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On 3-isoregularity of multicirculants

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Abstract

A graph is said to be *k-isoregular* if any two vertex subsets of cardinality at most k, that induce subgraphs of the same isomorphism type, have the same number of neighbors. It is shown that no 3-isoregular bicirculant (and more generally, no locally 3-isoregular bicirculant) of order twice an odd number exists. Further, partial results for bicirculants of order twice an even number as well as tricirculants of specific orders, are also obtained. Since 3-isoregular graphs are necessarily strongly regular, a motivation for the above result about bicirculants is that it brings us a step closer to obtaining a direct proof of a classical consequence of the Classification of Finite Simple Groups, that no simply primitive group of degree twice a prime exists for primes greater than 5.

1 Introductory remarks

As a meeting point of combinatorics, geometry and algebra, strongly regular graphs have been in the interest of the mathematical community for quite a long time, with the first related publications dating back over 60 years ago [2]. Recall that a *strongly regular graph* Γ *with parameters* (n, k, λ, μ), alternatively an (n, k, λ, μ)-*strongly regular graph*, is a regular graph of order n and valency k, such that the number λ of 3-cliques K_3 in Γ containing a given edge is independent of the choice of edge, and the number μ of 2-claws $K_{1,2}$ in Γ containing a given non-edge is independent of the

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choice of non-edge. Note that the complement $\overline{\Gamma}$ of Γ is also a strongly regular graph, with parameters

$$(n, k', \lambda', \mu') = (n, n-k-1, n-2-2k+\mu, n-2k+\lambda). \qquad (1)$$

A strongly regular graph Γ is said to be *non-trivial* if $n \geq 2$ and both Γ and its complement $\overline{\Gamma}$ are connected graphs. Hereafter, we restrict ourselves to non-trivial strongly regular graphs. Also, the following equality for an (n, k, λ, μ)- strongly regular graph is well known:

$$k(k-\lambda-1) = \mu(n-1-k). \qquad (2)$$

Strongly regular graphs exist in such an abundance that a complete classification seems hopeless (see for example [13, 33, 38]). It is therefore sensible to restrict oneself to certain more tractable subclasses of strongly regular graphs. This explains an increased interest in strongly regular graphs satisfying certain additional regularity conditions. In this respect different directions have been taken. One approach is using the concept of a t-vertex condition. Following [17, 37] we say that a graph Γ satisfies the *t-vertex condition* for some $t > 1$, if for every positive integer $j \leq t$ the number of j-vertex subgraphs of Γ of a given (isomorphism) type containing a given pair (u, v) of vertices depends solely on whether u and v are equal, adjacent or non-adjacent. Clearly, the 2-vertex condition is satisfied if and only if the graph is regular, and the 3-vertex condition is satisfied if and only if the graph is strongly regular. Since all *rank* 3 *graphs* (that is, orbital graphs of transitive permutation groups of rank 3) satisfy the t-vertex condition for every $t > 1$, an ongoing research effort has been directed towards getting constructions of non-rank 3 strongly regular graphs with the t-vertex condition for $t \geq 4$ (see [6, 20, 21, 23, 35, 36, 37]). It was conjectured that there is a number t_0 such that no non-rank 3 graph satisfies the t_0-vertex condition (see [12]). Since the point graphs of generalized quadrangles $GQ(s, s^2)$, $s \geq 2$, are proved to satisfy the 7-vertex condition [37, Theorem 2], it follows that $t_0 \geq 8$.

Another important restriction imposed on strongly regular graphs is via the concept of isoregularity. For a graph Γ, let S be a subset of the vertex set $V(\Gamma)$. A vertex $v \in V(\Gamma)$ is said to be a *neighbor* of S if it is a neighbor of every vertex of S, and the number of neighbors of S is the *valency* of S. Given a positive integer $0 < k \leq |V(\Gamma)|$, if for any j-element subset S, $j \leq k$, the valency of S depends solely on the isomorphism type of the subgraph induced by S, we say that Γ is *k-isoregular*. Clearly, a graph is 1-isoregular if and only if its regular, and it is 2-isoregular if and only if it is strongly regular. The isoregularity condition is quite restrictive for it is known that every 5-isoregular finite graph is *homogeneous* (see [7, 14, 16]), that is, in such a graph every isomorphism between two subgraphs extends to an automorphism of the graph. This puts the main focus of interest on 3-isoregular and 4-isoregular graphs which are not homogeneous. While the McLaughlin graph on 275 vertices (see [35]) is the only known 4-isoregular graph, there are several constructions of 3-isoregular graphs, which are not homogeneous. For example, the above mentioned point graphs of generalized quadrangles $GQ(s, s^2)$, as well as certain triangle-free point graphs of partial quadrangles are 3-isoregular (see [35, 37] for more details). Note that generalized quadrangles are a subclass of partial quadrangles, where in the latter we are requiring that a point not on a line has at most one collinear point on this line as opposed to the generalized quadrangles where there is exactly one such point.

