AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **92(3)** (2025), Pages 357–381

2-Domination critical trees upon edge subdivision

Magda Dettlaff

Faculty of Mathematics, Physics and Informatics
University of Gdańsk, Poland
magda.dettlafff@ug.edu.pl

Michael A. Henning

Department of Mathematics and Applied Mathematics
University of Johannesburg, South Africa
mahenning@uj.ac.za

Magdalena Lemańska

Faculty of Mathematics, Physics and Informatics
Gdańsk University of Technology, Poland
magdalena.lemanska@pg.edu.pl

A. Roux

Department of Mathematical Sciences
Stellenbosch University, South Africa
rianaroux@sun.ac.za

Abstract

A set S of vertices in a graph G is a 2-dominating set of G if every vertex not in S has at least two neighbors in S, where two vertices are neighbors if they are adjacent. The 2-domination number of G, denoted by $\gamma_2(G)$, is the minimum cardinality among all 2-dominating sets of G. The graph G is γ_2-q-critical if the smallest subset of edges whose subdivision necessarily increases $\gamma_2(G)$ has cardinality q. We characterize the γ_2-2-critical trees.

1 Introduction

In this paper, we continue the study of 2-domination critical trees upon edge subdivision. A *dominating set* of a graph G is a set S of vertices of G such that every

ISSN: 2202-3518

vertex not in S has a neighbor in S, where two vertices are neighbors if they are adjacent. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. The notion of domination and its variations in graphs has been studied a great deal. A thorough treatise on dominating sets can be found in the so-called "domination books" [10, 11, 12, 13].

A 2-*dominating set* of a graph G is a set S of vertices of G such that every vertex not in S has at least two neighbors in S. The 2-*domination number* $\gamma_2(G)$ of G is the minimum cardinality among all 2-dominating sets of G. A γ_2-*set of* G is a 2-dominating set of G of cardinality $\gamma_2(G)$. We denote by $\mathcal{A}_2(G)$ and $\mathcal{N}_2(G)$ the set of vertices in G that belong to every or no γ_2-set of G, respectively. Hence a vertex in $\mathcal{A}_2(G)$ belongs to every γ_2-set of G, and a vertex in $\mathcal{N}_2(G)$ belongs to no γ_2-set of G. The concept of 2-domination in graphs, and more generally of k-domination in graphs, is very well studied (see, for example, [3, 4, 6, 7, 8]). An excellent survey on 2-domination in graphs can be found in the book chapter by Hansberg and Volkmann [9].

The *subdivision* of an edge $e = uv$ in a graph G consists of deleting the edge e from $E(G)$, adding a new vertex w to $V(G)$, and adding the new edges uw and vw to $E(G)$. In this case, we say that the edge e has been *subdivided*. Further, we denote the resulting graph G with the edge e subdivided by G_e. Thus, G_e is the graph obtained from G by subdividing the edge e. Moreover if e and f are two distinct edges of G, then we denote by $G_{e,f}$ the graph obtained from G by subdividing both edges e and f. The 2-*domination subdivision number* of G is the minimum number of edges which must be subdivided (where each edge can be subdivided at most once) in order to increase the 2-domination number. The 2-domination subdivision number was defined by Atapour, Sheikholeslami, Hansberg, Volkmann, and Khodkar [1], and studied, for example, in [2].

A graph G is γ_2-q-critical if the smallest subset of edges (where each edge in G can be subdivided at most once) whose subdivision necessarily increases $\gamma_2(G)$ has cardinality q. Our aim is to characterize γ_2-2-critical trees.

1.1 Notation and terminology

For graph theory notation and terminology, we generally follow [12]. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. Two vertices u and v of G are *adjacent* if $uv \in E(G)$, and are called *neighbors*. The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of neighbors of v, while the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N_G(v)$. In general, for a subset $S \subseteq V(G)$, its *open neighborhood* is the set $N_G(S) = \cup_{v \in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$.

The *degree* of a vertex v in G is the number of neighbors v in G, and is denoted by $\deg_G(v)$, and so $\deg_G(v) = |N_G(v)|$. The maximum (minimum) degree among the vertices of G is denoted by $\Delta(G)$ ($\delta(G)$, respectively). An *isolated vertex* is a vertex of degree 0, and a graph is *isolate-free* if it contains no isolated vertex. A vertex of degree 1 is called a *leaf*, and its (unique) neighbor is called a *support vertex*. The edge incident with a leaf is called a *pendant edge.* A *strong support vertex* is a vertex with at least two leaf neighbors, and a *weak support vertex* is a vertex with exactly one leaf neighbor. We denote the set of leaves of G by $L(G)$.

A *rooted tree* T distinguishes one vertex r called the *root.* Let T be a tree rooted at vertex r. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. The root r does not have a parent in T and all its neighbors are its children. A *descendant* of v is a vertex x such that the unique (r, x)-path contains v. Thus, every child of v is a descendant of v. Let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$.

For $k \geq 1$ an integer, we let $[k]$ denote the set $\{1, \ldots, k\}$.

2 Main result

Our aim is to characterize γ_2-2-critical trees. For this purpose, we define a family $\mathcal{F}_2$ of labeled trees (T, S) in Section 5, where T is a tree and S is a labeling that assigns to every vertex v a label. We shall prove the following result.

Theorem 2.1 *A tree T is γ_2-2-critical if and only if $(T, S) \in \mathcal{F}_2$ for some labeling S.*

We proceed as follows. In Section 3 we present a known result that characterizes γ_2-1-critical trees. In Section 4 we present some preliminary observations and results that we will need when proving our main result. In Section 5, we formally define the family $\mathcal{F}_2$ of labeled trees (T, S). Further, we establish important properties of trees in the family $\mathcal{F}_2$ and show, in particular, that every tree in the family $\mathcal{F}_2$ is a γ_2-2-critical tree. Thereafter in Section 6, we present a proof of our main result, namely the characterization of γ_2-2-critical trees given in the statement of Theorem 2.1.

3 Known results

The authors in [5] characterized γ_2-1-critical trees. In order to state the characterization, they defined a family $\mathcal{F}_1$ of labeled trees (T, S) where T is a tree and S is a labeling that assigns to every vertex v of T a label, called the *status* of v and denoted by $\text{sta}(v)$, where $\text{sta}(v) \in \{A, B\}$. They defined (T_1, S_1) as the *labeled base tree* of the family $\mathcal{F}_1$, where T_1 is a path of order 3 given by $v_1v_2v_3$ where the labeling S_1 assigns to the two leaves status A and to the central vertex status B. The labeled base tree (T_1, S_1) is illustrated in Figure 1.

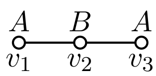


Figure 1: The labeled base tree (T_1, S_1)

A labeled tree (T, S) belongs to the family $\mathcal{F}_1$, if there is a sequence $(T_1, S_1), \dots, (T_k, S_k)$ of labeled trees where (T_1, S_1) is the labeled base tree defined earlier, $(T, S) = (T_k, S_k)$, and if $k \geq 2$, then the labeled tree (T_{i+1}, S_{i+1}) is obtained from the labeled tree (T_i, S_i) by applying one of the operations $\mathcal{T}_j$, $j \in [3]$, given below to a vertex $v \in V(T_i)$ for $i \in [k-1]$.

Operation $\mathcal{T}_1$. Assume $\mathrm{sta}(v) = B$ in (T_i, S_i). The labeled tree (T_{i+1}, S_{i+1}) is obtained from the labeled tree (T_i, S_i) by adding to it a new vertex x and the edge vx, and letting $\mathrm{sta}(x) = A$. Operation $\mathcal{T}_1$ is illustrated in Figure 2.

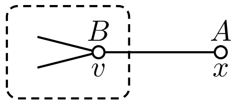


Figure 2: Operation $\mathcal{T}_1$

Operation $\mathcal{T}_2$. Assume $\mathrm{sta}(v) = A$ in (T_i, S_i). The labeled tree (T_{i+1}, S_{i+1}) is obtained from the labeled tree (T_i, S_i) by adding to it a path xy and the edge vx, and letting $\mathrm{sta}(x) = B$ and $\mathrm{sta}(y) = A$. Operation $\mathcal{T}_2$ is illustrated in Figure 3.

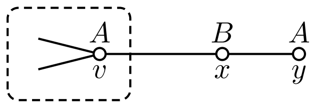


Figure 3: Operation $\mathcal{T}_2$

Operation $\mathcal{T}_3$. Assume $\mathrm{sta}(v) = B$ in (T_i, S_i). The labeled tree (T_{i+1}, S_{i+1}) is obtained from the labeled tree (T_i, S_i) by adding to it a path xyz and the edge vy, and letting $\mathrm{sta}(x) = \mathrm{sta}(z) = A$ and $\mathrm{sta}(y) = B$. Operation $\mathcal{T}_3$ is illustrated in Figure 4.

