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Quasiperiods of magic labeling quasipolynomials

Margaret Bayer

University of Kansas, Lawrence, KS, U.S.A.
bayer@ku.edu

Amanda Burcroff

Harvard University, Cambridge, MA, U.S.A.
aburcroff@math.harvard.edu

Tyrrell B. McAllister

University of Wyoming, Laramie, WY, U.S.A.
tmcallis@uwyo.edu

Leilani Pai

Denison University, Granville, OH, U.S.A.
pail@denison.edu

Abstract

A magic labeling of a graph is a labeling of the edges by nonnegative integers such that the label sum over the edges incident to every vertex is the same. This common label sum is known as the index. We count magic labelings by maximum edge label, rather than index, using an Ehrhart-theoretic approach. In contrast to Stanley's 1973 work showing that the function counting magic labelings with bounded index is a quasipolynomial with quasiperiod 2, we show by construction that the minimum quasiperiod of the quasipolynomial counting magic labelings with bounded maximum label can be arbitrarily large, even for planar bipartite graphs. Unfortunately, this rules out a certain Ehrhart-theoretic approach to proving Hartsfield and Ringel's Antimagic Graph Conjecture. However, we show that this quasipolynomial is in fact a polynomial for any bipartite graph with matching preclusion number at most 1, which includes any bipartite graph with a leaf.

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1 Introduction

A **magic labeling** of a graph is a function assigning to each edge of the graph a nonnegative integer so that the sum of the labels on the edges containing each vertex is the same. This common sum at each vertex is called the **index** of the magic labeling. The study of magic labelings of graphs was initiated by a problem proposed by Sedláček in 1966 [9], and the first paper devoted to the topic was by Stewart [12]. Interest in this topic grew significantly after the publication of Stanley's paper, "Linear homogeneous Diophantine equations and magic labelings of graphs" [11]. In that paper, Stanley showed that the number of magic labelings with index k is a quasipolynomial function of k with minimum quasiperiod at most 2.

In this paper, we count the number of magic labelings of a graph by the maximum label used, rather than by the index. The function that counts the number of magic labelings with maximum label at most k is again a quasipolynomial. Our main result (Theorem 2.7 below) is that, in contrast to Stanley's result, the minimum quasiperiod of this quasipolynomial is unbounded. We show this by constructing, for each positive integer n, a planar bipartite graph for which the minimum quasiperiod is n.

This result is motivated by work of Beck and Farahmand in [1] on *anti*magic labelings, which are labelings of the edges of a graph by distinct labels in $\{1, 2, \ldots, |E|\}$ such that the sums of the labels at each vertex are *distinct*. A famous open problem from 1990 posed in [5] is whether all connected graphs (except for K_2) have an antimagic labeling. See [3, Chapter 6] and [6] for comprehensive summaries of progress on this problem.

Beck and Farahmand pursued a strategy for proving a weakened form of this conjecture, namely that for some fixed $s \geq 1$, every connected graph (except for K_2) has a labeling using only labels in $\{1, \ldots, s|E|\}$ (allowing repeated labels) where the sums of the labels at each vertex are distinct. As shown in [1], this claim would follow if it were known that the function that counts the number of magic labelings with maximum label k has minimum quasiperiod at most s for every graph.

Unfortunately, our main result (Theorem 2.7 below) shows that no such bound on this minimum quasiperiod exists for general graphs. While a claim to the contrary appears in [1, Theorem 4], the authors of [1] report in private correspondence that the proof of their Theorem 4 contained an error and that an erratum is forthcoming. As a consequence of our Theorem 2.7, the approach explored in [1] cannot succeed without significant modification.

While this minimum quasiperiod is unbounded in general, we show that the minimum quasiperiod is bounded for certain classes of graphs (see Section 5). In particular, we show that, if the graph has a leaf, then this quasipolynomial has minimum quasiperiod at most 2. Furthermore, if the graph is additionally bipartite, then this quasipolynomial is in fact a polynomial. This suggests that Beck and Farahmand's approach may be adapted for special families of graphs.

2 Preliminaries

Throughout this paper we let $G = (V, E)$ be an undirected graph with vertex set V and edge set E, allowing multiple edges and loops.

An (**edge**) **labeling** of G is a function $E \to \mathbb{Z}_{\geq 0}$. We view the labelings of G as the points of the integer lattice $\mathbb{Z}^E$ that are in the positive orthant $\mathbb{R}^E_{\geq 0}$ of the vector space $\mathbb{R}^E$. For each $L \in \mathbb{R}^E$, write

$$s_L(v) \coloneqq \sum_{\substack{e \in E,\\ e \text{ incident to } v}} L(e)$$

for the sum of the labels of the edges incident to v. A labeling L is **magic** if the value of $s_L(v)$ is the same for each vertex v of G. Thus, the magic labelings are the lattice points in the polyhedral cone $C_G \subseteq \mathbb{R}^E$ defined by

$$C_G \coloneqq \left\{ L \in \mathbb{R}^E_{\geq 0} \,:\, s_L(v) = s_L(w) \text{ for all } v, w \in V \right\}.$$

See [3, Chapter 5] for a survey of results concerning magic labelings and related notions.

The primary object of study in this paper is the rational polytope $P_G \subseteq \mathbb{R}^E$ defined by

$$P_G \coloneqq C_G \cap [0, 1]^E.$$

For $k \in \mathbb{Z}_{\geq 0}$, a labeling L is a k**-labeling** if $L(e) \leq k$ for all edges $e \in E$. Thus, the magic k-labelings of G are precisely the integer-lattice points in the k^{th} dilate $kP_G \coloneqq \{kL \,:\, L \in P_G\}$ of P_G. We are in particular interested in the function

$$M_G(k) \coloneqq \left| kP_G \cap \mathbb{Z}^E \right|$$

that counts the number of magic k-labelings of G.