This brings us to the main purpose of this paper, where we approach the 3-isoregularity restriction on strongly regular graphs via semiregular automorphisms. Recall that a nonidentity automorphism of a graph on mn vertices is (m, n)-*semiregular*, if it has m cycles of length n in its cycle decomposition. It is worth mentioning a long standing conjecture regarding vertex-transitive graphs admitting semiregular automorphisms (see [1, 15, 24, 30] for more details and updates). A graph admitting a semiregular automorphism is referred to as a *multicirculant* (and n-*multicirculant* when the size of the corresponding cycles in the cycle decompositions, hereafter called *orbits*, needs to be specified). In particular, for $m = 1, 2, 3$ and 4, the corresponding multicirculants are called, respectively, n-*circulants*, n-*bicirculants*, n-*tricirculants* and n-*tetracirculants*. While everything is known about strong regularity of circulants (see [4]), the situation with bicirculants and tricirculants is still quite open, save for some structural results and some constructions (see [11, 25, 26, 28]). It is well known that a strongly regular circulant is necessarily a Paley graph, and it is easily checked that, with the exception of the 5-cycle, Paley graphs are not 3-isoregular. In this paper we show that there are no 3-isoregular graphs among n-bicirculants, n odd, and that there are no 3-isoregular graphs among n-tricirculants, n prime or coprime to a particular number depending on the corresponding strongly regular parameters (see Corollaries 4.4 and 6.4). In fact we prove a somewhat more general result, that is, the non-existence of the so-called locally 3-isoregular bicirculants and tricirculants with the above order restrictions (see Theorems 4.3 and 6.3, and Section 3 for the definition of locally 3-isoregular graphs). Let us also mention that, apart from the fact that our results hold for local 3-isoregularity, the proofs do not depend on the 3-isoregularity results from [9, Theorem 6.5.]. As for 3-isoregular n-bicirculants, n even, they do not exist unless n is divisible by 8. Furthermore, there are two such graphs of order 16, which happens to be the smallest admissible order for 3-isoregular graphs. One of them is the Clebsch graph, which is the folded 5-cube graph and also the point graph of the partial quadrangle $PQ(4, 1, 2)$, with parameters $(16, 5, 0, 2)$. The second graph is the Cartesian product $K_4 \square K_4$, also known as the line graph of the complete bipartite graph $K_{4,4}$, with parameters $(16, 6, 2, 2)$. By extension these two graphs are also examples of 3-isoregular 4-tetracirculants. This makes both the class of n-bicirculants, n even (see [26] for admissible parameters of these graphs), and tetracirculants as the next natural target in the investigation of 3-isoregular graphs.

As a final remark, since 3-isoregular graphs are a subclass of strongly regular graphs that satisfy the 4-vertex condition – and the latter is always satisfied for rank 3 graphs – the above mentioned result about non-existence of locally 3-isoregular n-bicirculants, n odd, brings us a step closer to obtaining a direct proof of a classical consequence of the Classification of Finite Simple Groups (CFSG), that no simply primitive group (i.e., a primitive group which is not doubly transitive) of degree twice a prime exists for primes greater than 5 (see [8, 27, 34, 39]). This would be achieved if one manages to show that no n-bicirculant, $n > 5$ a prime, satisfies the 4-vertex condition. In fact, we believe that this statement holds for all $n > 5$ odd.

2 Strongly regular multicirculants

As outlined in the introductory section, in this paper our main interest is in strongly regular and, more specifically, 3-isoregular multicirculants with semiregular automorphisms having a small number of orbits. Strongly regular circulants have been classified, they are necessarily Paley graphs (see Proposition 2.1). As for strongly regular bicirculants and tricirculants, the situation is quite a bit more complicated, as explained below.

2.1 Circulants

The complete classification of strongly regular circulants was was independently achieved by Bridges and Mena [4] (extensively using the results of Kelly [22]), Hughes, van Lint and Wilson [18], Ma [28], and partially by Marušič [32]. Using this classification and applying the characterization of 3-isoregular graphs in terms of the induced subgraphs on the neighbors' and non-neighbors' sets (see Proposition 2.2), one can easily see that the 5-cycle is the only nontrivial 3-isoregular strongly regular circulant (see Corollary 2.3).

Proposition 2.1 [4, 18, 28] *If Γ is a nontrivial strongly regular circulant, then Γ is isomorphic to a Paley graph $P(p)$ for some prime $p \equiv 1 \pmod 4$.*

For a graph Γ, a vertex v and an integer i, the *i-th subconstituent* $\Gamma_i(v)$ is defined as the subgraph of Γ induced by all vertices at distance i from v.

Proposition 2.2 [37, Proposition 4] *A strongly regular graph Γ is 3-isoregular if and only if the subconstituents $\Gamma_1(v)$ and $\Gamma_2(v)$ are strongly regular with parameters which do not depend on the choice of vertex v.*

In the next result we will show that the only nontrivial 3-isoregular strongly regular circulant graph is the five cycle C_5. The argument makes a direct use of the above two propositions and the well-known Hoffman's bound, which we now briefly recall for the case of strongly regular graphs (see for example [5, Proposition 1.3.2] or [10, Formula (3.22)]). If C is a clique in a strongly regular graph Γ with valency k and smallest eigenvalue $-\ell$, then the Hoffman's bound states that

$$|C| \leq 1 + \frac{k}{\ell}. \tag{3}$$

Corollary 2.3 *Let* $\Gamma = \mathrm{Circ}(n, S)$ *be a nontrivial strongly regular circulant with parameters* (n, k, λ, μ)*. Then* Γ *is not* 3*-isoregular unless* $\Gamma \cong C_5$*.*

PROOF: Let Γ be a 3-isoregular circulant. First of all, by Proposition 2.1, Γ is a Paley graph of prime order $p \equiv 1 \pmod 4$. It is clear that C_5 is 3-isoregular, so assume that $p \geq 13$. Pick a vertex v of Γ. The corresponding subconstituents $\Gamma_1(v)$ and $\Gamma_2(v)$ are both circulants of order $(p-1)/2$, and they are strongly regular by Proposition 2.2. Since $(p-1)/2$ is not a prime, $\Gamma_1(v)$ and $\Gamma_2(v)$ need to be trivial strongly regular graphs (a disjoint union of complete graphs mK_t, or a complete multipartite graph $K_{m\times t}$). Observe also that the valency of $\Gamma_1(v)$ is $(p-5)/4$. If $\Gamma_1(v) \cong K_{m\times t}$, then using $mt = (p-1)/2$ and $(m-1)t = (p-5)/4$ we get $m = (2p-2)/(p+3) = 2 - 8/(p+3)$, contradicting the integrality of m. If $\Gamma_1(v) \cong mK_t$, then we have $t - 1 = (p-5)/4$, and so Γ contains a clique of size $t + 1 = (p+3)/4$, contradicting (3) (recall that the valency and the smallest eigenvalue of Γ are $(p-1)/2$ and $(-1-\sqrt{p})/2$, respectively). □

2.2 Bicirculants

Let n be a positive integer, $n \geq 2$, and let $\mathbb{Z}_n^*$ and $\mathbb{Z}_n^\#$ denote, respectively, the group of all invertible elements and the set of all nonzero elements in $\mathbb{Z}_n$. Further, for a subset A of $\mathbb{Z}_n$ we let $A^\# = A \setminus \{0\}$, $A^c = \mathbb{Z}_n \setminus A$ and we let $\widehat{A} = \mathbb{Z}_n^\# \setminus A = (A^c)^\#$.

