussed in Section 3.4 is a special case of the piecewise-linear rowmotion for the relevant poset (i.e., the shape λ); see Figures 3.15 and 3.16.

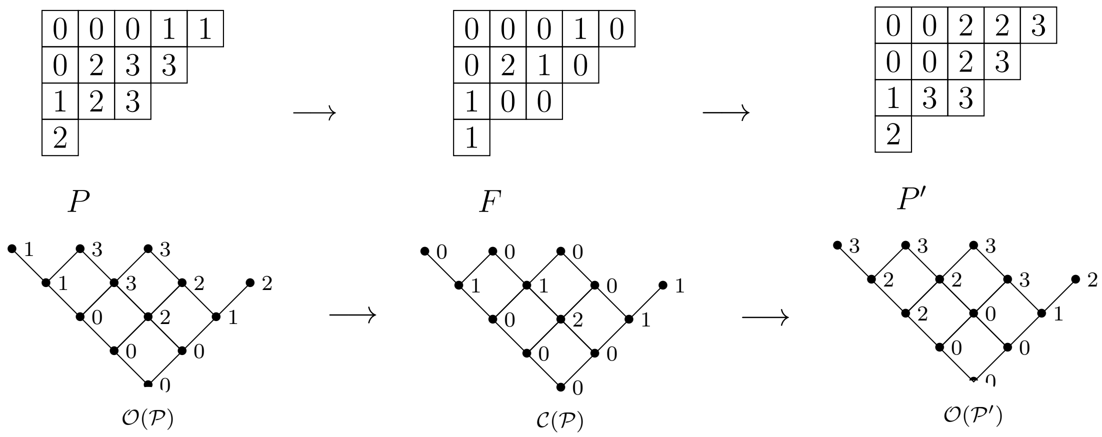


Figure 3.15: The key step for Ω in Figure 3.11.

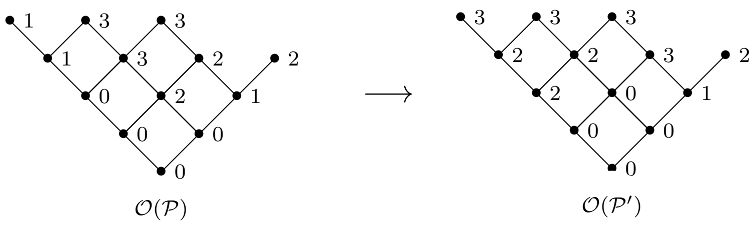


Figure 3.16: The piecewise-linear rowmotion.

Therefore, we can provide another proof of Theorem 2.2 using the theory of piecewise linear rowmotion.

4 Generating polynomial for valleys

In this section, we investigate the number $N_k(n,i)$ of k-tuples of nonintersecting Dyck paths of semilength n with i valleys. By Theorem 2.2, $N_k(n,i)$ is also the number of k-tuples of nonintersecting Dyck paths of semilength n with i high peaks. For any fixed n and k, we show that the sequence $\{N_k(n,i)\}_{i\geq 0}$ is symmetric. As an application, we give an alternative proof of a conjecture due to Reiner, Tenner and Yong [24] on barely set-valued tableaux for a staircase shape.

4.1 Symmetry of $\{N_k(n,i)\}_{i\geq 0}$

Write $N_{n,k}(q)$ for the generating polynomial of $N_k(n,i)$, namely,

$$N_{n,k}(q) = \sum_{i=0}^{k(n-1)} N_k(n,i)q^i.$$

Note that when $k=1$, $N_{n,1}(q) = N_n(q)$ reduces to the Narayana polynomial, which has the following expression

$$N_n(q) = \sum_{i=0}^{n-1} \frac{1}{n}\binom{n}{i}\binom{n}{i+1} q^i. \tag{4.1}$$

Hwang, Kim, Yoo and Yun [15] established the following determinantal formula for $N_{n,k}(q)$ in terms of Narayana polynomials:

$$N_{n,k}(q) = q^{-\binom{k}{2}} \det\left(N_{n+i+j-2}(q)\right)_{i,j=1}^{k}. \tag{4.2}$$

Suppose the highest degree of a polynomial is m; we say a polynomial is *symmetric* if the coefficients of a polynomial with degree i are equal to the coefficients of a polynomial with degree $m-i$. Using (4.1), we establish the following symmetry property of $N_{n,k}(q)$.

