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Palindromic periodicities

Jamie Simpson

130 Preston Point Rd., East Fremantle
WA 6158, Australia
jamiesimpson320@gmail.com

Abstract

A palindromic periodicity is a factor of an infinite word $(ps)^\omega$ where p and s are palindromes and the factor has length at least $|ps|$, for example *accabaccab*. In this paper we describe several ways in which a palindromic periodicity may arise through the interaction of palindromes and periodicity, the simplest case being when a palindrome is itself periodic. We then consider what happens when a word is a palindromic periodicity in two ways, a situation similar to that considered in the Fine and Wilf Lemma [4], and obtain something slightly stronger than that lemma. The paper ends with suggestions for further work.

1 Introduction

We are concerned with words which are factors of infinite words of the form $(ps)^\omega$ where p and s are palindromes and which have length at least $|ps|$. We call such words *palindromic periodicities*. Our interest in these objects arises from the facts that they arise naturally in several ways and that they have properties that are not obvious from their definition. As will be seen shortly, our discussion will also apply to powers of a single palindrome. The paper is set out as follows. After reviewing notation we discuss some of the simple properties of these words. We then describe, in Section 2, several ways in which palindromic periodicities can arise. In the third section we consider ways in which these words interact and obtain an analogy of Fine and Wilf's Periodicity Lemma [4] for these words. The Fine and Wilf Lemma does, of course, already apply to our periodic words but their extra structure means a slightly stronger result is possible. We also obtain a result about a palindrome embedded in a palindromic periodicity. We end the paper with some discussion and suggestions for further investigation.

We use the usual notation for combinatorics of words. A word containing n letters is $w = w[1 \dots n]$, with $w[i]$ being the ith letter and $w[i \dots j]$ the *factor* beginning at position i and ending at position j. If $i = 1$ then the factor is a *prefix* and if $j = n$ it is a *suffix*. If a prefix and a suffix of w both equal a word x then x is a *border* of w. If $w = tuv$ where t, u and v are factors, we say that w is the *union* of tu and uv.

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A factor which is neither a suffix nor a prefix is a *proper factor*[1]. The *length* of w, written $|w|$, is the number of letters that w contains. If $w = uv$ where u and v are words then vu is a *conjugate* of w. The *empty word* ϵ is a word with length 0. A word or factor w is *periodic* with period p if $w[i] = w[i+p]$ for all i such that both $w[i]$ and $w[i+p]$ are in w. This means that w has period p if and only $w = u^\omega = uuu\ldots$, the infinite concatenation of u with itself, where $|u| = p$. We then say that w is a *periodicity.*

Remark 1.1 It is easy to see that if x is a border of w then w has period $|w| - |x|$.

We will use the following well-known propositions.

Lemma 1.2 (Periodicity Lemma of Fine and Wilf, [4]) *Let w be a word having two periods p and q. If $|w| \geq p + q - \gcd(p, q)$ then w also has period* $\gcd(p, q)$.

Lemma 1.3 (Lemma 2.1 of [2] and Lemma 8.1.1 of [10]) *Let w be a word having two periods p and q with $p > q$. Then the suffix and prefix of length $|w| - q$ both have period $p - q$.*

Lemma 1.4 (Lemma 8.1.3 of [10]) *Let w be a word with period q which has a factor u with $|u| \geq q$ that has period r, where r divides q. Then w has period r.*

Lemma 1.5 (Lemma 8.1.2 of [10]) *Let u, v and w be words such that uv and vw both have period p and $|v| \geq p$. Then the word uvw has period p.*

Lemma 1.6 *If a word w has period p and $w[i + 1..i + q] = w[j + 1..j + q]$ where $i + 1 < j < i + q$ and $q \geq p$ then w has period* $\gcd(p, j - i)$.

Proof: The factor $w[i + 1..j + q]$ has a border of length q and therefore, by Remark 1.1, has period $j - i$. It also has period p and length $j + q - i$. Since $q \geq p$ the periodicity lemma applies and the factor has period $\gcd(j - i, p)$. By Lemma 1.4 this periodicity extends to the whole of w. □

The *reverse* of a word $w[1\ldots n]$ is the word $R(w) = w[n]w[n-1]\ldots w[1]$. The word w is a *palindrome* if $w = R(w)$. Thus the empty word ϵ is a palindrome. A palindrome is *odd* or *even* if its length is, respectively, odd or even. If $w[i\ldots j]$ is a palindrome we say it has *centre* $c = (i + j)/2$ and *radius* $r = (j - i)/2$. Note that c and r are integers if the palindrome is odd and each is an integer plus 1/2 if the palindrome is even.