Let Γ be an n-bicirculant, and ρ a $(2, n)$-semiregular automorphism of Γ with orbits U and W. We can represent Γ by a triple of subsets of $\mathbb{Z}_n$ in the following way. Let $u \in U$ and $w \in W$. Let $S = \{s \in \mathbb{Z}_n^\# \mid u \sim \rho^s(u)\}$ be the symbol of the n-circulant induced on U (relative to ρ). Similarly, let $S' = \{r \in Z_n^\# \mid w \sim \rho^r(w)\}$ be the symbol of the n-circulant induced on W (relative to ρ). Moreover, let $T = \{t \in \mathbb{Z}_n \mid u \sim \rho^t(w)\}$ define the adjacencies between the two orbits U and W. The ordered triple $[S, S', T]$ is called the *symbol* of Γ relative to the triple (ρ, u, w). Note that $S = -S$ and $S' = -S'$ are symmetric, that is, inverse-closed subsets of $\mathbb{Z}_n^\#$, and are independent of the particular choice of vertices u and w.

The above "triple-representation" is particularly nice in case of strongly regular n-bicirculants, n odd. Namely, in this case the above two sets S and S' are complementary in $\mathbb{Z}_n^\#$, a fact that proves to be crucial for the purpose of this paper. Using finite Fourier transform it was first proved in [31] for the case when n is a prime, and later extended also to the case when n is odd in [11]. The result below is a transcription of [11, Theorem 4.4] from their group rings terminology into our graph-theoretic language. We would also like to remark that the result about the parameters of strongly regular graphs is explicitly stated in [11, Theorem 4.4], whereas the fact that the two symbols S and S' are complementary can be deduced from the proof itself.

Theorem 2.4 *Let* n *be odd and let* Γ *be a non-trivial strongly regular* n*-bicirculant with parameters* $(2n, k, \lambda, \mu)$*. Then* $2n = (2m+1)^2+1$ *for some positive integer* m *and (up to taking complements) we have* $k = m(2m+1)$*,* $\lambda = m^2-1$ *and* $\mu = m^2$*. Further,* *there exist* $S = -S \subseteq \mathbb{Z}_n^{\#}$ *and* $T \subseteq \mathbb{Z}_n$ *such that the symbol of* Γ *is of the form* $[S, \widehat{S}, T]$*, and moreover* $|S| = m(m+1) = (n-1)/2$ *and* $|T| = m^2 = (n - \sqrt{2n-1})/2$.

Of course, the Petersen graph and its complement are examples of strongly regular bicirculants (arising from the action of S_5 on the 10 pairs of a 5-element set). It is a direct consequence of CFSG that no other rank 3 strongly regular p-bicirculant, p a prime, exists. There are known constructions of strongly regular n-bicirculants, n odd, for $n \in \{13, 25, 41, 61\}$ (two for $n = 41$) which do not arise from groups of rank 3 (see [29]). With the exception of $n = 13$ and one of the constructions for $n = 41$, all other graphs are vertex-transitive.

As expected from Corollary 4.4 of this paper none of these graphs is 3-isoregular. In the example below this is shown for the Petersen graph.

Example 2.5 Note that as a rank 3-graph, the Petersen graph automatically satisfies the t-vertex condition for every $t \leq 10$. But it is not 3-isoregular; the argument is fairly straightforward. Namely, since its girth is 5, it is clear that any subset of three vertices with at least one edge (in the induced subgraph) has no neighbors. On the other hand, independent subsets with three vertices are of two different types. Adopting one of the usual notations consistent with the above mentioned action of S_5 on the pairs from $\{1, 2, 3, 4, 5\}$ with adjacency following the empty intersection rule, we have that the independent subsets $\{12, 13, 23\}$ and $\{12, 13, 14\}$ have one neighbor and no neighbors, respectively.

As mentioned in the introductory section there are two 3-isoregular 8-bicirculants. There is an additional strongly regular 8-bicirculant, known as the Shrikhande graph, but it is not 3-isoregular. Their respective symbols are given in the example below.

Example 2.6 The (bicirculant) symbol of the Clebsch graph is $[\{\pm1, 4\}, \{\pm3, 4\}, \{0, 2\}]$. To see that it is 3-isoregular note that $\lambda = 0$ implies that it is a triangle-free graph and so, with the exception of three independent vertices, any other triple of vertices has valency zero. It can be checked that the valency of three independent vertices is one.

Next, the symbol of $K_4 \square K_4$ is $[\{\pm1\}, \{\pm3\}, \{0, 1, 3, 4\}]$. To see that it is 3-isoregular we use the fact that it is vertex-transitive, arc-transitive and distance-transitive, and compute the valencies of the subgraphs isomorphic to K_3, $K_{1,2}$, $K_2 + K_1$ and $\bar{K}_3$ to be respectively 1, 0, 1 and 0.

Finaly, the Shrikhande graph has two different representations as an 8-bicirculant with respective symbols being

$$[\{\pm1\}, \{\pm3\}, \{0, \pm1, 4\}] \text{ and } [\{\pm1, \pm2\}, \{\pm2, \pm3\}, \{1, 3\}].$$

It is not 3-isoregular as it admits induced subgraphs $K_{1,2}$ of different valencies, namely 0 and 1.

2.3 Tricirculants

Strong regularity of tricirculants has also been a matter of research interest. In [25] certain structural results for such graphs were obtained. In short, based on arithmetic conditions on the valency of subgraphs induced on the three orbits of a $(3, n)$-semiregular automorphism, two types of such graphs were identified (up to taking the complement). The first type is characterized by the fact that the induced subgraphs on the three orbits are of the same valency, while in the second type this is not the case.