A better-studied counting function in the context of magic labelings is the function $S_G(k)$ that counts the magic labelings of G with index exactly k, where the **index** of a magic labeling L of G is the common value of $s_L(v)$ for all $v \in V$. This function corresponds to the polytope

$$Q_G \coloneqq \left\{ L \in \mathbb{R}^E_{\geq 0} \,:\, s_L(v) = 1 \text{ for all } v \in V \right\},$$

since the number of magic labelings of G with index k is

$$S_G(k) = \left| kQ_G \cap \mathbb{Z}^E \right|.$$

Note that, since labelings are nonnegative, a magic labeling with index k is in particular a magic k-labeling. The corresponding statement regarding the polytopes is that $Q_G \subseteq P_G$. However, the dimension of Q_G is always strictly less than that of P_G.

From the point of view of Ehrhart theory, the definitions of M_G and S_G immediately imply that they are the *Ehrhart quasipolynomials* of the polytopes P_G and Q_G, respectively. We briefly explain this connection here, but we refer the reader to [2] for a thorough introduction to Ehrhart theory, including the properties of Ehrhart quasipolynomials stated here.

A function $F\colon \mathbb{Z} \to \mathbb{C}$ (or $F\colon \mathbb{Z}_{\geq 0} \to \mathbb{C}$) is a **quasipolynomial** of **degree** d if there exist an integer $s \in \mathbb{Z}_{\geq 1}$ and polynomials $\phi_1, \dots, \phi_s \in \mathbb{C}[x]$, called the **constituents** of F, such that $d = \max\{\deg(\phi_1), \dots, \deg(\phi_s)\}$ and $F(t) = \phi_r(t)$ whenever $t \equiv r \pmod{s}$. Such a positive integer s is a **quasiperiod** of F. The quasiperiods of F are precisely the positive integer multiples of the *minimum* quasiperiod of F, which we denote $\mathrm{mqp}(F)$. Alternatively, F is a quasipolynomial of degree d if and only if $F(t) = \sum_{i=0}^{d} c_i(t)\, t^i$ for some sequence $c_0, \dots, c_d$ of periodic **coefficient functions** $\mathbb{Z} \to \mathbb{C}$ with c_d not identically zero. Writing s_i for the minimum period of c_i for $0 \leq i \leq d$, we then have that $\mathrm{mqp}(F) = \mathrm{lcm}\{s_0, \dots, s_d\}$. Note that a constant function c_i has period 1. In particular, for $i > d$, we set $c_i = 0$ and $s_i = 1$.

Now let $P \subseteq \mathbb{R}^N$ be a d-dimensional **rational** polytope, meaning that $\mathrm{vert}(P) \subseteq \mathbb{Q}^N$, where $\mathrm{vert}(P)$ denotes the set of vertices of P. By a celebrated theorem of Eugène Ehrhart, the function $\mathrm{ehr}_P(t) \coloneqq \left|tP \cap \mathbb{Z}^N\right|$ for $t \in \mathbb{Z}_{\geq 1}$ is a degree-d quasipolynomial called the **Ehrhart quasipolynomial** of P. Moreover, the *minimum* quasiperiod of ehr_P divides the **denominator** of P, which is defined to be $\mathrm{den}(P) \coloneqq \min\left\{t \in \mathbb{Z}_{\geq 1} : \mathrm{vert}(tP) \subseteq \mathbb{Z}^N\right\}$. Thus, an upper bound on the denominator of P is also an upper bound on $\mathrm{mqp}(\mathrm{ehr}_P)$.

As discussed in the introduction, Beck and Farahmand showed in [1] that a proof of an upper bound on $\mathrm{mqp}(M_G)$ independent of G would suffice to prove a weakened version of an open problem regarding antimagic graph labelings. However, we find below that no such upper bound on $\mathrm{mqp}(M_G)$ exists.

In order to compute the denominator and minimum quasiperiod of a rational polytope $P \subseteq \mathbb{R}^N$, we study a related semigroup in $\mathbb{R}^{N+1}$. The **semigroup of** P, denoted by $\Phi(P)$, has elements (L, k) where k is a nonnegative integer and L is a lattice point in kP. That is, the semigroup $\Phi(P)$ consists of the integer points in the **homogenized cone over** P, which is the cone generated by $P \times \{1\}$ in $\mathbb{R}^{N+1}$. The binary operation in the semigroup is entry-wise addition.

Example 2.1 Let G be the graph with one node v and two loops at v. Then we can identify $\mathbb{R}^E$ with $\mathbb{R}^2$. Under this identification, P_G is the unit square $[0,1]^2$, and Q_G is the line segment with endpoints $(1,0)$ and $(0,1)$. In Figure 1, the semigroup $\Phi(P_G)$ is the set of integer-lattice points (not shown) in the blue cone, and $\Phi(Q_G)$ is the set of such points in the red cone.

Definition 2.2 A nonzero element a of an additive semigroup Φ is **completely fundamental** if, for all $b, c \in \Phi$, if $b + c = ma$ for some positive integer m, then b and c are both nonnegative integer multiples of a.