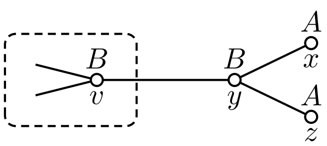


Figure 4: Operation $\mathcal{T}_3$

We are now in a position to state the characterization of γ_2-1-critical trees given in [5].

Theorem 3.1 ([5]) *A tree T is γ_2-1-critical if and only if $(T, S) \in \mathcal{F}_1$ for some labeling S.*

4 Preliminary observations and results

In this section, we present some preliminary results that we will need in order to prove our characterization of γ_2-2-critical trees. Recall that if T is a tree and $e \in E(T)$, then T_e denotes the tree obtained from T by subdividing the edge e. Furthermore if $\{e, f\} \subseteq E(T)$, then $T_{e,f}$ denotes the tree obtained from T by subdividing both edges e and f. Moreover recall that $\mathcal{A}_2(T)$ (respectively, $\mathcal{N}_2(T)$) denotes the set of vertices of T that belong to all (respectively, to no) γ_2-set of T. Since every leaf in a tree T belongs to every 2-dominating set of T, we have the following observation.

Observation 4.1 *If v is a leaf of a tree T, then $v \in \mathcal{A}_2(T)$.*

Lemma 4.2 *If v is a strong support vertex in a γ_2-2-critical tree T, then $v \in \mathcal{N}_2(T)$.*

Proof. Let v be a strong support vertex in a γ_2-2-critical tree T. Suppose, to the contrary, that v belongs to some γ_2-set, S, of T. Let v_1 and v_2 be two distinct leaf neighbors of v in T, and let $e_i = vv_i$ for $i \in [2]$. Since T is γ_2-2-critical, subdividing any two arbitrary distinct edges in T increases the 2-domination number. In particular, $\gamma_2(T_{e_1,e_2}) > \gamma_2(T)$. However by supposition and by Observation 4.1, we have $\{v, v_1, v_2\} \subseteq S$. Thus the set S is a 2-dominating set of T_{e_1,e_2}, and so $\gamma_2(T_{e_1,e_2}) \leq |S| = \gamma_2(T)$, a contradiction. Hence, no γ_2-set of T contains v, that is, $v \in \mathcal{N}_2(T)$. □

Lemma 4.3 *If T is a tree that contains a strong support vertex v with at least three leaf neighbors, then $\gamma_2(T) = \gamma_2(T - v') + 1$, where v' is an arbitrary leaf neighbor of v in T.*

Proof. Let T be a tree that contains a strong support vertex v with at least three leaf neighbors, and let $T' = T - v'$, where v' is an arbitrary leaf neighbor of v in T. The trees T and T' are illustrated in Figure 5.

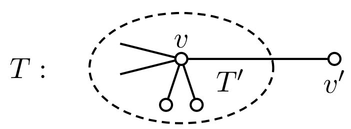


Figure 5: A tree T in the statement of Lemma 4.3

Let S be a γ_2-set of T. By Observation 4.1, all leaf neighbors of v belong to the set S. Thus since v has at least three leaf neighbors in T, one of which is the leaf neighbor v', we infer that the set $S \setminus \{v'\}$ is a 2-dominating set of T'. Thus, $\gamma_2(T') \leq |S| - 1 = \gamma_2(T) - 1$. Conversely, every 2-dominating set of T' can be extended to a 2-dominating set of T by adding to it the vertex v', implying that $\gamma_2(T) \leq \gamma_2(T') + 1$. Consequently, $\gamma_2(T) = \gamma_2(T') + 1$. □

Lemma 4.4 *If T is a tree that contains a strong support vertex v with at least three leaf neighbors, then the tree T is γ_2-2-critical if and only if the tree $T - v'$ is γ_2-2-critical where v' is an arbitrary leaf neighbor of v in T.*

Proof. Let T be a tree and let v be a strong support vertex in T with at least three leaf neighbors. Let $T' = T - v'$. Suppose firstly that T is a γ_2-2-critical tree. Thus there exists an edge $e \in E(T)$ such that $\gamma_2(T) = \gamma_2(T_e)$. Since v has at least three leaf neighbours in T, the vertex v is a strong support vertex in T_e, and so by Lemma 4.2 we have $v \in \mathcal{N}_2(T_e)$. From this we infer that subdividing an edge incident with a leaf neighbor of v increases the 2-domination number of T. Thus the edge e is not incident with a leaf neighbor of v, and so $e \in E(T')$ and every leaf neighbor of v in T remains a leaf neighbor of v in T_e. Let S_e be an arbitrary γ_2-set of T_e. By Observation 4.1, all leaf neighbors of v in T belong to the γ_2-set S_e. The set $S_e \setminus \{v'\}$ is a 2-dominating set of T'_e, and so $\gamma_2(T') \le \gamma_2(T'_e) \le |S_e| - 1 = \gamma_2(T_e) - 1 = \gamma_2(T) - 1 = \gamma_2(T')$. Consequently, we must have equality throughout this inequality chain. In particular, $\gamma_2(T') = \gamma_2(T'_e)$.

Let e_1 and e_2 be two arbitrary distinct edges in T'. Since T is a γ_2-2-critical tree and $\{e_1, e_2\} \subset E(T)$, we have $\gamma_2(T) < \gamma_2(T_{e_1,e_2})$. Every γ_2-set of T'_{e_1,e_2} can be extended to a 2-dominating set of T_{e_1,e_2} by adding to it the vertex v', and so $\gamma_2(T) < \gamma_2(T_{e_1,e_2}) \le \gamma_2(T'_{e_1,e_2}) + 1$. Thus, $\gamma_2(T'_{e_1,e_2}) > \gamma_2(T) - 1 = \gamma_2(T')$. Hence subdividing any two arbitrary distinct edges in T' increases the 2-domination number. As observed earlier, there exists an edge e' in T', namely the edge $e' = e$, such that $\gamma_2(T) = \gamma_2(T_{e'})$. These observations imply that the tree T' is γ_2-2-critical.

Conversely, suppose that T' is a γ_2-2-critical tree. Thus there exists an edge $f \in E(T')$ such that $\gamma_2(T') = \gamma_2(T'_f)$. Every γ_2-set of T'_f can be extended to a γ_2-set of T_f by adding to it the vertex v', and so $\gamma_2(T) \le \gamma_2(T_f) \le \gamma_2(T'_f) + 1 = \gamma_2(T') + 1 = \gamma_2(T)$. Hence we must have equality throughout this inequality chain. In particular, $\gamma_2(T) = \gamma_2(T_f)$.

Let e_1 and e_2 be two arbitrary distinct edges in T. We note that $\gamma_2(T) \le \gamma_2(T_{e_1,e_2})$. We show that $\gamma_2(T) < \gamma_2(T_{e_1,e_2})$. If at least one of e_1 and e_2 is incident with a leaf neighbor of v, then $\gamma_2(T) < \gamma_2(T_{e_1,e_2})$, as desired. Hence we may assume that neither e_1 nor e_2 is incident with a leaf neighbor of v. In particular, both e_1 and e_2 are edges in T' (and neither is incident with a leaf neighbor of v in T'). Thus, the vertex v is a strong support vertex in T_{e_1,e_2} with at least three leaf neighbors, and is therefore a strong support vertex in $T'_{e_1,e_2} = T_{e_1,e_2} - v'$ with at least two leaf neighbors. By Lemma 4.3, $\gamma_2(T_{e_1,e_2}) = \gamma_2(T'_{e_1,e_2}) + 1$. Suppose, to the contrary, that $\gamma_2(T) = \gamma_2(T_{e_1,e_2})$. In this case, $\gamma_2(T') + 1 = \gamma_2(T) = \gamma_2(T_{e_1,e_2}) = \gamma_2(T'_{e_1,e_2}) + 1$, and so $\gamma_2(T') = \gamma_2(T'_{e_1,e_2})$, contradicting the fact that T' is a γ_2-2-critical tree. Hence, $\gamma_2(T) < \gamma_2(T_{e_1,e_2})$. As observed earlier, there exists an edge in T, namely the edge f, such that $\gamma_2(T) = \gamma_2(T_f)$. These observations imply that the tree T is γ_2-2-critical. This completes the proof of Lemma 4.4. □

Lemma 4.5 *If T is obtained from a nontrivial tree T' by adding to it a path xyz and the edge vy where v is an arbitrary vertex of T', then $\gamma_2(T) = \gamma_2(T') + 2$.*

Proof. Let T be a tree obtained from a nontrivial tree T' by adding to it a path xyz and the edge vy where v is an arbitrary vertex of T'. The trees T and T' are illustrated in Figure 6.