Proposition 2.7 [25, Theorem 5.3] *Let* Γ *denote a strongly regular* n*-tricirculant with parameters* $(3n, k, \lambda, \mu)$*, where either* n *is a prime number or* n *is coprime to* $6\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}$*. Assume further that there is at least one edge in* Γ *with both endvertices in the same orbit of the* $(3, n)$*-semiregular automorphism* σ *of* Γ*, and that the number of edges of* Γ *with both endvertices in the same orbit of* σ *in* Γ *is smaller than or equal to the number of edges with both endvertices in the same orbit of* σ *in* $\overline{\Gamma}$*. Then there exists an integer* s*, such that either*

$$(3n, k, \lambda, \mu) = (3(12s^2 + 9s + 2), (4s + 1)(3s + 1), s(4s + 3), s(4s + 1)),$$

or

$$(3n, k, \lambda, \mu) = (3(3s^2 - 3s + 1), s(3s - 1), s^2 + s - 1, s^2).$$

In [25] constructions of graphs were given for $n \in \{5, 7, 19\}$. None of these graphs is 3-isoregular, as checked by Magma [3]. A brief explanation is given below for the case $n = 5$.

Example 2.8 The graph on 15 vertices is the point graph of the generalized quadrangle $GQ(2, 2)$. Its valency is 6. Alternatively, it can be thought of as the complement of the line graph $L(K_6)$ of the complete graph on 6 vertices, or, as a third possibility, as the line graph of the Petersen graph with added triangles inside each of the 5 blocks of imprimitivity (of size 3).

With regards to the number of neighbors, there are two kinds of independent sets with three vertices, respectively, with one neighbor and with three neighbors. We show this using the “the complement of $L(K_6)$” representation. Assume that the vertex set of K_6 is $\{1, 2, 3, 4, 5, 6\}$, and let Γ denote the complement of $L(K_6)$. Observe that $\{12, 13, 23\}$ and $\{12, 13, 14\}$ are independent subsets of Γ. However, the former one has three neighbours (namely 45, 46 and 56), while the later one has just one neighbour (namely 56).

3 Locally 3-isoregular graphs

In this section we introduce the concept of locally 3-isoregular graphs. Let Γ denote a finite, simple, connected graph. We say that a pair of distinct vertices

(x, y) of Γ is 3-*isoregular* if and only if the valency of any set $T = \{x, y, z\}$, $z \neq x, y$, depends solely on the isomorphism type of the subgraph induced on T. Furthermore, we say that Γ is *locally* 3-*isoregular at vertex* x, if there exist an edge (x, y) and a non-edge (x, z), both 3-isoregular. And moreover, we say that Γ is *locally* 3-*isoregular* if there exists a vertex x of Γ, such that G is locally 3-isoregular at x. Clearly, in a 3-isoregular graph all edges and non-edges are 3-isoregular. Note that if Γ has a 3-isoregular edge (x, y), then its complement $\overline{\Gamma}$ has a 3-isoregular non-edge (x, y). It follows that Γ is locally 3-isoregular at vertex x if and only if $\overline{\Gamma}$ is locally 3-isoregular at x.

Assume now that the edge (x, y) of Γ is 3-isoregular. Then there exist non-negative integers Q, R, W which are the valencies of a set $\{x, y, z\}$, $z \in V(\Gamma)$, depending, respectively, on whether the subgraph induced on $\{x, y, z\}$ is isomorphic to the triangle K_3, the 2-claw $K_{1,2}$, or the graph $K_2 + K_1$. We say that Q, R, W are *parameters associated with* the edge (x, y). The next result gives two relations on the parameters Q, R and W.

Proposition 3.1 *Let Γ be a strongly regular graph with parameters (n, k, λ, μ) and let (x, y) be a 3-isoregular edge of Γ with Q, R, W as the associated parameters. Then the following (i)–(ii) hold:*

(i) $\lambda(\lambda - Q - 1) = R(k - \lambda - 1)$,

(ii) $\lambda\mu(k - 2\lambda + Q) = W(k - \mu)(k - \lambda - 1)$.

Consequently, $W(k - \mu) = \mu(\lambda - R)$.

PROOF: For a vertex z of Γ and for a non-negative integer i, we set $\Gamma_i(z)$ to denote the set of vertices of Γ at distance i from z. Furthermore, for $0 \leq i, j \leq 2$, we set

$$D^i_j = D^i_j(x, y) = \Gamma_i(x) \cap \Gamma_j(y).$$

Note that $D^0_1 = \{x\}$, $D^1_0 = \{y\}$, $D^0_0 = \emptyset$, and that we have

$$|D^1_1| = \lambda, \quad |D^1_2| = |D^2_1| = k - \lambda - 1,$$

$$|D^2_2| = n - \sum_{(i,j) \neq (2,2)} |D^j_i| = n - k - 1 = \frac{(k - \mu)(k - \lambda - 1)}{\mu}.$$

In order to prove (i), observe that by the definition of parameter R, every vertex $z \in D^2_1$ has exactly R neighbors in D^1_1, and consequently exactly $\lambda - Q - 1$ neighbors in D^2_1. Now if $D^1_1 = \emptyset$, then $R = \lambda = 0$, and the above equality holds. Assume therefore that $D^1_1 \neq \emptyset$. Then, by the definition of parameter Q, every vertex $z \in D^1_1$ has exactly Q neighbors in D^1_1. Now counting the number of edges between D^1_1 and D^2_1 in two different ways (taking into account the above formulae for the cardinalities of these sets), we get the desired equality.

As for (ii), observe that by the definition of parameter W, every vertex $z \in D_2^2$ has exactly W neighbors in D_1^1. Similarly as above, counting the number of edges between D_1^1 and D_2^2 in two different ways (taking into account the above formulae for the cardinalities of these sets), we get the desired equality.

The last claim of the Proposition 3.1 now follows immediately from (i), (ii) above. □

Assume now that a non-edge (x, z) $(x \neq z)$ of Γ is 3-isoregular. Then there exist non-negative integers R', W', V which are the valencies of a set $\{x, z, u\}$, $u \in V(\Gamma)$, depending, respectively, on whether the subgraph induced on $\{x, z, u\}$ is isomorphic to the 2-claw $K_{1,2}$, the graph $K_2 + K_1$, or the graph $3K_1$. In this case we say that R', W', V are *parameters associated with* the non-edge (x, z). The next result gives two relations on the parameters R', W' and V.