We remark that, when $\Phi = \Phi(P)$ is the semigroup of a rational polytope P, then the completely fundamental elements of Φ are precisely the points $(d_v v, d_v)$ where v

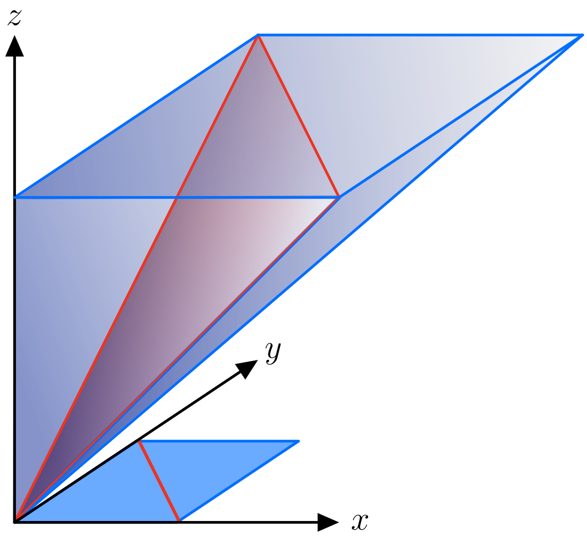


Figure 1: The polytopes P_G and Q_G from Example 2.1 are shown in the xy-plane in blue and red, respectively. A portion of the homogenized cone over each polytope is shown in the corresponding color.

is a vertex of P and $d_v := \mathrm{den}(v)$. Thus, the denominator of P can be expressed in terms of the completely fundamental elements of $\Phi(P)$.

Proposition 2.3 *The denominator of a polytope P is equal to the least common multiple of the final coordinates of the completely fundamental elements of $\Phi(P)$.*

When $P \in \{P_G, Q_G\}$, we say that a magic labeling L is a **completely fundamental (magic) labeling of** $\Phi(P)$ if (L, k) is a completely fundamental element of $\Phi(P)$ for some $k \in \mathbb{Z}_{\geq 0}$. An important subtlety is that which labelings are "completely fundamental" depends upon whether one is considering the polytope P_G of all magic labelings with labels $\leq k$ or the polytope Q_G of magic labelings of index exactly k.

Example 2.4 Let G be as in Example 2.1, with P_G and Q_G shown in Figure 1. The completely fundamental elements of $\Phi(P_G)$ are $(0, 0, 1)$, $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 1)$, corresponding to the four vertices of P_G. The completely fundamental elements of $\Phi(Q_G)$ are $(0, 1, 1)$ and $(1, 0, 1)$, corresponding to the two vertices of Q_G.

In 1973, Stanley proved a strong bound on the denominator of the polytope Q_G, which in turn implies the same bound on the minimum quasiperiod of the quasipolynomial $S_G(k)$ that counts the index-k magic labelings of G. Stanley stated his result in terms of the "completely fundamental magic labelings of G", by which he meant, in our nomenclature, the completely fundamental labelings of $\Phi(Q_G)$ specifically.

Proposition 2.5 ([11, Proposition 2.7]) *For a finite graph G, every completely fundamental magic labeling of $\Phi(Q_G)$ has index 1 or 2. If G is additionally bipartite, then every completely fundamental magic labeling of $\Phi(Q_G)$ has index 1.*

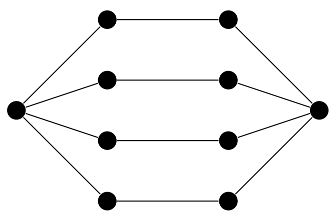
Applying Proposition 2.3 yields the following uniform bounds.

Corollary 2.6 ([11, Corollary 2.8]) *For all graphs G, the denominator of the polytope Q_G is at most 2. In particular, the minimum quasiperiod of $S_G(k)$ is at most 2.*

Our main theorem is that *no such bounds exist* on either the denominator of the polytope P_G or on the minimum quasiperiod of the quasipolynomial $M_G(k)$ that counts the magic k-labelings.

Theorem 2.7 *There exist graphs G for which the minimum quasiperiod of M_G is arbitrarily large. In particular, for each $n \in \mathbb{Z}_{\geq 2}$, there exists a graph G_n (on $2n+2$ vertices and $3n$ edges) such that P_{G_n} has a vertex with denominator $n-1$, and the minimum quasiperiod of the quasipolynomial M_{G_n} is $n-1$.*

Example 2.8 This example is the $n = 4$ case of the general construction of G_n given in Definition 3.1. Let G_4 be the following graph on 10 vertices.



The Ehrhart quasipolynomial $M_{G_4}(k)$ has minimum quasiperiod 3 and is given as follows:

$$M_{G_4}(k) = \begin{cases} \frac{1}{18}k^4 + \frac{4}{9}k^3 + \frac{25}{18}k^2 + 2k + 1 & \text{if } k \equiv 0 \pmod 3, \\ \frac{1}{18}k^4 + \frac{4}{9}k^3 + \frac{25}{18}k^2 + 2k + \frac{10}{9} & \text{if } k \equiv 1 \pmod 3, \\ \frac{1}{18}k^4 + \frac{4}{9}k^3 + \frac{25}{18}k^2 + 2k + 1 & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

The fact that only the degree-0 coefficient varies is explained by Theorem 4.8 below.

Since this graph is bipartite, Proposition 2.5 implies that $S_{G_4}(k)$ is a polynomial. Indeed, one can calculate explicitly that $S_{G_4}(k) = \frac{1}{6}k^3 + k^2 + \frac{11}{6}k + 1$. Note that the degree of $S_{G_4}(k)$ is less than that of $M_{G_4}(k)$ because $\dim(Q_{G_4}) < \dim(P_{G_4})$.

The proof of Theorem 2.7 appears in Section 4. The outline of the argument is as follows. The construction of G_n is at the beginning of Section 3, and the fact that P_{G_n} has a vertex with denominator $n-1$ is a direct consequence of Theorem 3.6. Now, since the denominator is only an upper bound on the minimum quasiperiod of M_G, the strategy explored in [1] might still be salvaged if P_G exhibited so-called “period collapse” [7]. However, in Propositions 4.1 and 4.2, we use a standard generating-function approach to establish that the Ehrhart quasipolynomial M_{G_n} of P_{G_n} has “full period”. That is, the quasiperiod of the quasipolynomial attains the upper bound in Proposition 2.3. In particular, the minimum quasiperiod of M_{G_n} is $n-1$, as claimed in Theorem 2.7.