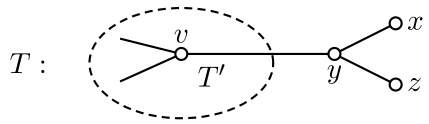


Figure 6: A tree T in the statement of Lemma 4.5

Let S be a γ_2-set of T. By Observation 4.1, all leaves in T belong to the set S. In particular, $\{x, z\} \subset S$. If $y \in S$, then we can simply replace the vertex y in S with the vertex v to produce a new γ_2-set of T. Hence we may choose the set S so that $y \notin S$. With this choice of the set S, the set $S \setminus \{x, z\}$ is a 2-dominating set of T', and so $\gamma_2(T') \le |S| - 2 = \gamma_2(T) - 2$. Conversely, every 2-dominating set of T' can be extended to a 2-dominating set of T by adding to it the vertices x and z, implying that $\gamma_2(T) \le \gamma_2(T') + 2$. Consequently, $\gamma_2(T) = \gamma_2(T') + 2$. □

Lemma 4.6 *Let T be obtained from a nontrivial tree T' by adding to it a path xyz and the edge vy where v is an arbitrary vertex of T'. If the tree T is γ_2-2-critical, then the tree T' is γ_2-2-critical.*

Proof. Suppose that T is a γ_2-2-critical tree. Thus there exists an edge $e \in E(T)$ such that $\gamma_2(T) = \gamma_2(T_e)$. Let S_e be a γ_2-set of T_e. The vertex y is a strong support vertex in T, and so by Lemma 4.2 we have $y \in \mathcal{N}_2(T)$. Thus, $y \notin S_e$. From this we infer that subdividing the edge xy or the edge yz increases the 2-domination number of T. Thus the edge e is incident with neither x nor z. If $e = vy$ and if w is the new vertex resulting from subdividing the edge e, then S_e contains y or w. As observed earlier, $y \notin S_e$, and so $w \in S_e$. However in this case, the set $S = (S_e \setminus \{w\}) \cup \{y\}$ is a 2-dominating set of T that contains the vertex y. As observed earlier, $y \in \mathcal{N}_2(T)$. From this we infer that the set S is not a γ_2-set of T, and so $\gamma_2(T) < |S| = |S_e| = \gamma_2(T_e)$, a contradiction. Hence, the edge e is not incident with the vertex y, implying that $e \in E(T')$. By Lemma 4.5, $\gamma_2(T) = \gamma_2(T') + 2$. The set $S_e \setminus \{x, z\}$ is a 2-dominating set of T'_e, and so $\gamma_2(T') \le \gamma_2(T'_e) \le |S_e| - 2 = \gamma_2(T_e) - 2 = \gamma_2(T) - 2 = \gamma_2(T')$. Consequently, we must have equality throughout this inequality chain. In particular, $\gamma_2(T') = \gamma_2(T'_e)$.

Let e_1 and e_2 be two arbitrary distinct edges in T'. Since T is a γ_2-2-critical tree and $\{e_1, e_2\} \subset E(T)$, we have $\gamma_2(T) < \gamma_2(T_{e_1,e_2})$. Every γ_2-set of T'_{e_1,e_2} can be extended to a 2-dominating set of T_{e_1,e_2} by adding to it the vertices x and z, and so $\gamma_2(T) < \gamma_2(T_{e_1,e_2}) \le \gamma_2(T'_{e_1,e_2}) + 2$. Thus, $\gamma_2(T'_{e_1,e_2}) > \gamma_2(T) - 2 = \gamma_2(T')$. Hence subdividing any two arbitrary distinct edges in T' increases the 2-domination number. As observed earlier, there exists an edge e' in T', namely the edge $e' = e$, such that $\gamma_2(T) = \gamma_2(T_{e'})$. These observations imply that the tree T' is γ_2-2-critical. □

These observations imply that the tree T is γ_2-2-critical. This completes the proof of Lemma 4.6. □

5 The family $\mathcal{F}_2$

In this section, we define a family $\mathcal{F}_2$ of labeled trees (T, S) where T is a tree and S is a labeling that assigns to every vertex v of T a label, called the *status* of v and denoted by $\mathrm{sta}(v)$, where $\mathrm{sta}(v) \in \{A, B, Y, Z\}$. We define (T_1, S_1) as the *labeled base tree* of the family $\mathcal{F}_2$, where T_1 is a path of order 4 given by $v_1v_2v_3v_4$ where the labeling S_1 assigns to the two end-vertices status A and where $\mathrm{sta}(v_2) = X$ with $X \in \{Y, Z\}$ and $\mathrm{sta}(v_3) = \overline{X}$ with $\overline{X} \in \{Y, Z\} \setminus \{X\}$. The labeled base tree (T_1, S_1) is illustrated in Figure 7.

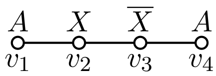


Figure 7: The labeled base tree (T_1, S_1)

A labeled tree (T, S) belongs to the family $\mathcal{F}_2$, if there is a sequence $(T_1, S_1), \ldots,$ (T_k, S_k) of labeled trees where (T_1, S_1) is the labeled base tree defined earlier, $(T, S) =$ (T_k, S_k), and if $k \geq 2$, then the labeled tree (T_{i+1}, S_{i+1}) is obtained from the labeled tree (T_i, S_i) by applying one of the operations $\mathcal{O}_j$, $j \in [5]$, given below to a vertex $v \in V(T_i)$ for $i \in [k-1]$.

Operation $\mathcal{O}_1$. Assume $\mathrm{sta}(v) = A$ and the vertex v has degree 1 with (unique) a neighbor x where either $\mathrm{sta}(x) = X$ and $X \in \{Y, Z\}$ or $\mathrm{sta}(x) = A$ in the labeled tree (T_i, S_i). The labeled tree (T_{i+1}, S_{i+1}) is formed from (T_i, S_i) by deleting the edge vx and adding the new vertices y and z and adding the new edges vy, yz, and zx (so that $vyzx$ is a path in T_{i+1}), and letting $\mathrm{sta}(y) = X$ and $\mathrm{sta}(z) = \overline{X}$. Operation $\mathcal{O}_1$ in the case when $\mathrm{sta}(x) = X$ is illustrated in Figure 8(a), while Operation $\mathcal{O}_1$ in the case when $\mathrm{sta}(x) = A$ is illustrated in Figure 8(b).

Operation $\mathcal{O}_2$. Assume $\mathrm{sta}(v) = X$ where $X \in \{Y, Z\}$ in (T_i, S_i). Let H be the subgraph of T_i induced by the vertices labeled X or $\overline{X}$ and let H_v be the component of $H - v$ of odd cardinality. The labeled tree (T_{i+1}, S_{i+1}) is obtained from the labeled tree (T_i, S_i) by adding to it a path xy and the edge vx. For each vertex $w \in V(H_v)$, if $\mathrm{sta}(w) = X$ in (T_i, S_i), then we let $\mathrm{sta}(w) = A$ in (T_{i+1}, S_{i+1}), while if $\mathrm{sta}(w) = \overline{X}$ in (T_i, S_i), then we let $\mathrm{sta}(w) = B$ in (T_{i+1}, S_{i+1}). Moreover, we change the status of v and let $\mathrm{sta}(v) = A$, and we let $\mathrm{sta}(x) = B$ and $\mathrm{sta}(y) = A$. Operation $\mathcal{O}_2$ is illustrated in Figure 9. We call the vertex v the *link vertex* of the operation.

Operation $\mathcal{O}_3$. Assume $\mathrm{sta}(v) = A$ and the vertex v has at least one neighbor with status B in (T_i, S_i). The labeled tree (T_{i+1}, S_{i+1}) is obtained from the labeled

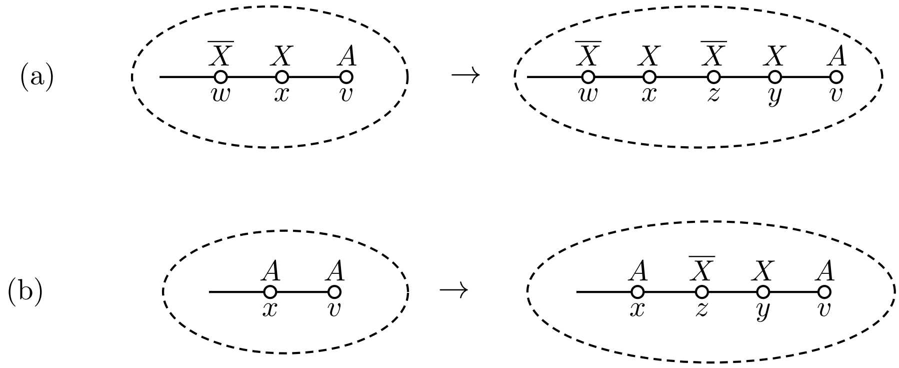


Figure 8: Operation $\mathcal{O}_1$

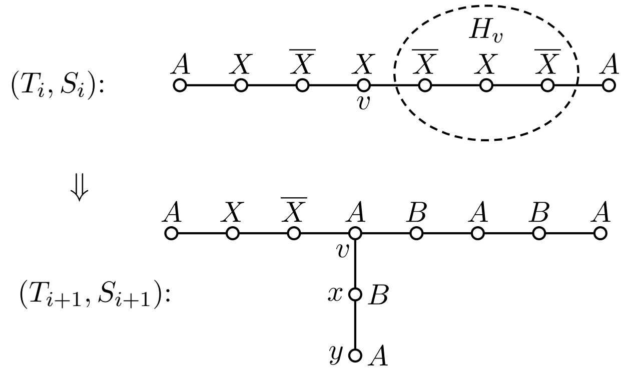


Figure 9: Operation $\mathcal{O}_2$

tree (T_i, S_i) by adding to it a path xy and the edge vx, and letting $\text{sta}(x) = B$ and $\text{sta}(y) = A$. Operation $\mathcal{O}_3$ is illustrated in Figure 10.