Theorem 4.1 *The polynomial $N_{n,k}(q)$ is symmetric.*

Proof. To derive its symmetry, we only need to prove

$$N_{n,k}(q) = q^{k(n-1)} N_{n,k}(q^{-1}). \tag{4.3}$$

By (4.1), it is easily checked that $N_n(q)$ is a symmetric polynomial, namely,

$$N_n(q) = q^{n-1} N_n(q^{-1}). \tag{4.4}$$

Combining (4.2) and (4.4), we see that

$$\begin{aligned} N_{n,k}(q^{-1}) &= q^{\binom{k}{2}} \det(N_{n+i+j-2}(q^{-1}))_{i,j=1}^{k} \\ &= q^{\binom{k}{2}} \det(q^{-(n+i+j-2-1)} N_{n+i+j-2}(q))_{i,j=1}^{k} \\ &= q^{\binom{k}{2}+3k} \det(q^{-(n+i+j)} N_{n+i+j-2}(q))_{i,j=1}^{k}. \end{aligned} \tag{4.5}$$

We proceed to apply a row and column transformation for the determinant in (4.5).

Extracting the factor $q^{-(n+i)}$ from row i, we obtain that

$$\det(q^{-(n+i+j)} N_{n+i+j-2}(q))_{i,j=1}^{k} = q^{-\sum_{i=1}^{k}(n+i)} \det(q^{-j} N_{n+i+j-2}(q))_{i,j=1}^{k}$$

$$= q^{-kn-\binom{k+1}{2}} \det(q^{-j} N_{n+i+j-2}(q))_{i,j=1}^{k}. \tag{4.6}$$

For the determinant appearing in (4.6), extracting the factor q^{-j} from each column j yields that

$$\begin{aligned}\det(q^{-j} N_{n+i+j-2}(q))_{i,j=1}^{k} &= q^{-(1+2+\cdots+k)} \det(N_{n+i+j-2}(q))_{i,j=1}^{k} \\ &= q^{-\binom{k+1}{2}} \det(N_{n+i+j-2}(q))_{i,j=1}^{k}.\end{aligned} \tag{4.7}$$

Combining (4.5), (4.6) and (4.7), we have

$$\begin{aligned}N_{n,k}(q^{-1}) &= q^{\binom{k}{2}+3k-kn-\binom{k+1}{2}-\binom{k+1}{2}} \det(N_{n+i+j-2}(q))_{i,j=1}^{k} \\ &= q^{-\binom{k}{2}-k(n-1)} \det(N_{n+i+j-2}(q))_{i,j=1}^{k} \\ &= q^{-k(n-1)} q^{-\binom{k}{2}} \det(N_{n+i+j-2}(q))_{i,j=1}^{k},\end{aligned}$$

which, together with (4.2), leads to the relation (4.3). This completes the proof. ∎

For example, if $n = 4$ and $k = 2$, then $N_{4,2}(q) = q^6+6q^5+21q^4+28q^3+21q^2+6q+1$. Obviously, $N_{4,2}(q)$ is symmetric.

In fact there is another Dyck path bijection which is called the Lalanne-Kreweras involution (because it was considered by Kreweras [19] and Lalanne [20]). The Lalanne-Kreweras involution bijectively exhibits the symmetry of the number of valleys statistic on Dyck paths of semilength n because it sends a Dyck path with k valleys to one with $n-1-k$ valleys.

Later, Hopkins and Joseph [14] constructed a piecewise-linear extension of the Lalanne-Kreweras involution, which in particular gives a combinatorial proof of Theorem 4.1 for the general k. The specific operation is, the Dyck paths of semilength n are in simple bijection with the antichains of the Type A_{n-1} root poset, and under this bijection, the number of valleys becomes the cardinality of the antichain. Moreover, recall that the indicator functions of antichains of a finite poset are the vertices of its order polytope. In this way, the k-tuples of nonintersecting Dyck paths of semilength n are in bijection with the rational points in the order polytope of the Type A_{n-1} root poset whose denominators divide k. That is, the points in $\frac{1}{k}\mathbb{Z}^{\triangle^{n-1}} \cap \mathcal{C}(\triangle^{n-1})$ correspond to k-tuples of nonintersecting Dyck paths. Under this bijection, the number of valleys of the k-tuple becomes k times the sum of the coordinates of the corresponding point. The piecewise-linear extension of the Lalanne-Kreweras involution implies that the generating function over these k-tuples for the number of valleys statistic is still symmetric.

4.2 A conjecture of Reiner-Tenner-Yong

Let us begin by recalling a conjecture of Reiner, Tenner and Yong [24] concerning the enumeration of barely set-valued semistandard Young tableaux. A semistandard Young tableau of shape λ is a filling of positive integers into the squares of λ such that the entries in each row are weakly increasing, and the entries along each column are strictly increasing. A set-valued semistandard Young tableau of shape λ is an assignment of finite sets of positive integers into the squares of λ such that the sets in each row (respectively, column) are weakly (respectively, strictly) increasing [2]. Here, for two sets A and B of positive integers, write $A \leq B$ if $\max A \leq \min B$ and $A < B$ if $\max A < \min B$. Clearly, when the set in each square exactly contains a single integer, a set-valued semistandard Young tableau becomes an ordinary semistandard Young tableau.