Furthermore, it is possible, with nearly the same amount of work, to get much "higher resolution" information about the quasiperiodicity of M_{G_n}. In particular, in Theorem 4.8, we use a result of Sam and Woods (Theorem 4.5 below) to show that all of the quasiperiodicity of M_{G_n} is contained in its degree-0 coefficient, as seen in the $n = 4$ case in Example 2.8. This section of our paper may serve as an advertisement for the Sam–Woods approach to quasipolynomials developed in [8], which takes little more effort than the standard generating-function approach, while giving far more detailed information about quasiperiods.

The contrast between our results and Stanley's results is rooted in the difference between $\Phi(P_G)$ and $\Phi(Q_G)$. It follows from Theorem 2.7 and Proposition 2.3 that the analogue of Proposition 2.5 is false if we replace $\Phi(Q_G)$ with $\Phi(P_G)$, and moreover no uniform bound can be placed on the index of completely fundamental magic labelings of $\Phi(P_G)$.

However, an analogue of Proposition 2.5 does hold for certain types of graphs. We show in Section 5 that if G has an edge that attains the maximum label in every index-2 magic labeling, then the completely fundamental magic labelings of $\Phi(P_G)$ have index at most 2. Many graphs satisfy this property, including any graph with a leaf. Moreover, if G is additionally bipartite, then the completely fundamental magic labelings of $\Phi(P_G)$ have index at most 1. In particular, if G is any bipartite graph with a leaf, it follows that $M_G(k)$ is a polynomial.

Remark 2.9 The quasipolynomials $M_{G_n}(k)$ and their *partially magic* analogs are used by Beck and Farahmand to study $A_G(k)$, the quasipolynomial counting *weakly antimagic* labelings [1]. The quasipolynomial $A_G(k)$ can be written as a sum of the magic and partially magic quasipolynomials using inclusion-exclusion. As we will show, one of these summands can have arbitrarily large minimum quasiperiod. It is open whether the minimum quasiperiod of $A_G(k)$ can be arbitrarily large (the possiblility of unexpected cancellations must be ruled out), though we expect this to be the case. If G is the *butterfly graph* on 5 vertices, consisting of two 3 cycles that share a vertex, then one can explicitly compute $A_G(k)$ to see that it has minimum quasiperiod 6. This contradicts [1, Theorem 2]), which claims that the minimum quasiperiod is at most 2.

3 Unbounded completely fundamental labelings

In this section, we construct a family $\{G_n\}_{n\geq 2}$ of graphs for which there are completely fundamental magic labelings of $\Phi(P_{G_n})$ with arbitrarily large maximum label. This means that there is no uniform upper bound on the denominator of P_G.

Definition 3.1 For each integer $n \geq 2$, let G_n be the graph on the vertex set $\{a_1, \ldots, a_n, b_1, \ldots, b_n, x, y\}$ with the following $3n$ edges:

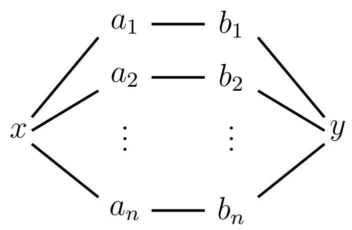


Figure 2: The construction of the graph G_n.

- an edge from a_i to b_i for each $i \in \{1, \dots, n\}$,
- an edge from x to a_i for each $i \in \{1, \dots, n\}$, and
- an edge from y to b_i for each $i \in \{1, \dots, n\}$.

(See Figure 2.)

We will construct a completely fundamental magic labeling of $\Phi(P_{G_n})$ with maximum edge label $n-1$.

Definition 3.2 Let L_* be the labeling on the edges of G_n where edges from a_i to b_i have label $n-1$ and all other edges of label 1.

It is straightforward to check that L_* is a magic labeling of index n, but it remains to show that this is a completely fundamental labeling of $\Phi(P_{G_n})$. In order to show this, we first consider the perfect matchings on G_n. A **perfect matching** in G is a subset J of the edges such that each vertex in G is incident to exactly one edge of J. Note that a magic labeling of G of index 1 can be identified with a perfect matching of G by taking the set of edges of G with label 1.

Definition 3.3 For $1 \le i \le n$, let L_i be the perfect matching on G_n formed by taking the edge from x to a_i, from y to b_i, and all edges from a_j to b_j for $j \ne i$.

Proposition 3.4 *Every perfect matchings on G_n is one of $L_1, \dots, L_n$.*

Let $\max(L)$ denote the maximum label appearing in a labeling L, and write $\vec{0}$ for the trivial magic labeling under which every edge is labeled 0.

Lemma 3.5 *Any element of $\Phi(P_{G_n})$ can be written as a nonnegative integer combination of the elements $(L_1, 1)$, $(L_2, 1)$, $\dots$, $(L_n, 1)$, $(\vec{0}, 1)$, and $(L_*, n-1)$.*

Proof. Fix (L, k) in $\Phi(P_{G_n})$. By subtracting off copies of $(\vec{0}, 1)$, we can assume $k = \max(L)$. For $1 \le i \le n$, let $u_i = L(xa_i)$, i.e., the label of the edge between x and a_i in L. The index of L is then $\sum_{i=1}^n u_i$. In order to have the correct sum at a_j, we must have $L(a_jb_j) = (\sum_{i=1}^n u_i) - u_j$. Similarly, $L(b_jy) = u_j$ in order to have the correct sum at b_j. The maximum label in L is then

$$\max(L) = \left(\sum_{i=1}^{n} u_i\right) - \min_{1 \le i \le n} u_i.$$

Let m be such that $u_m = \min_{1\leq i\leq n} u_i$. It is then straightforward to check that

$$(L, \max(L)) = \left(\sum_{i=1}^{n}(u_i - u_m)\cdot(L_i, 1)\right) + u_m \cdot (L_*, n-1).$$

Thus, we can conclude that $(L, \max(L))$ can be decomposed as desired. □

Note that the magic labeling L_* is equal to the sum of the magic labelings L_i for $1 \leq i \leq n$. However, if we additionally choose an appropriate bound for the maximum label, this no longer holds. That is, the element $(L_*, n-1)$ is not a sum of the elements $(L_i, 1)$ in $\Phi(P_{G_n})$, which is a consequence of the following result.