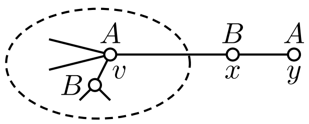


Figure 10: Operation $\mathcal{O}_3$

Operation $\mathcal{O}_4$. Assume $\text{sta}(v) = A$ and the vertex v has at least one neighbor with status B or $\text{sta}(v) \in \{B, X\}$ where $X \in \{Y, Z\}$ in (T_i, S_i). The labeled tree (T_{i+1}, S_{i+1}) is obtained from the labeled tree (T_i, S_i) by adding to it a path xyz and the edge vy, and letting $\text{sta}(x) = \text{sta}(z) = A$ and $\text{sta}(y) = B$. Operation $\mathcal{O}_4$ is illustrated in Figure 11.

Operation $\mathcal{O}_5$. Assume $\text{sta}(v) = B$ and v is a strong support vertex in (T_i, S_i). The labeled tree (T_{i+1}, S_{i+1}) is obtained from the labeled tree (T_i, S_i) by adding to it a new vertex x and the edge vx, and letting $\text{sta}(x) = A$. Operation $\mathcal{O}_5$ is illustrated in Figure 12.

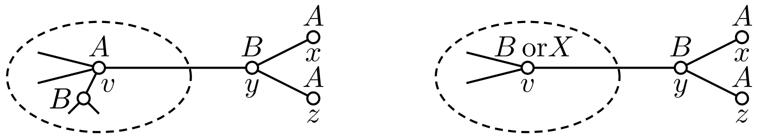


Figure 11: Operation $\mathcal{O}_4$

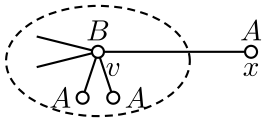


Figure 12: Operation $\mathcal{O}_5$

To illustrate operations $\mathcal{O}_1$ through to $\mathcal{O}_5$, let $(T_1, S_1), (T_2, S_2), \ldots, (T_6, S_6)$ be the labelled trees illustrated in Figure 13. The labeled tree (T_1, S_1) is the base tree illustrated in Figure 7, and the labeled tree (T_{i+1}, S_{i+1}) is obtained from the labeled tree (T_i, S_i) by applying operation $\mathcal{O}_i$ for $i \in [5]$. Thus, $(T_1, S_1), (T_2, S_2), \ldots, (T_6, S_6)$ is a sequence of labelled trees, each of which belong to the family $\mathcal{F}_2$. In particular, the labeled tree $(T, S) = (T_6, S_6)$ belongs to the family $\mathcal{F}_2$.

If $(T, S) \in \mathcal{F}_2$, we let $S_A(T)$, $S_B(T)$, $S_Y(T)$ and $S_Z(T)$ be the sets of vertices of status A, B, Y and Z, respectively, in the labeled tree (T, S). The following observation is immediate from the way in which each tree in the family $\mathcal{F}_2$ is constructed.

Observation 5.1 *If $(T, S) \in \mathcal{F}_2$, then the following properties hold.*

(a) *$L(T) \subseteq S_A(T)$.*
(b) *If $v \in S_B(T)$, then $|N_T(v) \cap S_A(T)| \geq 2$.*
(c) *For $X \in \{Y, Z\}$, the subgraph of T induced by all vertices labeled X and $\overline{X}$ is a path of even order. Furthermore, the labels of consecutive vertices on this path alternative between label X and $\overline{X}$, and so $S_X(T)$ is an independent set and $|S_X(T)| = |S_{\overline{X}}(T)|$.*
(d) *If $v \in S_X(T)$ where $X \in \{Y, Z\}$, then the neighbors of v may have status $A, \overline{X}$, or B. Apart from the neighbours of status B, v has either two neighbors of status $\overline{X}$ or one neighbor of status $\overline{X}$ and one neighbor of status A.*
(e) *The set $S_A(T) \cup S_X(T)$ is a 2-dominating set of T, and so $\gamma_2(T) \leq |S_A(T)| + |S_X(T)|$, where $X \in \{Y, Z\}$.*
(f) *If two adjacent vertices both have status A, then no vertex has status X or $\overline{X}$ where $X \in \{Y, Z\}$.*
(g) *A vertex of status A has at most one neighbor of status X or $\overline{X}$ where $X \in \{Y, Z\}$.*
(h) *Every strong support vertex has status B in the labeled tree (T, S).*

Proof. Observations (a)–(d) and (g) directly follow from the definition of the operations. From observations (b) and (d) it follow that $S_A(T) \cup S_X(T)$ is a 2-dominating set of T, proving (e). The only possiblity to eliminate the vertices of status X and

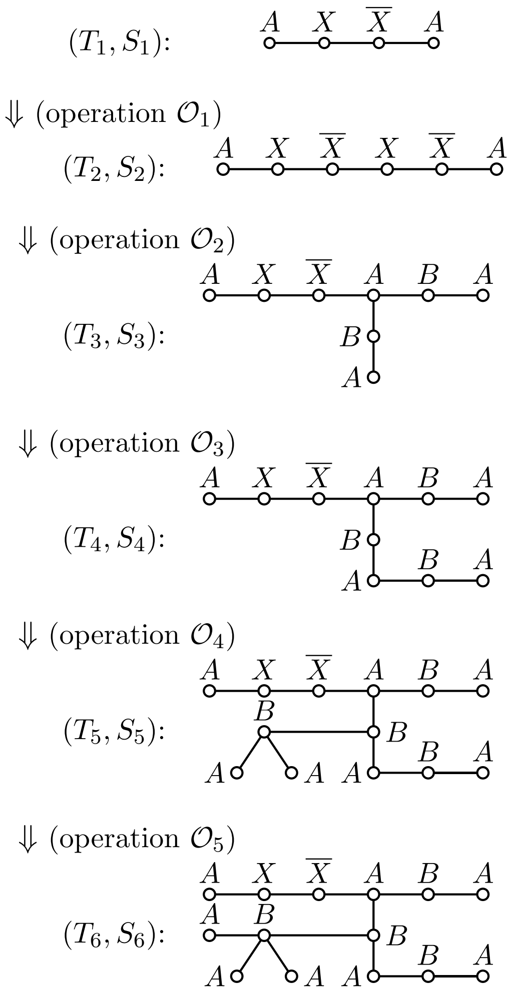


Figure 13: A labeled tree $(T, S) = (T_6, S_6)$ in the family $\mathcal{F}_2$

$\overline{X}$ is to repeatedly apply operation $\mathcal{O}_2$. This will result in two adjacent vertices of status A, proving (f). The only way to create a strong support is by applying operation $\mathcal{O}_4$, the strong support obtains status B proving (h). □

Theorem 5.2 *If $(T, S) \in \mathcal{F}_2$ and $X \in \{Y, Z\}$, then the following properties hold:*

(a) $\gamma_2(T) = |S_A(T)| + |S_X(T)|$.
(b) $S_A(T) \cup S_X(T)$ *and* $S_A(T) \cup S_{\overline{X}}(T)$ *are* γ_2*-sets of* T.
(c) $S_A(T) \subseteq S$ *and* $S_B(T) \cap S = \varnothing$ *for every* γ_2*-set* S *of* T.
(d) *If* $p \in S_X(T)$, *then* $\gamma_2(T - p) = \gamma_2(T)$.
(e) *There exists an edge* $e \in E(T)$ *such that* $\gamma_2(T) = \gamma_2(T_e)$ *and* $S_A(T) \subseteq S_e$ *for some* γ_2*-set* S_e *of* T_e.
(f) *The tree* T *is* γ_2*-2-critical.*

Proof. Let $(T, S) \in \mathcal{F}_2$ and let $X \in \{Y, Z\}$. Thus, there is a sequence $(T_1, S_1), \ldots,$ (T_k, S_k) of labeled trees where (T_1, S_1) is the labeled base tree in Figure 7, $(T, S) =$

(T_k, S_k), and if $k \geq 2$, then the labeled tree (T_{i+1}, S_{i+1}) is obtained from the labeled tree (T_i, S_i) by applying one of the operations $\mathcal{O}_j$, $j \in [5]$, to a vertex $v \in V(T_i)$ for $i \in [k-1]$. We proceed by induction on number k of trees used to construct the tree T. If $k = 1$, then $(T, S) = (T_1, S_1)$. In this case, it is straightforward to check that the desired properties (a)–(f) hold. This establishes the base case.