Much of what follows concerns centres of palindromes and it will be useful to have the following notation:

$$\mathbb{Z}/2 = \{n/2 : n \in \mathbb{Z}\}$$

[1]In linguistics a proper factor is called an "infix". Linguists use $\emptyset$ to mean empty set though they do not call it an empty set and iff as we do but they pronounce it, at least in the Linguistics Department of the University of Western Australia, as "if if".

so that c and r are in $\mathbb{Z}/2$ whereas $2c$, $2r$ and $c+r$ are in $\mathbb{Z}$. If w contains a palindrome with centre c and radius r and $c-r \leq i < j \leq c+r$ then

$$w[i] = w[2c-i] \tag{1.1}$$

and

$$w[i \dots j] = R(w[2c-j..2c-i]). \tag{1.2}$$

When using (1.1) we say that $w[i]$ is *reflected* in c. Having centres which are not integers means we have to decide whether $w[5/2]$ is in $w[1..2]$ or in $w[3..4]$, or in both or in neither. "Neither" seems wrong since it is in the concatenation of the two factors and "both" would also be inconvenient. We therefore adopt the convention that if a and b are integers then $w[a-1/2]$ is in $w[a..b]$ but $w[b+1/2]$ is not[2].

If $w[i \dots j]$ is a palindrome with $j \geq i+2$ then so is $w[i+1 \dots j-1]$. If $w[c-r \dots c+r]$ is a palindrome but none of $w[c-r-1 \dots c+r+1]$, $w[c-r \dots c+r+1]$, and $w[c-r-1 \dots c+r]$ is, then we say $w[c-r \dots c+r]$ is a *maximal palindrome*. The second and third cases here mean that we do not, for example, consider aa to be maximal in $baaac$. If $w[c-r \dots c+r]$ is maximal then the palindromes $w[c-r+i \dots c+r-i]$, $i = 1, \dots, \lfloor r \rfloor$, are *nested* in $w[c-r \dots c+r]$. If a palindrome is even, respectively odd, then its nested palindromes are even, respectively odd.

We now make some observations about palindromic periodicities. Recall that a palindromic periodicity is a factor of length at least $|ps|$ of the infinite word $(ps)^\omega$ where p and s are palindromes. For example $baccaba$ is a factor of $(ps)^\omega$ where $p = aba$ and $s = cc$. It is also a factor of $(ps)^\omega$ when $p = b$ and $s = acca$, or when $p = cabac$ and $s = \epsilon$. From similar considerations we obtain the following theorem

Theorem 1.7 *Consider a palindromic periodicity taken from $(ps)^\omega$ where p and s are palindromes.*

- *(a) Except in the case where both p and s are odd, ps can be replaced by a single palindrome. When they are both odd, one of them can be replaced by a single character.*
- *(b) Any palindromic periodicity is a prefix of some infinite word $(uv)^\omega$ where u and v are palindromes.*

Proof: (a) Say that w is a factor of $(ps)^\omega$ where s is even. Then $s = uR(u)$ for some word u and w is a factor of $(R(u)pu)^\omega$, which equals $R(u)(ps)^\omega$, and $R(u)pu$ is a palindrome. The argument when both $|p|$ and $|s|$ are odd is similar.
(b) Again let w be a factor of $(ps)^\omega$. Say it equals $s''px$ where $s = s's''$ and x is a prefix of $(sp)^\omega$. We suppose that $|s''| > |s'|$ (the $=$ and $<$ cases are similar). Then $s'' = vR(s')$ where v is a palindrome and $s''px$ is a prefix of $(vR(s')ps')^\omega$. Since v and $R(s')ps'$ are palindromes we are done. □

[2]We could adopt a more elaborate notation here, say with both $w[1/2]$ and $w[5/2]$ in $w(1..2)$ and neither in $w[1..2]$, but this is different from the notation for open and closed intervals where $(1,2)$ is contained in $[1,2]$.