Theorem 3.6 *The completely fundamental elements of* $\Phi(P_{G_n})$ *are* $(L_1, 1)$, $(L_2, 1)$, $\ldots$, $(L_n, 1)$, $(\vec{0}, 1)$, *and* $(L_*, n-1)$.

Proof. By Lemma 3.5, it is enough to show that no multiple of one of these elements can be written as a positive integer combination of the others.

This clearly holds for $(\vec{0}, 1)$, since all other elements have a positive edge label. This also holds for $(L_*, n-1)$ since $(\vec{0}, 1)$ and the $(L_i, 1)$ satisfy the property that the last entry is at least the index of the labeling, and this property is preserved under sums. Lastly, the theorem statement holds for each element $(L_i, 1)$, as the only other element with label 0 on the edge between a_i and b_i is $(\vec{0}, 1)$ and no multiple of $(\vec{0}, 1)$ is equal to a multiple of $(L_i, 1)$. □

Remark 3.7 It follows from Lemma 3.5 and Theorem 3.6 that

$$\left\{(L_1, 1), \ldots, (L_n, 1), (\vec{0}, 1), (L_*, n-1)\right\}$$

is the (minimal) Hilbert basis of $\Phi(P_{G_n})$.

Remark 3.8 Following Stanley's result (Proposition 2.5), the completely fundamental elements of $\Phi(Q_{G_n})$ have entries in $\{0, 1\}$, and moreover $S_{G_n}(k)$ is a polynomial. Similar reasoning to that in Theorem 3.6 yields that the completely fundamental elements of $\Phi(Q_{G_n})$ are precisely $(L_1, 1)$, $(L_2, 1)$, $\ldots$, $(L_n, 1)$.

Remark 3.9 Each graph G_n can be generalized to a family of graphs $G_{n,p}$ for $p \geq 1$ as follows, while retaining the same Ehrhart quasipolynomial to count the magic labelings.

Let $G_{n,p}$ denote the graph obtained by taking two vertices x and y and connecting them by n distinct paths of length $2p+1$. Since this graph is bipartite, any magic labeling can be decomposed as a sum of perfect matchings. Moreover, a perfect matching of $G_{n,p}$ is determined by a choice of edge incident to x, as is the case for G_n. This induces a bijection between perfect matchings on G_n and perfect matchings on $G_{n,p}$, and it is straightforward to show that this bijection can be extended additively to magic labelings. Thus, $M_{G_n}(k) = M_{G_{n,p}}(k)$.

4 The Ehrhart quasipolynomial of P_{G_n}

This section is focused on studying the Ehrhart quasipolynomial of P_{G_n}, where G_n is the graph constructed in Section 3. In Subsection 4.1, we give an explicit formula for this Ehrhart quasipolynomial as a sum of certain binomial coefficients, and we use a generating function to find the minimum quasiperiod of this quasipolynomial, thus proving the main result (Theorem 2.7). In Subsection 4.2, we present the approach of Sam and Woods [8] for studying quasipolynomials via the difference operator. Finally, in Subsection 4.3, we apply this approach to find the period of each coefficient of the Ehrhart quasipolynomial of P_{G_n}, as given in Corollary 4.9.

4.1 The Ehrhart quasipolynomial of P_{G_n} and its quasiperiod

Let $M_n(k) \coloneqq M_{G_n}(k)$ be the Ehrhart quasipolynomial of P_{G_n}. Thus, $M_n(k)$ is the number of integral magic labelings of G_n with maximum label at most k. For all $n \geq 1$, define the function $F_n \colon \mathbb{Z}_{\geq 0} \to \mathbb{C}$ by

$$F_n(k) \coloneqq \sum_{\substack{j \in [0,k]_{\mathbb{Z}} \\ j \equiv k \pmod{n}}} \binom{j}{n}.$$

Proposition 4.1 *For $n \geq 2$ and nonnegative k,*

$$M_n(k) = \binom{k+n}{n} + F_{n-1}(k)\,.$$

Proof. As in the proof of Lemma 3.5, each magic labeling L of G_n is determined by the labels u_j that L assigns to the edges xa_j for $1 \leq j \leq n$. Moreover, the labeling L is a k-labeling if and only if the maximum label $\sum_{j=1}^{n} u_j - \min_{1 \leq \ell \leq n} u_\ell$ is at most k. Thus, we can directly calculate $M_n(k)$ as follows:

$$\begin{aligned}
M_n(k) &= \#\left\{(u_1, \ldots, u_n) \in \mathbb{Z}^n_{\geq 0} \,:\, \sum_{j=1}^{n} u_j - u_\ell \leq k \text{ for all } \ell\right\} \\
&= \#\left\{(u_1, \ldots, u_n) \in \mathbb{Z}^n_{\geq 0} \,:\, \sum_{j=1}^{n} u_j \leq k\right\} \\
&\quad + \sum_{i \geq 1} \#\left\{(u_1, \ldots, u_n) \in \mathbb{Z}^n_{\geq 0} \,:\, \sum_{j=1}^{n} u_j = k + i \text{ and } u_\ell \geq i \text{ for all } \ell\right\} \\
&= \binom{k+n}{n} + \sum_{i \geq 1} \binom{k-(i-1)(n-1)}{n-1} \\
&= \binom{k+n}{n} + \sum_{i \geq 0} \binom{k-i(n-1)}{n-1}.
\end{aligned}$$