Proposition 3.2 *Let Γ be a strongly regular graph with parameters (n, k, λ, μ) and let (x, z) be a 3-isoregular non-edge of Γ with R', W', V as the associated parameters. Then the following (i)–(ii) hold:*

(i) $\mu(\lambda - R') = (k - \mu)W'$,

(ii) $\mu(k - 2 - 2\lambda + R') = V\left(\frac{k(k-\lambda-1)}{\mu} - k + \mu - 1\right)$.

PROOF: The argument is analogous to the one used in the proof of Proposition 3.1. □

Proposition 3.3 *Let Γ be a strongly regular graph with parameters (n, k, λ, μ) and assume that Γ is locally 3-isoregular at vertex x. Let y be a neighbor of x such that (x, y) is 3-isoregular and Q, R, W are the associated parameters, and let z be a non-neighbor of x such that (x, z) is 3-isoregular and R', W', V are the associated parameters. Then $R = R'$ and $W = W'$.*

PROOF: For $0 \leq i, j \leq 2$ we set

$$D_j^i = D_j^i(x, z) = \Gamma_i(x) \cap \Gamma_j(z).$$

Note that we have

$$|D_1^1| = \mu, \quad |D_2^1| = |D_1^2| = k - \mu, \quad |D_2^2| = n - \sum_{(i,j) \neq (2,2)} |D_i^j| = \frac{k(k-\lambda-1)}{\mu} - k + \mu - 1.$$

Observe that as (x, z) is 3-isoregular, every vertex $v \in D_1^1$ has exactly R' neighbors in D_1^1, every vertex $v \in D_2^1 \cup D_1^2$ has exactly W' neighbors in D_1^1, and every vertex $v \in D_2^2$ has exactly V neighbors in D_1^1. We split the proof into two cases, depending on weather y is a neighbor of z (that is, $y \in D_1^1$), or a non-neighbor of z (that is, $y \in D_2^1$).

Assume first that $y \in D_1^1$. By the comments above, y must have R' neighbors in D_1^1. However, since (x,y) is 3-isoregular, x,y,z have R common neighbors, implying that $R = R'$. By Proposition 3.2(i) we therefore have

$$\mu(\lambda - R) = (k - \mu)W'.$$

The equality $W = W'$ now follows from Proposition 3.1.

If $y \in D_2^1$ then we have that $W = W'$ using an argument analogous to the one that was used in the previous case to obtain $R = R'$. By Proposition 3.2(i) we therefore have

$$\mu(\lambda - R') = (k - \mu)W,$$

and again using Proposition 3.1 we get $R = R'$. □

4 Non-existence of local 3-isoregular strongly regular n-bicirculants, n odd

Let Γ denote a strongly regular n-bicirculant of order $2n$, n odd. Recall that, by Theorem 2.4, parameters of Γ satisfies

$$(2n, k, \lambda, \mu) = (2(2m^2 + 2m + 1), m(2m + 1), m^2 - 1, m^2),$$

or

$$(2n, k, \lambda, \mu) = (2(2m^2 + 2m + 1), (m + 1)(2m + 1), m(m + 2), (m + 1)^2),$$

for some positive integer m (see also [11, 31, 39, 40]). In this section we show that such a graph is not locally 3-isoregular at any vertex x. Note that if Γ has parameters $(2(2m^2 + 2m + 1), m(2m + 1), m^2 - 1, m^2)$, then $\overline{\Gamma}$ has parameters $(2(2m^2 + 2m + 1), (m + 1)(2m + 1), m(m + 2), (m + 1)^2)$, and viceversa. Hence, in our discussion, we can focus on just one of the two parameters choice above.

If $m = 1$, then the above two parameter sets correspond to the Petersen graph and its complement. Note that every edge of the Petersen graph is 3-isoregular (and so every non-edge of its complement is also 3-isoregular). On the other hand, none of the non-edges of the Petersen graph is 3-isoregular (and so none of the edges of its complement is 3-isoregular). Therefore, for the rest of this section we may assume $m \geq 2$.

Proposition 4.1 *Let Γ be a strongly regular bicirculant with parameters $(2(2m^2 + 2m + 1), m(2m + 1), m^2 - 1, m^2)$, where $m \geq 2$, and let (x,y) be a 3-isoregular edge of Γ with the associated parameters Q, R, W. Then m is odd and*

$$Q = \frac{m^2 - m - 4}{2}, \quad R = \frac{m^2 - 1}{2}, \quad W = \frac{m(m - 1)}{2}.$$

PROOF: Using Proposition 3.1(i) we find that

$$R = \frac{(m-1)(m^2-Q-2)}{m},$$

and so $Q+2 = \alpha m$ for some positive integer α (recall that $Q+2 \geq 2$). It follows that $R = (m-1)(m-\alpha)$. Since $R \geq 0$, we have that $1 \leq \alpha \leq m$.

We claim that $\alpha < m$. Assume to the contrary that $\alpha = m$. Then $Q = m^2 - 2$ and $R = 0$. Let $D_1^1 = \Gamma_1(x) \cap \Gamma_1(y)$ and note that $|D_1^1| = \lambda = m^2 - 1$. As $Q = m^2 - 2$, this shows that $D_1^1 \cup \{x, y\}$ is a clique of cardinality m^2+1 in Γ. But this contradicts the Hoffman bound (3). This shows that $\alpha < m$.

Using Proposition 3.1(ii) we find that

$$W = \frac{(\alpha+1)(m-1)m}{m+1}.$$

If m is even then $\gcd(m-1, m+1) = 1$. It follows that $m+1$ divides $\alpha+1$, which is impossible as $1 \leq \alpha \leq m-1$. If m is odd then $\gcd(m-1, m+1) = 2$. It follows that $(m+1)/2$ divides $\alpha+1$, and so $\alpha = (m-1)/2$, which gives us that $R = (m^2-1)/2$. The expressions for Q and W then follow using Proposition 3.1. □

We next prove that graph Γ from Proposition 4.1 is not locally 3-isoregular at any vertex.