While p and s are not specified for a given palindromic periodicity, their centres are fixed, and the distance between these centres is fixed and equals $|ps|/2$, which we call the *half-period* of the palindromic periodicity. This will sometimes be a more convenient parameter than the period. Note that the half-period is in $\mathbb{Z}/2$. The palindromic periodicity *abbcbbadabbcbbad* is a factor of $(ps)^\omega$ with $p = ada$ and $s = bbcbb$ with centres d and c. We call such centres *essential palindromic centres*, or just *essential centres*. That is, a palindromic periodicity being a factor of $(ps)^\omega$ has essential palindromic centres at the centres of the occurrences of p and s. A palindromic periodicity can contain other palindromic centres. For example the palindromic periodicity above contains the palindrome *bb* whose centre is not the centre of p or s, and so it is not essential.

We note the following:

Remark 1.8 Any essential centre of a palindromic periodicity w is the centre of a palindromic prefix or a palindromic suffix of w. This is not true of non-essential centres.

Following these observations we see that the following is an alternative definition of a palindromic periodicity: word w is a palindromic periodicity with length n, offset r and half-period h if each position $r, r+h, \ldots, r+\lfloor (n-r)/h \rfloor h$ is the palindromic centre of a palindromic prefix or suffix, and further, that the sum of the lengths of the longest such prefix and the longest such suffix is at least $|w|$. The centres of the longest palindromic prefix and longest palindromic suffix are, respectively

$$r + \left\lfloor \frac{\left\lfloor \frac{n-r}{h} \right\rfloor}{2} \right\rfloor h \text{ and } r + \left\lceil \frac{\left\lfloor \frac{n-r}{h} \right\rfloor}{2} \right\rceil h.$$

Remark 1.9 If we know that a word has period π then knowing any factor of length π and its position in the word and the length of the word will determine the rest of the word. In the case of a palindromic periodicity a factor containing two essential centres will determine the rest of the word. The maximum alphabet size of a word with period π is π. It is not hard to see that the maximum alphabet size for a palindromic periodicity with half-period h is h, $h + 1/2$ or $h + 1$ if the essential centres are centres of palindromes which are respectively all even, alternately odd and even or all odd.

2 Creating a Palindromic Periodicity

Palindromic periodicities arise naturally through the interaction of palindromes and periodicities. In this section we describe some ways in which this can happen.

Theorem 2.1 *If w is a palindrome which is periodic with period π and $|w| \geq 2\pi + 1$ then w is a palindromic periodicity with period π.*

Proof: Say that the palindrome's centre is c. If the palindrome is even then w is a factor of v^ω where v is the length π factor that follows c. By periodicity v equals the length π factor immediately preceding c and by palindromicity it equals the reverse of this factor. Therefore v is a palindrome and w is a palindromic periodicity.

If both the palindrome and π are odd then w is a factor of v^ω where v is the length π palindrome centred at c.

If the palindrome is odd and π is even then w is a factor of $(st)^\omega$ where s is the length $\pi - 1$ palindrome with centre c and t is a single letter. □

Theorem 2.2 *Let u be a finite word, v be its reverse and n its length. Then if w is a prefix of $u'u^\omega$ and also of $v'v^\omega$, where u' is a suffix of u and v' is a suffix of v, then w is a palindromic periodicity with period n.*

Proof: Without loss of generality assume that $|u'| \geq |v'|$ and consider $w[|v'|+1..|u'|]$. This is a suffix of u and a prefix of v. But since v is the reverse of u it is also the reverse of a suffix of u. Thus it equals the reverse of itself and is therefore a palindrome. Call it s. Similarly we find that $w[|u'| + 1..|v'| + n]$ is a palindrome which we call t. Thus $w[|v'| + 1..|v'| + n]$ equals st. It also equals v. Since w is a factor of v^ω it is a palindromic periodicity. □

Theorem 2.3 *Let p_1 and p_2 be palindromes with centres c_1 and c_2 respectively, and radii r_1 and r_2 respectively in a word w and $c_2 > c_1$. Then if at least one of p_1 and p_2 contains the other's centre and if they are such that neither is a proper factor of the other then their union is a palindromic periodicity with period $2(c_2 - c_1)$.*