Let $k \geq 2$ and assume that if $(T', S') \in \mathcal{F}_2$ and (T', S') can be built from a sequence of k' trees in the family $\mathcal{F}_2$ where $1 \leq k' < k$, then the desired properties (a)–(f) hold for the labeled tree (T', S'). Let $(T, S) \in \mathcal{F}_2$ and let $(T_1, S_1), \ldots, (T_k, S_k)$ be a sequence of labeled trees used to build the labeled tree (T, S), where (T_1, S_1) is the labeled base tree and $(T, S) = (T_k, S_k)$. Let $(T', S') = (T_{k-1}, S_{k-1})$. Thus, $(T', S') \in \mathcal{F}_2$ and the labeled tree (T, S) is obtained from the labeled tree (T', S') by applying one of the five operations $\mathcal{O}_j$, $j \in [5]$, to a vertex $v \in V(T')$ for $i \in [k-1]$. We consider five cases, depending on which of the five operations the labeled tree (T, S) is built from the labeled tree (T', S'). In all cases, we let D be a γ_2-set of T and we let D' be the restriction of D to T', and so $D' = D \cap V(T')$. Since $(T', S') \in \mathcal{F}_2$, we note by Observation 5.1(a) that every leaf in (T', S') has status A.

Case 1. (T, S) is obtained from (T', S') by operation $\mathcal{O}_1$. In this case the vertex v of degree 1 in T' has status A in (T', S') and has a neighbor x where either $\text{sta}(x) = X$ and $X \in \{Y, Z\}$ or $\text{sta}(x) = A$ in the labeled tree (T', S'). Suppose firstly that $\text{sta}(x) = X$ where $X \in \{Y, Z\}$. Let w be the neighbor of x in (T', S') of status $\overline{X}$. Since $(T', S') \in \mathcal{F}_2$, we note by Observation 5.1(d) that all neighbors of x, if any, in (T', S') that are different from v and w have status B. The tree (T, S) is formed from (T', S') by deleting the edge vx and adding new vertices y and z and adding the new edges vy, yz, and zx, and letting $\text{sta}(y) = X$ and $\text{sta}(z) = \overline{X}$, as illustrated in Figure 14.

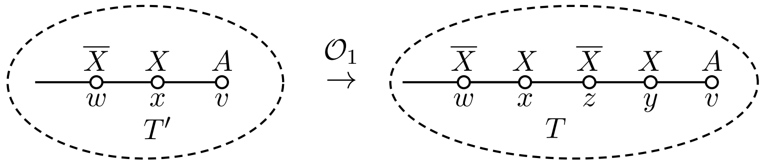


Figure 14: Operation $\mathcal{O}_1$

Since every leaf belongs to every 2-dominating set, we note that $v \in D$. Since D is a γ_2-set of T, the set D contains exactly one of y and z. Suppose that $y \in D$. In this case, $z \notin D$, implying that $x \in D$ in order to 2-dominate the vertex z. Let $D' = D \setminus \{y\}$. The set D' is a 2-dominating set of T', and so $\gamma_2(T') \leq |D'| = |D| - 1$. Applying the inductive hypothesis to $(T', S') \in \mathcal{F}_2$, we have $\gamma_2(T') = |S_A(T')| + |S_X(T')| = |S_A(T)| + |S_X(T)| - 1$. Therefore, $\gamma_2(T) = |D| \geq \gamma_2(T') + 1 = |S_A(T)| + |S_X(T)|$. By Observation 5.1(e), $\gamma_2(T) \leq |S_A(T)| + |S_X(T)|$. Consequently, $\gamma_2(T) = |S_A(T)| + |S_X(T)|$. Analogous arguments yield $\gamma_2(T) = |S_A(T)| + |S_{\overline{X}}(T)|$. Suppose that $y \notin D$, and so $z \in D$. In this case, let $D' = D \setminus \{z\}$. The set D' is a 2-dominating set of T', and so $\gamma_2(T') \leq |D'| = |D| - 1$. Analogous arguments as before show that $\gamma_2(T) = |S_A(T)| + |S_X(T)|$ and $\gamma_2(T) = |S_A(T)| + |S_{\overline{X}}(T)|$. Thus, property (a) holds in the tree T.

Moreover, the inequalities in the previous paragraph are all equalities. In particular, $\gamma_2(T') = \gamma_2(T) - 1$. Further, $\gamma_2(T') = |D'|$, implying that D' is a γ_2-set of T'. Thus, from property (a) and Observation 5.1(e) we infer that property (b) holds in the labeled tree (T, S).

Since D' is a γ_2-set of T' and since the labeled tree (T', S') has property (c), we note that $S_A(T') \subseteq D'$ and $S_B(T') \cap D' = \varnothing$. Thus since either $D = D' \cup \{y\}$ where the vertex y has status X or $D = D' \cup \{z\}$ where the vertex z has status $\overline{X}$, we infer that property (c) holds in the labeled tree (T, S).

Let $p \in S_X(T)$. If $p \in V(T')$, then since (T', S') has property (d) and since every γ_2-set of $T' - p$ can be extended to a 2-dominating set of T by adding exactly one additional vertex, we infer that $\gamma_2(T - p) = \gamma_2(T)$. Suppose that $p \in \{y, z\}$. In this case, we consider the tree $T' - v$. The set $(S_A(T') \setminus \{v\}) \cup S_X(T')$ is a 2-dominating set of $T' - v$ and can be extended to a 2-dominating set of $T - p$ by adding to it the vertex v and adding either y (if $p = z$) or z (if $p = y$). Thus, $\gamma_2(T - p) \le \gamma_2(T' - v) + 2 = (\gamma_2(T') - 1) + 2 = \gamma_2(T') + 1 = \gamma_2(T)$. Since $\gamma_2(T) \le \gamma_2(T - p)$, we therefore infer that $\gamma_2(T - p) = \gamma_2(T)$. Thus, property (d) holds in the labeled tree (T, S).

Since $(T', S') \in \mathcal{F}_2$, property (e) holds in the labeled tree (T', S'). Thus there exists an edge $e' \in E(T')$ such that $\gamma_2(T') = \gamma_2(T'_e)$ and $S_A(T') \subseteq S'$ for some γ_2-set S'_e of T'_e. The set S'_e can be extended to a γ_2-set of T_e by adding to it the vertex z. Thus, $\gamma_2(T_e) \le |S'_e| + 1 = \gamma_2(T'_e) + 1 = \gamma_2(T') + 1 = \gamma_2(T)$. Clearly, $\gamma_2(T) \le \gamma_2(T_e)$. Consequently, $\gamma_2(T) = \gamma_2(T_e)$. Moreover, the set $S_e = S'_e \cup \{z\}$ is a γ_2-set of T_e and $S_A(T) \subseteq S_e$. Thus, property (e) holds in the labeled tree (T, S).

We show next that T is γ_2-2-critical. Since $(T', S') \in \mathcal{F}_2$, there is a path P' that starts at the vertex v (of degree 1 with status A) and ends at a vertex v^* of status A, where all internal vertices alternative with status X and status $\overline{X}$. We note that the first few vertices on the path P' are v, x, and w. Let P be the path in T obtained from P' by deleting the edge vx and adding new vertices y and z and adding the new edges vy, yz, and zx. We note that the path P starts at the vertex v and ends at a vertex v^* of status A. Furthermore, all internal vertices alternative with status X and status $\overline{X}$.

As observed earlier, property (e) holds in the labeled tree (T, S). Since T' is γ_2-2-critical, there exist an edge $e' \in E(T')$ whose subdivision does not increase the 2-domination number of T'. Subdividing such an edge e' will necessarily not increase the 2-domination number of T. Let $F = \{f_1, f_2\} \subset E(T)$. If $F \subset E(T')$, then since T' is γ_2-2-critical, subdividing the two edges in F increases the 2-domination number of T', and therefore also increase the 2-domination number of T. If both f_1 and f_2 are edges from the set $E(T) \setminus E(T') = \{zx, zy, yv\}$, then subdividing the two edges in F increases the 2-domination number of T by 1. Hence we may assume that exactly one of f_1 and f_2, say f_2, is not an edge of T'. Thus, $f_1 \in E(T')$ and f_2 is one of the edges zx, zy and yv.