Proposition 4.2 *Let Γ be a strongly regular bicirculant with parameters $(2(2m^2 + 2m+1), m(2m+1), m^2-1, m^2)$ for some $m \geq 1$, and let x be a vertex of Γ. Then Γ is not locally 3-isoregular at x.*

PROOF: If $m = 1$, then Γ is the Petersen graph, and the result follows in view of the discussion in the beginning of this section. Assume therefore that $m \geq 2$ and that Γ is locally 3-isoregular at x. Let $y, z \in V(\Gamma)$ such that (x, y) is a 3-isoregular edge, and (x, z) is a 3-isoregular non-edge of Γ. Then by Proposition 3.3 there exist nonnegative integers Q, R, W, V such that the triples Q, R, W and R, W, V are the respective associated parameters for (x, y) and (x, z). Then, by Proposition 4.1, it follows that m is odd and

$$Q = (m^2-m-4)/2, \ R = (m^2-1)/2, \ W = (m(m-1))/2.$$

Finally, using Proposition 3.2(ii) we now find that

$$V = \frac{m(m^2+2m-1)}{2(m+2)}.$$

Since m is odd, this implies that $m+2$ divides $m^2+2m-1 = m(m+2)-1$, a contradiction. □

Theorem 4.3 *Let Γ be a strongly regular n-bicirculant, n odd, and let x be a vertex of Γ. Then Γ is not locally 3-isoregular at x.*

PROOF: Recall that Γ is locally 3-isoregular at x if and only if $\overline{\Gamma}$ is locally 3-isoregular at x. Since either Γ or $\overline{\Gamma}$ has the strong regularity parameters $(2(2m^2+2m+1), m(2m+1), m^2-1, m^2)$, $m \geq 1$, the result follows from Proposition 4.2. □

The result about non-existence of 3-isoregular n-bicirculants in the n odd case, is now immediate.

Corollary 4.4 *There exists no 3-isoregular n-bicirculant for n odd.*

5 On non-existence of 3-isoregular n-bicirculants for n even

The aim of this section is to lay out the ground for future research on 3-isoregularity and more generally strong regularity of n-bicirculants, n even. The goal of this section is to prove that no locally 3-isoregular n-bicirculant, $n = 2m^2$, exists for m odd.

The following result of Leung and Ma [26] is essential in this respect. Note that the 'partial difference triple' refers to the triple (C, D, D') where $C = T$, $D = S$ and $D' = S'$ in our notation for bicirculants in Subsection 2.2. Also, $c = |C| = |T|$, $d = |D| = |D'| = |S| = |S'|$.

Proposition 5.1 [26, Theorem 3.1] *Up to complementation, the parameters for any non-trivial partial difference triples in cyclic groups are the following:*

(a) $(n; c, d; \lambda, \mu) = (2m^2+2m+1; m^2, m^2+m; m^2-1, m^2)$ *where* $m \geq 1$.

(b) $(n; c, d; \lambda, \mu) = (2m^2; m^2, m^2-m; m^2-m, m^2-m)$ *where* $m \geq 2$.

(c) $(n; c, d; \lambda, \mu) = (2m^2; m^2, m^2+m; m^2+m, m^2+m)$ *where* $m \geq 3$.

(d) $(n; c, d; \lambda, \mu) = (2m^2; m^2 \pm m, m^2; m^2 \pm m, m^2 \pm m)$ *where* $m \geq 2$.

By Proposition 5.1 there are four families of strongly regular n-bicirculants for n even (essentially two when one only takes into account parameters k, λ and μ and disregards the cardinalities of sets C, D and D' in their notation, respectively, T, S and S' in our notation). Namely, both (b) and (c) in Proposition 5.1 have their counterparts in the two variations of (d).

Using Propositions 3.1 and 3.2 we get the following expressions for parameters Q, W and V in terms of R, k and λ (using that $\lambda = \mu$).

$$\lambda Q = \lambda(\lambda - 1) - (k - \lambda - 1)R.$$

$$(k - \lambda)W = \lambda(\lambda - R).$$

$$V\Big(\frac{k(k-\lambda-1)}{\lambda}-k+\mu-1\Big)=\lambda(k-2-2\lambda+R).$$

We deal first with case (b).

Proposition 5.2 *For m odd, there are no locally 3-isoregular graphs in family (b) of Proposition 5.1.*

Proof: We have that $k = 2m^2 - m$ and $\lambda = m^2 - m$, giving us the following expressions for the above parameters:

$$Q = m^2 - m - 1 - \frac{(m+1)R}{m}, \tag{4}$$

$$W = (m-1)^2 - \frac{(m-1)R}{m}, \tag{5}$$

$$V = \frac{m(m-2+R)}{m+2}. \tag{6}$$

Now since all of these parameters are non-negative integers we get that m divides R, and also that $R \le m^2 - m$ and that

$$R \le \frac{m(m^2-m-1)}{m+1} = (m-1)^2 - \frac{1}{m+1} < (m-1)^2.$$

Since the last inequality is sharp and R is an integer, we can rewrite it as $R \le m(m-2)$. Hence, we can say that there is a non-negative integer α such that $R = \alpha m$, and furthermore,

$$\alpha \le m - 2. \tag{7}$$

Substituting $R = \alpha m$ in (6), we obtain that

$$V = \frac{m^2-2m+mR}{m+2} = \frac{m^2+2m-4m+mR}{m+2} = m + \frac{m(R-4)}{m+2} = m + \frac{m(\alpha m-4)}{m+2}.$$

Since we assume that m is odd, we have that m and $m+2$ are coprime, and so $m+2$ is a divisor of $\alpha m - 4 = (\alpha m + 2\alpha - 2\alpha - 4)$, thus $m+2$ is a divisor of $2\alpha + 4$ and hence of $\alpha + 2$. In particular, we have that $m \le \alpha$ contradicting (7). □

We deal next with case (c).