Proof: We first suppose that p_1 contains c_2 but p_2 does not contain c_1, so we have

$$c_1 - r_1 < c_1 < c_2 - r_2 < c_2 \leq c_1 + r_1 \leq c_2 + r_2.$$

Equality is possible in the last inequality since we have assumed that p_2 is not a proper factor of p_1, so p_2 can be a suffix of p_1. Consider the factors

$$\begin{aligned} &w[c_1 - r_1..2c_1 - c_2 + r_2] \\ &w[2c_1 - c_2 + r_2 + 1..c_2 - r_2 - 1] \\ &w[c_2 - r_2..c_2 + r_2]. \end{aligned}$$

Note that the concatenation of these factors is $p_1 \cup p_2$. The third factor is p_2. The second is a palindrome nested in p_1 which we will call s. The first is the reflection in c_1 of $w[c_2 - r_2..c_1 + r_1]$ which is a prefix of p_2. So the first factor is equals a suffix of p_2 which we will call p_2'. Thus

$$p_1 \cup p_2 = p_2' s p_2$$

which is a palindromic periodicity.

Now assume that each palindrome contains the other's centre, and that

$$r_2 \leq r_1. \tag{2.1}$$

This involves no loss of generality since if it did not hold we could take the reverse of w. Since the palindromes contain each other's centres we have $c_2 - r_2 \leq c_1$ and $c_2 \leq c_1 + r_1$ and since neither palindrome is a proper factor of the other we have $c_1 - r_1 \leq c_2 - r_2$ and $c_1 + r_1 \leq c_2 + r_2$. Combining all this gives

$$c_1 - r_1 \leq c_2 - r_2 \leq c_1 < c_2 \leq c_1 + r_1 \leq c_2 + r_2. \tag{2.2}$$

so that $p_1 \cup p_2 = w[c_1 - r_1 \ldots c_2 + r_2]$. Suppose

$$i \in [c_1 - r_1, c_2 + r_2 - 2(c_2 - c_1)] = [c_1 - r_1, 2c_1 + r_2 - c_2]. \tag{2.3}$$

From (2.2) we have

$$2c_1 + r_2 - c_2 \leq c_1 + r_1$$

so that $w[i]$ is in p_1 and by (1.1) we have

$$w[i] = w[2c_1 - i].$$

Now

$$\begin{aligned} 2c_1 - i \in & [2c_1 - (2c_1 + r_2 - c_2), 2c_1 - (c_1 - r_1)] \\ &= [c_2 - r_2, c_1 + r_1] \end{aligned}$$

so, by (2.2), $w[2c_1 - i]$ is in p_2. By (1.1) again we have

$$w[2c_1 - i] = w[i + 2(c_2 - c_1)]$$

for any i satisfying (2.3) and the maximum value of i maps onto $c_2 + r_2$ which is the upper bound of $p_1 \cup p_2$, thus $p_1 \cup p_2$ has period $2(c_2 - c_1)$. We now show that $p_1 \cup p_2$ is a palindromic periodicity.

If c_1 is an integer then $s = w[c_1]$ and $t = w[c_1 + 1..2c_2 - c_1 - 1]$ are palindromes with centres c_1 and c_2 respectively. If c_1 is not an integer then $s = w[c_1 - 1/2..c_1 + 1/2]$ and

$$\begin{aligned} t &= w[c_1 + 3/2..c_1 + 3/2 + 2(c_2 - c_1) - 3] \\ &= w[c_1 + 3/2..2c_2 - c_1 - 3/2] \end{aligned}$$

are palindromes. In both cases $|s| + |t| = 2(c_2 - c_1)$ and s is nested in p_1 and t is nested in p_2. Using (2.2) it is easily shown that $|p_1 \cup p_2| \geq |s| + |t|$. As noted above, the union of p_1 and p_2 has period $2(c_2 - c_1)$ and so is a factor of $(st)^\omega$, and so is a palindromic periodicity. □

Remark 2.4 In the special case where $c_1 = n - 1/2$ and $c_2 = n + 1/2$ for some integer n the period is 1 as well as 2. If $c_1 = n$ and $c_2 = n + 1$ the period is 2 and need not be 1.