A barely set-valued semistandard Young tableau is a set-valued semistandard Young tableau such that exactly one square is assigned a set of two integers, and each of the remaining squares is assigned a set containing a single integer.

Given a partition λ and a positive integer k, let $\mathrm{BSSYT}(\lambda, k)$ and $\mathrm{SYT}(\lambda, k)$ denote the set of barely set-valued semistandard Young tableaux and ordinary semistandard Young tableaux of shape λ, respectively, where no integer in row i can exceed $k+i$. A partition is called a *rectangular staircase shape* $\delta_n(b^a)$ if the Young diagram is obtained from the staircase shape $\delta_n = (n-1, n-2, \ldots, 1)$ by replacing each square with an $a \times b$ rectangle. See Figure 4.17 as an example.

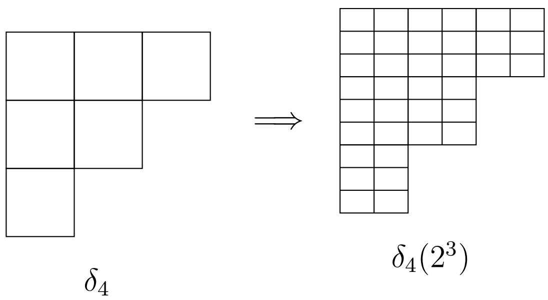


Figure 4.17: A rectangular staircase shape $\delta_4(2^3)$.

If λ is a rectangular staircase shape, Reiner, Tenner and Yong [24] posed the following conjecture.

Conjecture 4.2 (Reiner, Tenner and Yong [24, Conjecture 6.4′]) *For any positive integers a, b, n and k,*

$$|\mathrm{BSSYT}(\delta_n(b^a), k)| = \frac{kab(n-1)}{(a+b)} |\mathrm{SYT}(\delta_n(b^a), k)|. \tag{4.8}$$

This conjecture was proved by Fan, Guo and Sun [12] by applying results of Chan, Haddadan, Hopkins, and Moci on jaggedness of shapes [4]. Here, we shall explain that for the staircase shape (namely, $a = b = 1$), we can use Theorem 4.1 to give an alternative proof of (4.8).

When $a = b = 1$, relation (4.8) reduces to

$$|\mathrm{BSSYT}(\delta_n, k)| = \frac{k(n-1)}{2} |\mathrm{SYT}(\delta_n, k)|. \tag{4.9}$$

Employing the bijection in [12, Theorem 3.2], we have the following relation

$$|\mathrm{BSSYT}(\lambda, k)| = \sum_{i \geq 0} i \cdot N_k(\lambda, i), \tag{4.10}$$

where $N_k(\lambda, i)$ denote the number of k-tuples of nonintersecting paths in λ with i valleys.

Using (4.10) for the case $\lambda = \delta_n$ and together with Theorem 4.1, we see that

$$\begin{aligned} |\mathrm{BSSYT}(\delta_n, k)| &= \sum_{i=0}^{k(n-1)} i \cdot N_k(n, i) \\ &= \frac{\sum_{i=0}^{k(n-1)} i \cdot N_k(n, i) + \sum_{i=0}^{k(n-1)} (k(n-1) - i) \cdot N_k(n, i)}{2} \\ &= \frac{\sum_{i=0}^{k(n-1)} k(n-1) \cdot N_k(n, i)}{2} \\ &= \frac{k(n-1) \sum_{i=0}^{k(n-1)} N_k(n, i)}{2} \\ &= \frac{k(n-1) |\mathrm{Path}(\delta_n, k)|}{2}. \end{aligned} \tag{4.11}$$

To conclude (4.9), we still need to check that

$$|\mathrm{Path}(\delta_n, k)| = |\mathrm{SYT}(\delta_n, k)|. \tag{4.12}$$

Recall that $\mathrm{RPP}(\lambda, k)$ is the set of reverse plane partitions of shape λ with entries not exceeding k. Subtracting each entry in row i of $T \in \mathrm{SYT}(\delta_n, k)$ by i, we are led to a bijection between $\mathrm{SYT}(\delta_n, k)$ and $\mathrm{RPP}(\delta_n, k)$. In view of the bijection in Section 3.1 between $\mathrm{Path}(\lambda, k)$ and $\mathrm{RPP}(\lambda, k)$, we arrive at (4.12). Putting (4.12) into (4.11), we obtain (4.9).

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