□

Thus to compute the minimum quasiperiod of $M_n(k)$ it is enough to compute the quasiperiod of $F_n(k)$. For this we use a generating function approach with a theorem of Stanley.[1]

Proposition 4.2 ([10, Proposition 4.4.1]) *Let $F\colon \mathbb{Z}_{\geq 0} \to \mathbb{C}$ be a quasipolynomial. Then the generating function of F may be written as a rational function*

$$\sum_{t=0}^{\infty} F(t)z^t = \frac{p(z)}{q(z)}$$

in which $p, q \in \mathbb{C}[z]$, $\gcd(p, q) = 1$, and every root of q is a root of unity. Furthermore, the minimum quasiperiod of F is the minimum positive integer n such that every root of q is an n^{th} root of unity.

Proposition 4.3 *The generating function of F_n is*

$$\sum_{t=0}^{\infty} F_n(t)z^t = \frac{z^n}{(1-z^n)(1-z)^{n+1}}.$$

In particular, the minimum quasiperiod of F_n is n.

Proof. We compute the generating function as follows:

$$\begin{aligned}\sum_{t=0}^{\infty} F_n(t)z^t &= \sum_{t=0}^{\infty} \sum_{\substack{j\in[0,t]_{\mathbb{Z}} \\ j\equiv t \pmod n}} \binom{j}{n} z^t = \sum_{j=0}^{\infty} \sum_{\substack{t\geq j \\ t\equiv j \pmod n}} \binom{j}{n} z^t \\ &= \sum_{j=0}^{\infty} \binom{j}{n} \sum_{\ell=0}^{\infty} z^{j+\ell n} = \frac{1}{1-z^n} \sum_{j=n}^{\infty} \binom{j}{n} z^j \\ &= \frac{1}{1-z^n} \sum_{j=0}^{\infty} \binom{j+n}{n} z^{j+n} = \frac{z^n}{1-z^n} \cdot \frac{1}{(1-z)^{n+1}}.\end{aligned}$$

Since the roots of the denominator are precisely the nth roots of unity, the minimum quasiperiod of F_n is n. □

Corollary 4.4 *The minimum quasiperiod of the Ehrhart polynomial $M_n(k)$ of P_{G_n} is $n-1$.*

Thus, we have proved Theorem 2.7. However, in the following subsections, we see that a result of Sam and Woods yields not just the quasiperiod of $M_n(k)$, but also the precise period of each coefficient function of this quasipolynomial.

[1]We thank an anonymous referee for pointing out the relevance of this result.

4.2 Minimum quasiperiods and finite differences

In this subsection, we review a result of Sam and Woods [8] and use it to show that the minimum quasiperiod of a quasipolynomial F equals the minimum quasiperiod of the *first difference* of F. In the next subsection we use this result to analyze the periods of all of the coefficient functions of F_n, and hence of $M_{G_n}(k)$.

The **difference operator** $F \mapsto \Delta F$ is defined as follows. For any complex-valued function F defined on $\mathbb{Z}$, or on any interval $[a, \infty) \cap \mathbb{Z}$, the **first difference** of F is the function ΔF defined on the same domain by

$$\Delta F(t) \coloneqq F(t+1) - F(t).$$

For $i \in \mathbb{Z}_{\geq 2}$, the i^{th} **difference** of F is defined by $\Delta^i F \coloneqq \Delta(\Delta^{i-1} F)$. The operator Δ satisfies an analogue of the fundamental theorem of calculus: If f and F are functions defined on $[a, \infty)$, then $\Delta \sum_{x=a}^{t-1} f(x) = f(t)$ and $\sum_{x=a}^{t-1} \Delta F(x) = F(t) - F(a)$ for all $t > a$. This operator thus gives rise to a rich "calculus of finite differences" [4]. Sam and Woods use this calculus in [8] to give elementary proofs of several foundational results in Ehrhart theory.

We now show that the difference operator preserves the minimum quasiperiod of quasipolynomials:

$$\operatorname{mqp}(\Delta F) = \operatorname{mqp}(F). \tag{4.1}$$

The easy half of this equation is the inequality $\operatorname{mqp}(\Delta F) \leq \operatorname{mqp}(F)$. For, let $\phi_1, \ldots, \phi_s \in \mathbb{C}[x]$ be the constituents of F. Then, for each $r \in [s]$ and $t \equiv r \pmod{s}$, $\Delta F(t) = \phi_{r+1}(t+1) - \phi_r(t)$ (indices modulo s), which is a polynomial function of t. Thus, ΔF is a quasipolynomial, and s is a quasiperiod ΔF.

To complete the proof of Equation (4.1), it remains to prove that $\operatorname{mqp}(F) \leq \operatorname{mqp}(\Delta F)$. This inequality follows from [8, Lemma 2.1], a special case of which[2] is the following.

Theorem 4.5 ([8, Lemma 2.1]) *Let $f(t) = \sum_{i=0}^{d} c_i(t)\, t^i$ be a quasipolynomial, let s_i be the minimum period of c_i for all $i \geq 0$, and let $F(t) \coloneqq \sum_{j=0}^{t-1} f(j)$ for $t \geq 1$. Then $F(t) = \sum_{i=0}^{d+1} C_i(t)\, t^i$ for some periodic functions $C_0, C_1, \ldots, C_{d+1}$ such that the minimum period of C_i divides $\operatorname{lcm}\{s_i, s_{i+1}, \ldots, s_d\}$ for $0 \leq i \leq d$, and C_{d+1} is constant.*[3]

Equation (4.1) is now a straightforward corollary.