If the condition of the theorem that neither palindrome is a proper factor of the other does not hold we still have the following result.

Corollary 2.5 *Let $w[1..n]$ and $w[k+1..k+l]$ be palindromes with $1 < k+1 < k+l < n$, and with centers $c_1 = (1+n)/2$ and $c_2 = k + (l+1)/2$ respectively, that contain each others centres, so that $k+1 \leq (1+n)/2 \leq k+l$.*

(a) If $c_1 < c_2$ then $w[n+1-k-l..k+l]$ is a palindromic periodicity with period $2(c_2 - c_1) = 2k + l - n$.

(b) If $c_2 < c_1$ then $w[k+1..n-k]$ is a palindromic periodicity with period $2(c_1 - c_2) = n - 2k - l$.

Proof: We prove case (a), the other follows by symmetry. Note that $w[n+1-k-l..k+l]$ is a palindrome nested in $w[1..n]$. The palindrome $w[k+1..k+l]$ is a suffix of this so we can apply the theorem and the result follows. □

3 Towards a periodicity lemma for palindromic periodicities

We say that a word is a *double palindromic periodicity* if it is a palindromic periodicity in two ways each with its own offset and half-period. In this section we obtain something like Fine and Wilf's lemma but applied to words which are double palindromic periodicities. That is, we show that if a double palindromic periodicity with half-periods h_1 and h_2 is sufficiently long then it is a palindromic periodicity whose half-period is possibly less than both h_1 and h_2.

To do this we first consider the case of a single palindrome embedded in a single palindromic periodicity and show that if the embedded palindrome is sufficiently long the half-period of the palindromic periodicity is reduced. Here "sufficiently long" depends on the parameters of the palindromic periodicity and on the position of the centre of the embedded palindrome.

We now define something called a g-word. This will be the palindrome we embed in our double palindromic periodicity.

A *g-word* is a non-empty palindrome $w[i+1..i+n]$ with centre c which is embedded in a palindromic periodicity w with half-period h and offset r. Its length is $n = 2h-g$ where

$$g = \gcd(2|c-r|, 2h).$$

Saying that the palindrome is non-empty means we do not allow $2h$ to equal g or c to equal $r + kh$ for any k. For example,

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abcdexxedcbaabcdexxedcbaabcdexxedcbaabcdexxedcba
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is a g-word with parameters $i = 0, r = 13/2$, $c = 49/2$, $h = 30$, $n = 48$ and $g = 12$.

Remark 3.1 Notice that the length of a g-word is always divisible by g.

From its definition it is immediate that a g-word has period $2h$. The next results show that a g-word has period g and is a palindromic periodicity with that period. Clearly the word above does have period $g = 12$. The following lemma tells us a bit about the structure of a g-word.

Lemma 3.2 *A g-word $w[i+1..i+n]$ contains exactly two essential centres of the underlying palindromic periodicity,*

Proof: Let the underlying palindromic periodicity be w and have half period h and offset r. Let the g-word be $w[i+1..i+n]$ and its centre be c. We know that the g-word contains at least one essential palindromic centre since its length is greater than h. Say the first such centre is $w[r+kh]$ where $0 \leq r < h$. Since the g-word's length is less than $2h$ it can contain at most one other. Clearly $w[i+1..i+2h]$ contains a second centre at position $r+(k+1)h$. We must show that this does not lie in the factor $w[i+n+1..i+2h]$.

We have

$$\begin{aligned} g &= \gcd(2|c-r|, 2h) \\ n &= 2h - g \\ c &= i + (n+1)/2. \end{aligned}$$

In terms of c, g, i and r,

$$\begin{aligned} n &= 2c - 2i - 1 \\ h &= (n+g)/2 = c - i + (g-1)/2 \\ g &= \gcd(2|c-r|, 2(c-i) + g - 1) \\ &= \gcd(|2c-2r|, 2(c-i) - 1). \end{aligned}$$

We write r_0 for $r+kh$ and consider two cases. If $r_0 < c$ then $g = \gcd(2c-2r_0, 2(c-i)-1)$. Suppose, for the sake of contradiction, that $r_0 + h > i + n + 1/2$. Then

$$\begin{aligned} & r_0 + c - i + (g-1)/2 > i + 1 + 2c - 2i - 1 + 1/2 = 2c - i - 1/2 \\ \Rightarrow\ & r_0 + g/2 > c \\ \Rightarrow\ & g > 2(c - r_0) \end{aligned}$$

which is impossible since g divides $2(c-r_0)$.