Proposition 5.3 *For m odd, there are no locally 3-isoregular graphs in family (c) of Proposition 5.1.*

PROOF: Assume to the contrary, that there exists Γ in family (c) of Proposition 5.1, that is locally 3-isoregular at vertex x. Fix a neighbor y and a non-neighbor z of x, such that (x, y) and (x, z) are 3-isoregular. Let Q, R, W, V be the associated parameters. Recall that $k = 2m^2 + m$, and $\lambda = m^2 + m$, giving us the following expressions for the above parameters:

$$Q = m^2 + m - 1 - \frac{(m-1)R}{m}, \tag{8}$$

$$W = (m+1)^2 - \frac{(m+1)R}{m}, \tag{9}$$

$$V = \frac{m(R-m-2)}{m-2}. \tag{10}$$

Again, there exists a non-negative integer α such that $R = \alpha m$. Using this fact in the expression for V in (10), we have, with an analogous argument as in the previous proposition, that $m-2$ divides $2\alpha - 4$. Since m is odd this implies that $m-2$ divides $\alpha - 2$, and so $m \leq \alpha$. Furthermore, note that it follows from the definition of parameter V that $V \leq \mu$. Therefore, (10) implies that $R = \alpha m \leq m^2$, and consequently $\alpha \leq m$. The above two inequalities thus imply that $\alpha = m$, and so we have the following values for Q, R, W and V:

$$R = m^2, \quad Q = 2m-1, \quad W = m+1, \quad V = m(m+1).$$

We will now argue that this is not possible. First note that the eigenvalues of Γ are $k = 2m^2 + m$ and

$$r, s = \frac{(\lambda - \mu) \pm \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}{2} = \pm\sqrt{k-\mu} = \pm m.$$

By Hoffman's bound, any clique C in Γ satisfies

$$|C| \leq 1 + \frac{k}{m} = 2 + 2m.$$

Define the sets $D_1^1 = \Gamma(x) \cap \Gamma(z)$ and $D_2^1 = \Gamma(x) \cap \Gamma_2(z)$, and observe that $\Gamma(x) = D_1^1 \cup D_2^1$. Note that $|D_1^1| = \mu = m^2 + m$, and consequently $|D_2^1| = k - |D_1^1| = 2m^2 + m - m^2 - m = m^2$. Pick a vertex $w \in D_2^1$ and note that w has exactly $W = m+1$ neighbors in D_1^1. Since x and w must have exactly $\lambda = m^2 + m$ common neighbors, it follows that w has exactly $m^2 - 1$ neighbors in D_2^1. As $|D_2^1| = m^2$, this implies that the subgraph of Γ, induced on $\{x\} \cup D_2^1$, is a clique with cardinality $1 + m^2$. Using the above Hoffman's bound we now get that

$$1 + m^2 \leq 2 + 2m,$$

a contradiction as $m \geq 3$ by Proposition 5.1. □

Remark 5.4 Of course, the same approach follows also in the case m even but here we do have to consider also the possibility that $(m \pm 2)/2$, respectively, divide $\alpha \pm 2$ and so the computation for the parameters Q, R, W and V gives us the following, respectively for cases (b) and (c):

$$Q = W = (m^2 - m)/2, R = V = (m^2 - 2m)/2$$

and

$$Q = W = (m^2 + m)/2, R = V = (m^2 + 2m)/2.$$

Remark 5.5 In view of Propositions 5.2 and 5.3 a 3-isoregular n-bicirculant, n even, may only exist when $n = 2m^2$ for m even. As mentioned in Example 2.6 the Clebsch graph and $K_4 \Box K_4$ are 3-isoregular 8-bicirculants, fitting the above requirement for $m = 2$. The next possibility for such a 3-isoregular bicirculant could occur for $m = 4$. Note that in this case the parameters of a potential 3-isoregular bicirculant would be $(64, 28, 12, 12)$ or $(64, 36, 20, 20)$. While the exact number of strongly regular graphs with these parameters is not known, the results in [19] show that there is at least 11,063,360 strongly regular graphs with parameters $(64, 28, 12, 12)$, and at least 8,613,977 strongly regular graphs with parameters $(64, 36, 20, 20)$.

6 Local 3-isoregularity of strongly regular tricirculants

Let Γ denote a strongly regular n-tricirculant with parameters $(3n, k, \lambda, \mu)$, where either n is a prime number or n is coprime to $6\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}$. Assume further that there is at least one edge in Γ with both endvertices in the same orbit of the $(3, n)$-semiregular automorphism σ of Γ, and that the number of edges of Γ with both endvertices in the same orbit of σ in Γ is smaller than or equal to the number of edges with both endvertices in the same orbit of σ in $\overline{\Gamma}$. In this section, we show that Γ is not locally 3-isoregular with respect to any vertex x of Γ. First recall that, by Proposition 2.7, there exists an integer s, such that either

$$(3n, k, \lambda, \mu) = (3(12s^2 + 9s + 2), (4s + 1)(3s + 1), s(4s + 3), s(4s + 1)),$$

or

$$(3n, k, \lambda, \mu) = (3(3s^2 - 3s + 1), s(3s - 1), s^2 + s - 1, s^2).$$

Proposition 6.1 *With notation as above, assume that the parameters of Γ satisfy*

$$(3n, k, \lambda, \mu) = (3(12s^2 + 9s + 2), (4s + 1)(3s + 1), s(4s + 3), s(4s + 1)).$$

Then an edge (x, y) of Γ is 3-isoregular if and only if $s = -1$, in which case Γ is isomorphic to the complement of the triangular graph $T(6)$.