Corollary 4.6 *Let $F \colon \mathbb{Z} \to \mathbb{C}$ be a quasipolynomial. Then ΔF is also a quasipolynomial, and* $\operatorname{mqp}(\Delta F) = \operatorname{mqp}(F)$.

[2] The authors of [8] consider the more general case in which the upper bound of summation in the definition of $F(t)$ is itself a quasipolynomial function of t of the form $t \mapsto \lfloor at/b \rfloor$ for some $a, b \in \mathbb{Z}$.

[3] It may happen that $C_{d+1} = 0$, as for example when $f(t) \coloneqq (-1)^t$. In general, $C_{d+1} = 0$ if and only if $\sum_{x=1}^{s_d} c_d(x) = 0$. Thus, the "anti-difference" operator does not increase the degree of every quasipolynomial.

Proof. From the argument immediately following Equation (4.1), it remains only to prove that $\operatorname{mqp}(F) \le \operatorname{mqp}(\Delta F)$. Let

$$H(t) \coloneqq F(t) - F(0) = \sum_{j=0}^{t-1} \Delta F(j).$$

Write $H(t) \eqqcolon \sum_{i=0}^{d+1} C_i(t)\, t^i$, and let s'_i be the minimum period of C_i for $0 \le i \le d+1$. Put $\Delta F(t) \eqqcolon \sum_{i=0}^{d} c_i(t)\, t^i$ and let s_i be the minimum period of c_i for $0 \le i \le d$. Then, by Theorem 4.5,

$$\operatorname{mqp}(F) = \operatorname{mqp}(H) = \operatorname{lcm}\{s'_0, \ldots, s'_d\} \mid \operatorname{lcm}\{s_0, \ldots, s_d\} = \operatorname{mqp}(\Delta F).$$

In particular, $\operatorname{mqp}(F) \le \operatorname{mqp}(\Delta F)$. □

4.3 Minimum quasiperiod of M_n and its coefficients

We now apply the Sam–Woods result to the function F_n.

Lemma 4.7 *Let $n \ge i \in \mathbb{Z}_{\ge 0}$. The i^{th} difference of F_n satisfies*

$$\Delta^i F_n(t) = \sum_{\substack{j \in [0,t]_{\mathbb{Z}} \\ j \equiv t \,(\mathrm{mod}\, n)}} \binom{j}{n-i} \qquad \text{for } t \in \mathbb{Z}_{\ge 0}. \tag{4.2}$$

In particular,

$$\Delta^n F_n(t) = \left\lfloor \frac{t}{n} \right\rfloor + 1 \qquad \text{for } t \in \mathbb{Z},$$

and so $\Delta^n F_n$ is a quasipolynomial with minimum quasiperiod n.

Proof. The claim is trivial when $i = 0$. Proceeding by induction on i, we find that, for $n \ge i+1$ and $t \in \mathbb{Z}_{\ge 0}$,

$$\begin{aligned}
\Delta^{i+1} F_n(t) &= \Delta^i F_n(t+1) - \Delta^i F_n(t) \\
&= \sum_{\substack{j \in [0,t+1]_{\mathbb{Z}} \\ j \equiv t+1 \,(\mathrm{mod}\, n)}} \binom{j}{n-i} - \sum_{\substack{j \in [0,t]_{\mathbb{Z}} \\ j \equiv t \,(\mathrm{mod}\, n)}} \binom{j}{n-i} \\
&= \sum_{\substack{j \in [-1,t]_{\mathbb{Z}} \\ j \equiv t \,(\mathrm{mod}\, n)}} \binom{j+1}{n-i} - \sum_{\substack{j \in [0,t]_{\mathbb{Z}} \\ j \equiv t \,(\mathrm{mod}\, n)}} \binom{j}{n-i} \\
&= \sum_{\substack{j \in [0,t]_{\mathbb{Z}} \\ j \equiv t \,(\mathrm{mod}\, n)}} \binom{j}{n-(i+1)}.
\end{aligned}$$

(The condition that $n \geq i+1$ is used to eliminate the $j = -1$ term in the first sum on the right-hand side of the third equation.) Thus, Equation (4.2) is proved. In particular, for $t \in \mathbb{Z}_{\geq 0}$,

$$\Delta^n F_n(t) = \#\{j \in [0,t]_{\mathbb{Z}} : j \equiv t \,(\mathrm{mod}\, n)\} = \left\lfloor \frac{t}{n} \right\rfloor + 1.$$

Clearly $\Delta^n F_n(t) = \left\lfloor \frac{t}{n} \right\rfloor + 1$ has minimum quasiperiod n. □

Theorem 4.8 *The minimum quasiperiod of F_n is n. Furthermore, each coefficient function of F_n is constant, except for the degree-0 coefficient function, which has minimum period n.*

Proof. By Lemma 4.7, $\Delta^n F_n$ has minimum quasiperiod n. Therefore, by Corollary 4.6, F_n itself has minimum quasiperiod n. Now we consider the minimum periods of the coefficient functions of F_n.

Suppose f is a quasipolynomial whose ith coefficient function has minimum period s_i. Let F be a function such that $\Delta F = f$. Then F differs from the function $t \mapsto \sum_{j=0}^{t-1} f(j)$ by a constant, and so the minimum quasiperiod of the ith coefficient function of F divides $\mathrm{lcm}\{s_i, s_{i+1}, \ldots\}$ for all $i \geq 1$. Apply this to the quasipolynomial $f(t) = \left\lfloor \frac{t}{n} \right\rfloor + 1 = \frac{1}{n}t + c_0(t)$. Here $c_0(t) = -r_n(t)/n + 1$, where $r_n(t)$ denotes the remainder of t divided by n. Also, for $i > 0$, $s_i = 1$. Then, since $\Delta^n F_n(t) = f(t)$, by induction we get that all coefficients of F_n are constant except for the degree-0 coefficient. □

We now obtain an extension of Corollary 4.4 by determining the exact periods of all the coefficient functions of $M_{G_n}(k)$.