If $r_0 > c$ then $g = \gcd(2(r_0-c), 2h)$. Suppose, for the sake of contradiction, that $r_0 + h \leq i + 2h$, then

$$\begin{aligned} & r_0 \leq i + c - i + (g-1)/2 \\ \Rightarrow\ & 2(r_0 - c) < g \end{aligned}$$

which is impossible since g divides $2(r_0-c)$. Thus in each case we obtain a contradiction and conclude that $r_0 + h \leq i + n + 1/2$ and the statement of the lemma follows. □

We have five parameters to use in describing a g-word: r, c, h, n and g. These are not independent. We can determine all five if we know $c-r$ and h. However we will find it convenient to use all of them. It is possible that either r or $r+h$ coincides with c. In this case either $g = \gcd(0, 2h) = 2h$ or $r + h = c$ and $g = \gcd(2h, 2h) = 2h$. In either case $n = 2h - g = 0$ and w is empty. We will assume henceforth that neither r nor $r + h$ coincides with c and we have $r < c < r + h$.

Theorem 3.3 *(a) A g-word is a power of a length g palindrome, and*
(b) this periodicity extends to the whole of the embedding palindromic periodicity.

Proof: (a) Let w be a g-word as in Lemma 3.2 and without loss of generality set $i = 0$ so that $w = w[1..2h-g]$ with essential centers at r and $r+h$. Thus $w[1..2r-1]$ and $w[2(h+r)-(2h-g)..2h-g] = w[2r+g..2h-g]$ are, respectively, a palindromic prefix and a palindromic suffix of w. Since a g-word is a palindrome w also has a palindromic centre at c.

The palindrome centred at r is contained in w so we may apply Theorem 2.3 or Corollary 2.5 (depending on whether or not c is inside $w[1..2r-1]$) and conclude that w has period $2(c-r)$. Similarly, using the palindrome centred at $r+h$, we find that w also has period $2(h + r - c)$. We now apply the Lemma 1.2. The greatest common divisor of these periods is

$$(2c - 2r, 2h + 2r - 2c) = (2(c - r), 2h) = g$$

and their sum is $2h$. Since $|w| = 2h - g$ we may apply the lemma and conclude that w has period g. Since the length g prefix equals the reverse of the length g suffix and $|w|$ is divisible by g this prefix is a palindrome w is a power of this prefix.

(b) The embedding palindromic periodicity has period $2h$, the g-word has period g which divides $2h$ so part (b) follows from Lemma 1.4. □

The following example shows that the theorem is sharp in the sense that it would not hold if a g-word was defined to have some length less than $2h - g$.

Example. The first word below is a palindromic periodicity with offset 25/2, half-period 30 and length 60. In the second word we have inserted a g-word at the centre of this palindromic periodicity whose parameters are $r = 25/2$, $c = 49/2$, $g = 12$ and length is 48. This word has period 12 in agreement with Theorem 3.3 and the periodicity extends to the whole of the word. In the last word the central palindrome has length 46 which is two letters too short to be a g-word. In this case the central palindrome has period 24. This periodicity does not extend to the whole word. This shows that in some cases Theorem 3.3 is sharp.

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abcdefghijkllkjihgfedcbamnopqrstuvwxyzABCDDCBAzyxwvutsrqponm

abcdeffedcbaabcdeffedcbaabcdeffedcbaabcdeffedcbaabcdeffedcba

cdefgbbgfedccdefgbbgfedccdefgaagfedccdefgbbgfedccdefgaagfedc
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We now present our periodicity theorem for palindromic periodicities.

Theorem 3.4 *If w is a double palindromic periodicity with parameters (r_1, h_1) and (r_2, h_2) and with*

$$|w| \geq 2h_1 + 2h_2 - \gcd(2(r_2 - r_1), 2h_1, 2h_2) \tag{3.1}$$

then w has period $\gcd(2(r_2 - r_1), 2h_1, 2h_2)$.