We now consider the set $F' = \{f_1, f_2'\}$ where f_2' is the edge xv. In this case, the 2-domination number of T with the two edges in F subdivided is exactly one more than the 2-domination number of T' with the two edges in F' subdivided. Since subdividing the two edges in F' increases the 2-domination number of T', we therefore infer that subdividing the two edges in F increases the 2-domination number of T. From these properties we infer that the tree T is γ_2-2-critical. Thus, property (f) holds in the tree T.

Hence if $\text{sta}(x) = X$ where $X \in \{Y, Z\}$ in the labeled tree (T', S'), then properties (a) to (f) hold in the tree T. Analogous arguments show that if $\text{sta}(x) = A$ in the labeled tree (T', S'), then properties (a) to (f) hold in the tree T.

Case 2. (T, S) is obtained from (T', S') by operation $\mathcal{O}_2$. Let P' be the path in T' that starts and ends at vertices of status A, with all internal vertices alternating with status X and status $\overline{X}$. We adopt the notation in Operation $\mathcal{O}_2$. Thus, v is an internal vertex of P' of status X. Let H be the subgraph of T' induced by the internal vertices of P' (labeled X or $\overline{X}$) and let H_v be the component of $H - v$ of odd order. The labeled tree (T, S) is formed from the labeled tree (T', S') by adding to it a path xy and the edge vx. For each vertex $w \in V(H_v)$, if $\text{sta}(w) = X$ in T', then $\text{sta}(w) = A$ in T, while if $\text{sta}(w) = \overline{X}$ in T', then $\text{sta}(w) = B$ in T. Moreover, the status of v is changed from status X in T' to status A in T, and $\text{sta}(x) = B$ and $\text{sta}(y) = A$, as illustrated in Figure 9.

Since every leaf belongs to every 2-dominating set, we note that $y \in D$. Since D is a γ_2-set of T, the set D contains exactly one of v and x. We show that $v \in D$. Suppose, to the contrary, that $v \notin D$, and so $x \in D$. Let H_v be the path $v_1 \dots v_{2p+1}$, where v is adjacent to the vertex v_1. We note that in the tree T', the labels of the vertices on the path H_v alternate between $\overline{X}$ and X. Moreover, v_1 has label $\overline{X}$ in T'. We note that H_v contains $p+1$ vertices of label $\overline{X}$ and p vertices of label X. Moreover in the tree T, every vertex of status $\overline{X}$ in H_v changes to status B, while every vertex of status X in H_v changes to status A. By supposition, $v \notin D$.

Suppose that D contains a vertex of status B that does not belong to the path H_v. The set $D' = (D \setminus \{x, y\}) \cup \{v\}$ is a 2-dominating set of T' that therefore contains a vertex of status B, implying by the inductive hypothesis that D' is not a γ_2-set of T', and so $\gamma_2(T') < |D'|$. Thus, $\gamma_2(T') \le |D'| - 1 = |D| - 2 = \gamma_2(T) - 2$. However every γ_2-set of T' that contains the vertex v can be extended to a 2-dominating set of T by adding to it the vertex y, implying that $\gamma_2(T) \le \gamma_2(T') + 1$. This contradicts our earlier observation that $\gamma_2(T) \ge \gamma_2(T') + 2$.

Hence, every vertex of status B, if any, in D must belong to the path H_v. By Observation 5.1(d), vertices v and v_2 are the only neighbours of v_1. Since $v \notin D$, it follows that $v_1 \in D$. Since D is a 2-dominating set in T, we can choose D to contain all the vertices of status B in H_v. However, replacing the vertex x and the $p+1$ vertices of status B in H_v with the vertex v and the p vertices of status A in H_v produces a new 2-dominating set of T of cardinality $|D| - 1$, contradicting the minimality of the set D.

From the above arguments, we infer that $v \in D$, implying that $x \notin D$. In this case, we let $D' = D \setminus \{y\}$ where D is an arbitrary γ_2-set of T. The set D' is a 2-dominating set of T', and so $\gamma_2(T') \le |D'| = |D| - 1$. Applying the inductive hypothesis to $(T', S') \in \mathcal{F}_2$, we have $\gamma_2(T') = |S_A(T')| + |S_X(T')| = |S_A(T)| - 1 + |S_X(T)|$. Therefore, $\gamma_2(T) = |D| \ge \gamma_2(T') + 1 = |S_A(T)| + |S_X(T)|$. By Observation 5.1(e), $\gamma_2(T) \le |S_A(T)| + |S_X(T)|$. Consequently, $\gamma_2(T) = |S_A(T)| + |S_X(T)|$. Analogously, $\gamma_2(T) = |S_A(T)| + |S_{\overline{X}}(T)|$. Thus, property (a) holds in the labeled tree (T, S).

Moreover, the inequalities in the previous paragraph are all equalities. In particular, $\gamma_2(T') = \gamma_2(T) - 1$. Further, $\gamma_2(T') = |D'|$, implying that D' is a γ_2-set of T'. Thus, from property (a) and Observation 5.1(e) we infer that property (b) holds in the labeled tree (T, S).

Since D' is a γ_2-set of T' and since the labeled tree (T', S') has property (c), we note that $S_A(T') \subseteq D'$ and $S_B(T') \cap D' = \varnothing$. Thus since $D = D' \cup \{y\}$ and y has status A, we infer that property (c) holds in the labeled tree (T, S).

Let $p \in S_X(T)$. Thus, $p \in V(H) \setminus (V(H_v) \cup \{v\})$. In particular, $p \in V(T')$. Since (T', S') has property (d), $\gamma_2(T' - p) = \gamma_2(T')$. Moreover, we can choose a γ_2-set of $T' - p$ to contain the two vertices of status A on the path P'. Such a set can therefore be extended to a 2-dominating set of T by adding to it the vertex y, implying that $\gamma_2(T - p) \le \gamma_2(T' - p) + 1 = \gamma_2(T') + 1 = \gamma_2(T)$. Since $\gamma_2(T) \le \gamma_2(T - p)$, we therefore infer that $\gamma_2(T - p) = \gamma_2(T)$. Thus, property (d) holds in the labeled tree (T, S).

Since $(T', S') \in \mathcal{F}_2$, property (e) holds in the labeled tree (T', S'). Thus there exists an edge $e \in E(T')$ such that $\gamma_2(T') = \gamma_2(T'_e)$ and $S_A(T') \subseteq S'_e$ for some γ_2-set S'_e of T'_e. By the structure of the tree T' that contains the path P', we can choose such a set S'_e to contain the vertex v and all vertices of status X in H_v. The set $S_e = S'_e \cup \{y\}$ is a 2-dominating set of T. Thus, $\gamma_2(T_e) \le |S'_e| + 1 = \gamma_2(T'_e) + 1 = \gamma_2(T') + 1 = \gamma_2(T)$. Clearly, $\gamma_2(T) \le \gamma_2(T_e)$. Consequently, $\gamma_2(T) = \gamma_2(T_e)$. Moreover, the set S_e is a γ_2-set of T_e that contains all vertices of status A in (T, S) noting that the vertex v and the vertices of status X in H_v in the labeled tree (T', S') changed to status A in (T, S). Thus, property (e) holds in the labeled tree (T, S).

We show next that T is γ_2-2-critical. As observed earlier, property (e) holds in the labeled tree (T, S). Let $F = \{f_1, f_2\} \subset E(T)$. If $F \subset E(T')$, then since T' is γ_2-2-critical, subdividing the two edges in F increases the 2-domination number of T', and therefore also increases the 2-domination number of T. If f_1 is one of the added edges vx or xy, then subdividing the edge f_1 necessarily increases the 2-domination number of T. From these properties we infer that the tree T is γ_2-2-critical. Thus, property (f) holds in the tree T.

Case 3. *(T, S) is obtained from (T', S') by operation $\mathcal{O}_3$.* In this case the vertex $v \in V(T')$ is a vertex of status A in (T', S') with at least one neighbor with status B in (T', S'). Let w be a neighbor of v in T' of status B. The labeled tree (T, S) is formed from the labeled tree (T', S') by adding to it a path xy and the edge vx, and letting $\text{sta}(x) = B$ and $\text{sta}(y) = A$, as illustrated in Figure 15.