Proof: Assume that the edge (x, y) is 3-isoregular and let Q, R, W denote the associated parameters. Observe that if $s = 0$, then $3n = 6$, $k = 1$, and so Γ is isomorphic to $3K_2$, contradicting our assumptions that Γ is connected. Therefore, $s \neq 0$. Using Proposition 3.1(i) we get

$$R = \frac{(4s+3)(4s^2+3s-Q-1)}{4(2s+1)}.$$

Note that $(4s+3) - 2(2s+1) = 1$, and so for every integer s we have $\gcd(4s+3, 2s+1) = 1$. Now clearly $4s+3$ is odd, and so also $\gcd(4s+3, 4(2s+1)) = 1$. It follows that $4(2s+1)$ divides $4s^2+3s-Q-1$, that is,

$$Q = 4s^2 + 3s - 1 - 4\alpha(2s+1)$$

for some integer α. Note that this yields $R = \alpha(4s+3)$. Using Proposition 3.1(ii) we get

$$W = \frac{s(s-\alpha)(4s+3)}{2s+1}.$$

Observe that $\gcd(s(4s+3), 2s+1) = 1$, and so $2s+1$ divides $s - \alpha$. It follows that $\alpha = s - \beta(2s+1)$ for some integer β. This gives us

$$W = \beta s(4s+3), \quad R = -(4s+3)(2\beta s - s + \beta),$$

and so $W \geq 0$ implies $\beta \geq 0$. Now if $s \geq 1$, then $R \geq 0$ gives us $\beta \leq s/(2s+1) < 1$. Similarly, if $s \leq -1$, then again $R \geq 0$ gives us $\beta \leq s/(2s+1) \leq 1$. This shows that either $\beta = 0$, or $s = -1$ and $\beta = 1$. If $\beta = 0$ then $Q = -(4s^2+s+1)$, contradicting $Q \geq 0$. If $s = -1$ and $\beta = 1$, then Γ is the complement of the triangular graph $T(6)$, see [25, Section 6]. We now show that in this case every edge (x, y) of Γ is 3-isoregular. Since Γ is distance transitive, it follows that either all edges are 3-isoregular, or none is. But it is easy to check (for example using Magma [3]) that a particular edge (x, y) of Γ is indeed 3-isoregular with $Q = 0, R = 0, W = 1$. □

Proposition 6.2 *With notation as above, assume that the parameters of* Γ *satisfy*

$$(3n, k, \lambda, \mu) = (3(3s^2-3s+1), s(3s-1), s^2+s-1, s^2).$$

Then none of the edges of Γ *is* 3*-isoregular.*

Proof: Suppose to the contrary that an edge (x, y) is 3-isoregular and let Q, R, W denote the associated parameters. Observe that, if $s = 1$, then $3n = 3$, and so Γ is isomorphic to K_3, contradicting our assumptions that there exists an edge with both endvertices in the same orbit of σ. If $s \in \{0, -1\}$, then $\lambda = -1$, a contradiction. Therefore, $s \notin \{-1, 0, 1\}$. Using Proposition 3.1(i), we get

$$R = \frac{(s^2+s-1)(s^2+s-Q-2)}{2(s-1)s}.$$

Note that $(2s-1)(s^2+s-1)-(2s+3)s(s-1)=1$, and so for every integer s we have $\gcd(s^2+s-1, s(s-1))=1$. Now clearly s^2+s-1 is odd, and $\gcd(s^2+s-1, 2s(s-1))=1$. It follows that $2s(s-1)$ divides $s^2+s-Q-2$, that is,

$$Q = s^2 + s - 2 - 2\alpha s(s-1)$$

for some integer α. Note that this yields $R=\alpha(s^2+s-1)$. Since R is non-negative, we get that $\alpha \geq 0$. Using Proposition 3.1(ii), we get

$$W = -\frac{(\alpha-1)s(s^2+s-1)}{2s-1}.$$

Observe that since $s \notin \{0,-1\}$, we have that $s(s^2+s-1)/(2s-1)$ is positive. But as W is non-negative, this yields $\alpha \leq 1$. It follows that $\alpha \in \{0,1\}$.

If $\alpha = 1$, then $0 \leq Q = -(s-1)(s-2)$ implies $s=2$. In this case Γ is isomorphic to the triangular graph $T(7)$, see [25, Section 6]. We now show that in this case none of the edges of Γ is 3-isoregular. Since Γ is distance transitive, it follows that either all edges are 3-isoregular, or none is. But it is easy to check (for example using Magma [3]) that a particular edge (x,y) of Γ is indeed not 3-isoregular.

If on the other hand $\alpha = 0$, then we have $W = s(s^2+s-1)/(2s-1)$. It follows that $2s-1$ divides $s(s^2+s-1)$. But now

$$\frac{8s(s^2+s-1)}{2s-1} = 4s^2 + 6s - 1 - \frac{1}{2s-1}$$

implies that $2s-1$ divides 1, a contradiction. □

Theorem 6.3 *Let Γ be a strongly regular n-tricirculant with parameters $(3n,k,\lambda,\mu)$ where either n is a prime number or n is coprime to $6\sqrt{(\lambda-\mu)^2+4(k-\mu)}$. Then for every vertex $x \in V(\Gamma)$, it follows that Γ is not locally 3-isoregular at x.*

Proof: Assume first that Γ is the complement of the triangular graph $T(6)$. Then it is easy to see that none of the non-edges of Γ is 3-isoregular, implying that neither of Γ and $\overline{\Gamma}$ is locally 3-isoregular at x. Assume now that Γ is not isomorphic to the triangular graph $T(6)$ or its complement. If the number of edges of Γ that are contained in the orbits of a $(3,n)$-semiregular automorphism σ of Γ is smaller than or equal to the number of edges of $\overline{\Gamma}$ contained in the orbits of σ, then Propositions 6.1 and 6.2 shows that Γ is not 3-isoregular at x. If the number of edges of Γ that are contained in the orbits of σ is greater than the number of edges of $\overline{\Gamma}$ contained in the orbits of σ, then Propositions 6.1 and 6.2 shows that none of the non-edges (x,y) of Γ is 3-isoregular, and so again Γ is not 3-isoregular at x. □

Corollary 6.4 *Let Γ be a strongly regular n-tricirculant with parameters $(3n,k,\lambda,\mu)$ where either n is a prime number or n is coprime to $6\sqrt{(\lambda-\mu)^2+4(k-\mu)}$. Then Γ is not 3-isoregular.*

Remark 6.5 We would like to point out that Theorem 6.3 considers only strongly regular n-tricirculants, which parameters satisfy the condition stated in the statement of Theorem 6.3. For the proof of Theorem 6.3, the description of the parameters of these graphs that is given in Proposition 2.7, was crucial. A similar description of the parameters of strongly regular n-tricirculants for which the above mention condition is not satisfied is still to be obtained.

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