Proof: We write g for $\gcd(2(r_2 - r_1), 2h_1, 2h_2)$ and let $c = (|w| + 1)/2$ be the centre of w. By (3.1) we have

$$2c - 1 \geq 2h_1 + 2h_2 - g$$

so that

$$r_1 + \left(\frac{c - r_1}{h_1} - 1\right) h_1 \geq h_2 - \frac{g - 1}{2}.$$

But

$$\frac{c - r_1}{h_1} - 1 < \left\lfloor \frac{c - r_1}{h_1} \right\rfloor$$

so

$$r_1 + \left\lfloor \frac{c - r_1}{h_1} \right\rfloor h_1 > h_2 - \frac{g - 1}{2}.$$

The left hand side here is the position of largest essential centre of the first palindromic periodicity which is less than or equal to c. Writing i for $\lfloor \frac{c-r_1}{h_1} \rfloor$ we get

$$2(r_1 + ih_1) - 1 > 2h_2 - g.$$

The left hand side is the length of the palindromic prefix of w centred at $r_1 + ih_1$. Note that g divides $\gcd(2h_2, 2(r_1 + ih_1 - r_2))$ so that

$$2(r_1 + ih_1) - 1 \geq 2h_2 - \gcd(2h_2, 2(r_1 + ih_1 - r_2))$$

and we see that this palindromic prefix contains a g-word. Then Theorem 3.3 applies and $w[1..2(r_1 + ih_1) - 1]$ has period $\gcd(2(r_2 - (r_1 + i)h_1), 2h_2)$ and this periodicity extends to the whole of w. But w also has period $2h_1$. In order to apply Fine and Wilf's Lemma 1.2 we must show that the length of w is at least

$$\begin{aligned} & 2h_1 + \gcd(2(r_2 - (r_1 + i)h_1), 2h_2) - \gcd(2(r_2 - (r_1 + i)h_1), 2h_2, 2h_1) \\ = \; & 2h_1 + \gcd(2(r_2 - (r_1 + i)h_1), 2h_2) - g. \end{aligned}$$

Using (3.1) it is sufficient to show that

$$2h_1 + 2h_2 - \gcd(2(r_2 - r_1), 2h_1, 2h_2) \geq 2h_1 + \gcd(2(r_2 - (r_1 + i)h_1), 2h_2) - g.$$

But $\gcd(2(r_2 - r_1), 2h_1, 2h_2) = g$ so this is equivalent to showing that $2h_2 \geq \gcd(2(r_2 - (r_1 + i)h_1), 2h_2)$ which is clearly true. So w has period

$$\gcd(2(r_2 - (r_1 + ih_1)), 2h_1, 2h_2) = \gcd(2(r_2 - r_1), 2h_1, 2h_2)$$

as required. □

r_1	r_2	*lengths*	16	15	14	13	12	11	10	9	8	7	6
0	0		4	4	4	4	4	4	4	4	4	4	6
0	2		4	8	8	8	8	8	8	8	8	4	4
0	4		4	4	4	4	4	8	8	8	8	6	6
1	1		4	4	4	4	4	4	4	4	8	7	6
1	3		4	4	4	4	4	4	4	4	4	4	4
1	5		4	4	4	4	8	8	8	8	8	7	6
2	0		4	4	4	8	8	8	8	8	8	7	6
2	2		4	4	4	4	4	4	4	8	8	7	6
2	4		4	4	4	4	4	4	4	4	4	4	4
3	1		4	4	8	8	8	8	8	8	8	7	6
3	3		4	4	4	4	4	4	8	8	8	7	6
3	5		4	4	4	4	4	4	4	4	4	4	4

Table 1. Periods of double palindromic periodicities with parameters $h_1 = 4$, $h_2 = 6$ and offsets and lengths as shown. In each case $\gcd(2(r_2-r_1), 2h_1, 2h_2) = 4$.

Since $\gcd(2(r_2 - r_1), 2h_1, 2h_2)$ may be less than $\gcd(2h_1, 2h_2)$ our theorem may need a longer word to apply than does the Fine and Wilf Lemma. On the other hand our result may imply a shorter period than does Fine and Wilf.