Since every leaf belongs to every 2-dominating set, we note that $y \in D$. Since $(T', S') \in \mathcal{F}_2$, every γ_2-set of T' contains the vertex v of status A and can therefore

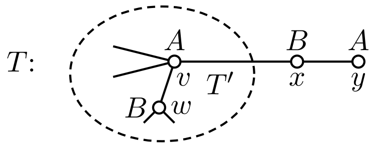


Figure 15: Operation $\mathcal{O}_3$

be extended to a 2-dominating set of T by adding to it the vertex y. Hence, $\gamma_2(T) \leq \gamma_2(T') + 1$. Since D is a γ_2-set of T, the set D contains exactly one of v and x. We show that $v \in D$. Suppose, to the contrary, that $v \notin D$, and so $x \in D$. If $w \notin D$, then let $D' = (D \setminus \{x, y\}) \cup \{w\}$. If $w \in D$, then let $D' = (D \setminus \{x, y\}) \cup \{v\}$. In both cases, $|D'| = |D| - 1$ and the set D' is a 2-dominating set of T' that contains at least one vertex of status B, implying that the set D' is not a γ_2-set of T' by induction. Therefore, $|D'| \geq \gamma_2(T') + 1$. Thus, $\gamma_2(T) = |D| = |D'| + 1 \geq \gamma_2(T') + 2$, contradicting our earlier observation that $\gamma_2(T) \leq \gamma_2(T') + 1$.

Hence, $v \in D$ (and $x \notin D$). We now let $D' = D \setminus \{y\}$. The set D' is a 2-dominating set of T', and so $\gamma_2(T') \leq |D'| = |D| - 1 = \gamma_2(T) - 1 \leq (\gamma_2(T') + 1) - 1 = \gamma_2(T')$. Hence, we must have equality throughout this inequality chain. In particular, $\gamma_2(T) = \gamma_2(T') + 1$. Applying the inductive hypothesis to $(T', S') \in \mathcal{F}_2$, we have $\gamma_2(T') = |S_A(T')| + |S_X(T')| = |S_A(T)| - 1 + |S_X(T)|$. Therefore, $\gamma_2(T) = \gamma_2(T') + 1 = |S_A(T)| + |S_X(T)|$. By Observation 5.1(e), $\gamma_2(T) \leq |S_A(T)| + |S_X(T)|$. Consequently, $\gamma_2(T) = |S_A(T)| + |S_X(T)|$. Analogously, $\gamma_2(T) = |S_A(T)| + |S_{\overline{X}}(T)|$. Thus, property (a) holds in the labeled tree (T, S).

Moreover, the inequalities in the previous paragraph are all equalities. In particular, $\gamma_2(T') = |D'|$, implying that D' is a γ_2-set of T'. Thus, from property (a) and Observation 5.1(e) we infer that property (b) holds in the labeled tree (T, S).

Since D' is a γ_2-set of T' and since the labeled tree (T', S') has property (c), we note that $S_A(T') \subseteq D'$ and $S_B(T') \cap D' = \varnothing$. Thus since $D = D' \cup \{y\}$ and y has status A, we infer that property (c) holds in the labeled tree (T, S).

Let $p \in S_X(T)$. Thus, $p \in V(T')$. Let Q_p be the path in T that starts and ends at vertices of status A, and whose internal vertices are all of status X and $\overline{X}$. Since (T', S') has property (d), $\gamma_2(T' - p) = \gamma_2(T')$. Moreover, we can choose a γ_2-set of $T' - p$ to contain the two vertices of status A on the path in T' that contain the vertex p. Indeed, we can choose a γ_2-set of $T' - p$ to contain all vertices in T' of status A. Such a set can therefore be extended to a 2-dominating set of T by adding to it the vertex y, implying that $\gamma_2(T - p) \leq \gamma_2(T' - p) + 1 = \gamma_2(T') + 1 = \gamma_2(T)$. Since $\gamma_2(T) \leq \gamma_2(T - p)$, we therefore infer that $\gamma_2(T - p) = \gamma_2(T)$. Thus, property (d) holds in the labeled tree (T, S).

Since $(T', S') \in \mathcal{F}_2$, property (e) holds in the labeled tree (T', S'). Thus there exists an edge $e \in E(T')$ such that $\gamma_2(T') = \gamma_2(T'_e)$ and $S_A(T') \subseteq S'_e$ for some γ_2-set S'_e of T'_e. In particular, the vertex v of status A belongs to the set S'_e. Thus the set S'_e can be extended to a γ_2-set of T_e by adding to it the vertex y of status A. Hence, $\gamma_2(T) = \gamma_2(T_e)$, and the resulting set $S_e = S'_e \cup \{y\}$ is a γ_2-set of T_e that contains all vertices of status A in T. Thus, property (e) holds in the labeled tree (T, S).

We show next that T is γ_2-2-critical. Let $F = \{f_1, f_2\} \subset E(T)$. If $F \subset E(T')$, then since T' is γ_2-2-critical, subdividing the two edges in F increases the 2-domination number of T', and therefore also increase the 2-domination number of T. If f_1 is one of the added edges vx or xy, then subdividing the edge f_1 necessarily increases the 2-domination number of T. From these properties we infer that the tree T is γ_2-2-critical. Thus, property (f) holds in the labeled tree (T, S).

Case 4. (T, S) is obtained from (T', S') by operation $\mathcal{O}_4$. In this case the vertex $v \in V(T')$ is a vertex of status A in (T', S') with at least one neighbor with status B in (T', S') or $\text{sta}(v) \in \{B, X\}$ where $X \in \{Y, Z\}$ in (T', S'). If v has status A in (T', S'), then let w be a neighbor of v of status B in (T', S'). The labeled tree (T, S) is formed from the labeled tree (T', S') by adding to it a path xyz and the edge vy, and letting $\text{sta}(x) = \text{sta}(z) = A$ and $\text{sta}(y) = B$, as illustrated in Figure 16.

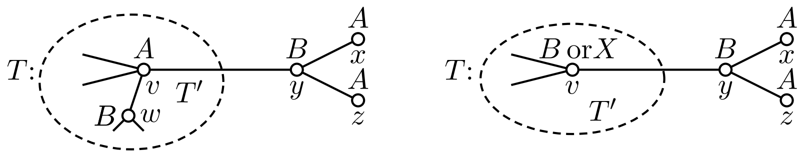


Figure 16: Operation $\mathcal{O}_4$

Since every leaf belongs to every 2-dominating set, we note that $\{x, z\} \subseteq D$. Every γ_2-set of T' can be extended to a 2-dominating set of T by adding to it the vertices x and z, and so $\gamma_2(T) \leq \gamma_2(T') + 2$.

We show that $y \notin D$. Suppose, to the contrary, that $y \in D$. By the minimality of the 2-dominating set D, the vertex $v \notin D$. Suppose that v has status A. If $w \in D$, then let $D' = (D \setminus \{x, y, z\}) \cup \{v\}$. If $w \notin D$, then let $D' = (D \setminus \{x, y, z\}) \cup \{w\}$. In both cases, $|D'| = |D| - 2$ and D' is a 2-dominating set of T' that contains at least one vertex of status B, implying that the set D' is not a γ_2-set of T' by induction. Therefore, $|D'| \geq \gamma_2(T')+1$. Thus, $\gamma_2(T) = |D| = |D'|+2 \geq \gamma_2(T')+3$, contradicting our earlier observation that $\gamma_2(T) \leq \gamma_2(T') + 2$. Suppose that v has status B. In this case, we let $D' = (D \setminus \{x, y, z\}) \cup \{v\}$. The resulting set D' is a 2-dominating set of T' that contains at least one vertex of status B, implying that the set D' is not a γ_2-set of T' by induction. Therefore, $|D'| \geq \gamma_2(T') + 1$, yielding a contradiction as before. Hence, the vertex v has status X. We now let $D' = D \setminus \{x, y, z\}$. Since $v \notin D$, the set D' is a 2-dominating set of $T' - v$. Since $(T', S') \in \mathcal{F}_2$, the labeled tree (T', S') has property (c). Thus since v is a vertex of status X in (T', S'), we infer that $\gamma_2(T') = \gamma_2(T' - v) \leq |D'| = |D| - 3 = \gamma_2(T) - 3$, and so $\gamma_2(T) \geq \gamma_2(T') + 3$, a contradiction. Therefore, $y \notin D$.

We now let $D' = D \setminus \{x, z\}$. Since $y \notin D$, the set D' is a 2-dominating set of T'. Thus, $\gamma_2(T') \leq |D'| = |D| - 2 = \gamma_2(T) - 2$, and so $\gamma_2(T) \geq \gamma_2(T') + 2$. As observed earlier, $\gamma_2(T) \leq \gamma_2(T')+2$. Consequently, $\gamma_2(T) = \gamma_2(T')+2$. Applying the inductive hypothesis to $(T', S') \in \mathcal{F}_2$, we have $\gamma_2(T') = |S_A(T')| + |S_X(T')| = |S_A(T)| - 2 +$

$|S_X(T)|$. Therefore, $\gamma_2(T) = \gamma_2(T') + 2 = |S_A(T)| + |S_X(T)|$. By Observation 5.1(e), $\gamma_2(T) \le |S_A(T)| + |S_X(T)|$. Consequently, $\gamma_2(T) = |S_A(T)| + |S_X(T)|$. Analogously, $\gamma_2(T) = |S_A(T)| + |S_{\overline{X}}(T)|$. Thus, property (a) holds in the labeled tree (T, S).