Corollary 4.9 *The Ehrhart quasipolynomial $M_{G_n}(k)$ of P_{G_n} has minumum quasiperiod $n-1$. More strongly. the coefficient functions of $M_{G_n}(k)$ are all constant, except for the degree-0 term, which has period $n-1$.*

Proof. Since $M_n(k) = M_{G_n}(k)$ is the sum of a polynomial and F_{n-1} by Proposition 4.1, the period of the degree-d term of $M_{G_n}(k)$ is equal to that of the degree-d term in F_{n-1}. Theorem 4.8 states that this period is $n-1$ for the degree-0 term, while all other terms have period 1. Therefore the minimum quasiperiod of $M_{G_n}(k)$ is $n-1$, as this is the least common multiple of the periods of all its coefficient functions. □

5 Graphs with Small Magic-Labeling Quasiperiods

Though Theorem 2.7 demonstrates that the quasipolynomial $M_G(k)$ can have arbitrarily large minimum quasiperiod, there are still large families of graphs for which we can give a uniform bound on the minimum quasiperiod. In this section, we show that for a large family of graphs, including any graph with a leaf, the minimum quasiperiod of this quasipolynomial is at most 2.

Proposition 5.1 *Let G be a graph with a leaf. Then the polytope P_G is the convex hull of the polytope Q_G and the origin $(0, 0, \ldots, 0)$. In particular, $M_G(k)$ is a quasipolynomial of minimum quasiperiod at most 2.*

Proof. In any magic labeling of G, the weight of the leaf edge must be equal to both the index of the magic labeling and the maximum label. Thus, the maximum label of any magic labeling is equal to its index. Since P_G corresponds to magic labelings with maximum label at most k while Q_G corresponds to magic labelings with index *exactly* k, we then obtain P_G by taking the convex hull of Q_G with the origin. The statement about quasiperiods then follows directly from Corollary 2.6 □

We can generalize the result about quasiperiods to a larger family of graphs as follows.

Theorem 5.2 *Suppose there is an edge e in the graph G that attains the maximum label in every magic labeling of G of index 2. Then $M_G(k)$ is a quasipolynomial of minimum quasiperiod at most 2. Furthermore, if G is bipartite, then $M_G(k)$ is a polynomial.*

Proof. Note that the statement holds trivially if the only magic labeling of G is the zero labeling, so we can suppose that G admits a nonzero magic labeling. Fix a magic labeling L of G of index greater than 2. By Proposition 2.5, we can decompose L as a sum of at least two magic labelings L_i of G, each of index at most 2. Since e attains the maximum label in each L_i, it must also attain the maximum label in L. Thus $\max(L) = \sum_i \max(L_i)$, and hence $(L, \max(L)) = \sum_i (L_i, \max(L_i))$ in $\Phi(P_G)$. So we can conclude that $(L, \max(L))$ is not completely fundamental in $\Phi(P_G)$. Therefore, any completely fundamental element of $\Phi(P_G)$ has index (and hence maximum label) at most 2. It follows from Proposition 2.3 that 2 is a quasiperiod of $M_G(k)$.

If G is bipartite, we can furthermore decompose L into a sum of magic labelings L_i of index 1 by Proposition 2.5. The same approach then applies to show that any completely fundamental labeling of $\Phi(P_G)$ has index 1. Thus, Proposition 2.3 implies that 1 is a quasiperiod of $M_G(k)$, i.e., $M_G(k)$ is a polynomial. □

Example 5.3 We now provide a family of graphs exhibiting that Theorem 5.2 is stronger than Proposition 5.1. For $k \geq 1$, consider the graph G_k constructed by connecting two copies of a $(2k+1)$-cycle C_{2k+1} by a single edge (connecting one vertex from each copy). Since odd cycles have no perfect matchings, the bridging edge between the two copies of C_{2k+1} must be included in any perfect matching on G_k. Thus G_k satisfies the hypotheses of Theorem 5.2, but not Proposition 5.1 since it has no leaves.

Remark 5.4 While the hypothesis of Theorem 5.2 only applies to magic labelings of index 2, it is in fact equivalent to assert that the hypothesis holds for all magic labelings. This follows from Proposition 2.5, as any magic labeling can be decomposed as a sum of magic labelings of index at most 2. Moreover, any magic labeling of index 1 can be doubled to yield a magic labeling of index 2.

Remark 5.5 Let G be a graph with an even number of vertices. The **matching preclusion number** $\mathrm{mp}(G)$ of G is the minimum cardinality of an edge set S such that $G - S$ has no perfect matching. If G is bipartite, then the condition on the edge e of Theorem 5.2 is equivalent to the condition that $\mathrm{mp}(G) \leq 1$.

Example 5.6 We now give a method for constructing a bipartite graph with matching preclusion number 1, generalizing Example 5.3. For $i \in \{1, 2\}$, let G_i be a bipartite graph with a vertex v_i such that $G_i \setminus \{v_i\}$ has a perfect matching. Note that such a graph G_i must have odd order. Then consider the graph formed by connecting G_1 and G_2 by a single edge between v_1 and v_2. The resulting graph has a perfect matching, since we can take the perfect matchings in $G_i \setminus \{v_i\}$ for $i \in \{1, 2\}$ along with the added edge. However, the graph obtained by removing the edge between v_1 and v_2 has no perfect matchings, since it consists of two components of odd size.

Thus we see that, while the quasiperiod of the quasipolynomial $M_G(k)$ is unbounded in general, it is small for specific families of graphs. It would be interesting to find other families of graphs for which this quasiperiod is bounded.

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