Note that if $2(r_2 - r_1)$ divides h_1 and h_2 then the theorem turns into the Fine and Wilf Lemma. Unlike Fine and Wilf's result, this theorem is usually not sharp. That is, for many combinations of h_1, h_2, r_1 and r_2 words shorter than $2h_1 + 2h_2 - \gcd(2(r_2 - r_1), 2h_1, 2h_2)$ will have period $\gcd(2(r_2 - r_1), 2h_1, 2h_2)$. This is illustrated in Table 1 which shows the least periods of double palindromic periodicities with parameters $h_1 = 4$, $h_2 = 6$ and all combinations of r_1 and r_2 for which r_1 and r_2 have the same parity. The last stipulation means that $2(r_2 - r_1)$ is divisible by 4 and $\gcd(2(r_2 - r_1), 2h_1, 2h_2) = 4$. The table shows the least period of words of decreasing length starting with length $2h_1 + 2h_2 - \gcd(2(r_2 - r_1), 2h_1, 2h_2) = 16$. A worthwhile periodicity lemma would predict all periods in this table, not just those in the third column. In Table 2, r_1 and r_2 have opposite parity so that $\gcd(2(r_2-r_1), 2h_1, 2h_2) = 2$ and lengths are decreasing from 18.

4 Discussion

There are several directions in which research into palindromic periodicities might proceed. One is to obtain a stronger version of Theorem 3.4. In an earlier version of this paper I suggested that another would be to look for and count occurrences of

r_1	r_2	*lengths*	18	17	16	15	14	13	12	11	10	9	8
0	1		2	2	2	2	2	2	2	2	2	6	6
0	3		2	2	2	2	2	2	2	2	2	8	8
0	5		2	2	2	2	2	2	2	2	2	2	2
1	0		2	2	2	2	2	2	2	2	2	2	8
1	2		2	2	2	2	2	2	2	2	6	6	6
1	4		2	2	2	2	2	2	2	2	2	2	2
2	1		2	2	2	2	2	2	2	2	2	8	8
2	3		2	2	2	2	2	2	2	6	6	6	6
2	5		2	2	2	2	2	2	2	2	2	2	2
3	0		2	2	2	2	2	2	2	2	2	2	6
3	2		2	2	2	2	2	2	2	2	2	8	8
3	4		2	2	2	2	2	2	2	2	2	2	2

Table 2. Periods of double palindromic periodicities with parameters $h_1 = 4$, $h_2 = 6$ and offsets and lengths as shown. In each case $\gcd(2(r_2 - r_1), 2h_1, 2h_2) = 2$. Here all words of length at least 12 have period 2.

palindromic periodicities in famous words. This has since been thoroughly investigated by Fici, Shallit and Simpson in [3].

My interest in palindromic periodicities began with the paper [13] in which an almost sharp bound was obtained for the maximum number of distinct palindromes in circular words. A major ingredient in that paper was a weaker version of Theorem 2.3. This version recognised that the union of a pair of palindromes containing each other's centres is periodic, but not that it was a palindromic periodicity. It might be that Theorem 2.3 might lead to a sharp bound. Similarly, paper [6] by Glen, Simpson and Smyth failed to obtain a sharp bound on the maximum number of distinct palindromes in an edge-labelled starlike tree.

Another direction would be to investigate the maximum number of maximal palindromic periodicities in a word. This would be analogous to the problem of determining the maximum number of maximal periodicities, also known as runs, that can occur in a word of length n. These are periodic factors, with length at least twice the period, which cannot be extended to the left or right without altering their periods. In 2000 Kolpakov and Kucherov [8] showed that the number was $O(n)$, without giving any information about the size of the implied constant. They conjectured that the number of runs was less than n. Then Rytter [11] showed the number was less than $5n$. This was followed by a sequence of increasingly long and complicated papers decreasing the bound. Then along came Bannai, I, Inenaga, Nakashima, Takeda, and Tsuruta who showed, with a very short and elegant proof, that the Kolpakov-Kucherov conjecture was correct. Since then the bound has been further decreased, best so far being 183/193 by Štěpán Holub [7].

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