Moreover, the inequalities in the previous paragraph are all equalities. In particular, $\gamma_2(T') = |D'|$, implying that D' is a γ_2-set of T'. Thus, from property (a) and Observation 5.1(e) we infer that property (b) holds in the labeled tree (T, S).

Since D' is a γ_2-set of T' and since the labeled tree (T', S') has property (c), we note that $S_A(T') \subseteq D'$ and $S_B(T') \cap D' = \varnothing$. Thus since $D = D' \cup \{x, z\}$ and both x and z have status A, we infer that property (c) holds in the labeled tree (T, S).

Let $p \in S_X(T)$. Thus, $p \in V(T')$. Since (T', S') has property (d), $\gamma_2(T' - p) = \gamma_2(T')$. Every γ_2-set of $T' - p$ can be extended to a 2-dominating set of T by adding to it the vertices x and z, implying that $\gamma_2(T - p) \le \gamma_2(T' - p) + 2 = \gamma_2(T') + 2 = \gamma_2(T)$. Since $\gamma_2(T) \le \gamma_2(T - p)$, we therefore infer that $\gamma_2(T - p) = \gamma_2(T)$. Thus, property (d) holds in the labeled tree (T, S).

Since $(T', S') \in \mathcal{F}_2$, property (e) holds in the labeled tree (T', S'). Thus there exists an edge $e \in E(T')$ such that $\gamma_2(T') = \gamma_2(T'_e)$ and $S_A(T') \subseteq S'_e$ for some γ_2-set S'_e of T'_e. The set $S_e = S'_e \cup \{x, z\}$ is a 2-dominating set of T_e, and so $\gamma_2(T_e) \le |S_e| = |S'_e| + 2 = \gamma_2(T'_e) + 2 = \gamma_2(T') + 2 = \gamma_2(T)$. Thus, property (e) holds in the labeled tree (T, S).

We show next that T is γ_2-2-critical. Since property (e) holds in the labeled tree (T, S), there exists an edge $e \in E(T)$ such that $\gamma_2(T_e) = \gamma_2(T)$. Let $F = \{f_1, f_2\} \subset E(T)$. If $F \subset E(T')$, then since T' is γ_2-2-critical, subdividing the two edges in F increases the 2-domination number of T', and therefore also increase the 2-domination number of T. If f_1 is one of the added edges vx or xy, then subdividing the edge f_1 necessarily increases the 2-domination number of T. From these properties we infer that the tree T is γ_2-2-critical. Thus, property (f) holds in the labeled tree (T, S).

Case 5. (T, S) is obtained from (T', S') by operation $\mathcal{O}_5$. In this case, $v \in S_B(T')$, and so v has status B in T'. Furthermore, v is a strong support vertex in T', and so v has at least two leaf neighbors in T'. By our earlier observations, every leaf in T' has status A. The labeled tree (T, S) is formed from the labeled tree (T', S') by adding to it a new vertex x and the edge vx, and letting $\mathrm{sta}(x) = A$, as illustrated in Figure 12.

Since every leaf belongs to every 2-dominating set, the added vertex $x \in D$. Let $D' = D \setminus \{x\}$. Since the set D' contains at least two neighbors of v in T', we infer that the set D' is a 2-dominating set of T', and so $\gamma_2(T') \le |D'| = |D| - 1$. Applying the inductive hypothesis to $(T', S') \in \mathcal{F}_2$, we have $\gamma_2(T') = |S_A(T')| + |S_X(T')| = |S_A(T)| - 1 + |S_X(T)|$ for $X \in \{Y, Z\}$. Therefore, $\gamma_2(T) = |D| \ge \gamma_2(T') + 1 = |S_A(T)| + |S_X(T)|$. By Observation 5.1(e), $\gamma_2(T) \le |S_A(T)| + |S_X(T)|$. Consequently, $\gamma_2(T) = |S_A(T)| + |S_X(T)|$. Thus, property (a) holds in the labeled tree (T, S).

Moreover, the inequalities in the previous paragraph are all equalities. In particular, $\gamma_2(T') = \gamma_2(T) - 1$. Further, $\gamma_2(T') = |D'|$, implying that D' is a γ_2-set of T'. Thus, from property (a) and Observation 5.1(e) we infer that property (b) holds in the labeled tree (T, S).

Since D' is a γ_2-set of T' and since the labeled tree (T', S') has property (c), we note that $S_A(T') \subseteq D'$ and $S_B(T') \cap D' = \varnothing$. Thus since $D = D' \cup \{x\}$ and since x has status A, we infer that property (c) holds in the labeled tree (T, S).

Let $p \in S_X(T)$. Thus, $p \in V(T')$. Since the labeled tree (T', S') has property (d), $\gamma_2(T'-p) = \gamma_2(T')$. Every γ_2-set of $T'-p$ can be extended to a 2-dominating set of T by adding to it the vertex x, implying that $\gamma_2(T-p) \le \gamma_2(T'-p)+1 = \gamma_2(T')+1 = \gamma_2(T)$. Since $\gamma_2(T) \le \gamma_2(T-p)$, we therefore infer that $\gamma_2(T-p) = \gamma_2(T)$. Thus, property (d) holds in the labeled tree (T, S).

Since $(T', S') \in \mathcal{F}_2$, property (e) holds in the tree (T', S'). Thus there exists an edge $e \in E(T')$ such that $\gamma_2(T') = \gamma_2(T'_e)$ and $S_A(T') \subseteq S'_e$ for some γ_2-set S'_e of T'_e. In particular, the leaf neighbors of the vertex v of status A in T' belongs to the set S'_e. Thus the set S'_e can be extended to a γ_2-set of T_e by adding to it the vertex x of status A. Hence, $\gamma_2(T) = \gamma_2(T_e)$, and the resulting set $S_e = S'_e \cup \{x\}$ is a γ_2-set of T_e that contains all vertices of status A in T. Thus, property (e) holds in the labeled tree (T, S).

We show next that T is γ_2-2-critical. Since property (e) holds in the labeled tree (T, S), there exists an edge $e \in E(T)$ such that $\gamma_2(T_e) = \gamma_2(T)$. Let $F = \{f_1, f_2\} \subset E(T)$. If $F \subset E(T')$, then since T' is γ_2-2-critical, subdividing the two edges in F increases the 2-domination number of T', and therefore also increase the 2-domination number of T. Hence it remains for us to consider the case when one of f_1 or f_2 is the edge vx that was added to T'. Let f be such an edge, and so $f = xv$, and let z be the resulting new vertex obtained by subdividing the edge f. Let D_f be a γ_2-set of T_f. We note that the leaf $x \in D_f$. If $z \in D_f$, then we can replace z in D_f with the vertex v. Hence, we can choose the set D_f to contain the vertices v and x. The set $D' = D_f \setminus \{x\}$ is a 2-dominating set of T', and so $\gamma_2(T') \le |D'| = |D_f| - 1$. However since the vertex v has status B in (T', S'), by the inductive hypothesis the set D' is not a γ_2-set of T', implying that $\gamma_2(T') \le |D'| - 1 = |D_f| - 2 = \gamma_2(T_f) - 2$. Thus, $\gamma_2(T_f) \ge \gamma_2(T') + 2 = \gamma_2(T) + 1$. Thus, subdividing the edge vx increases the 2-domination number of the tree T. From these properties we infer that the tree T is γ_2-2-critical. Thus, property (f) holds in the labeled tree (T, S). This completes the proof of the theorem. □

6 γ_2-2-Critical trees

In this section, we characterize γ_2-2-critical trees. Adopting our earlier notation, if T is a tree and $e \in E(T)$, then we denote by T_e the tree obtained from T by subdividing the edge e. Further, if $\{e, f\} \subset E(T)$, then we denote by $T_{e,f}$ the tree obtained from T by subdividing both edges e and f. We are now in a position to prove Theorem 2.1. Recall the statement of the theorem.

Theorem 2.1. *A tree T is γ_2-2-critical if and only if $(T, S) \in \mathcal{F}_2$ for some labeling S.*

Proof. If $(T, S) \in \mathcal{F}_2$, then by Theorem 5.2(f) the tree T is γ_2-2-critical. Hence it suffices for us to show that if T is γ_2-2-critical tree, then $(T, S) \in \mathcal{F}_2$ for some labeling S. We proceed by induction on the order n of a γ_2-2-critical tree T. If $n \in \{1, 2, 3\}$, then T is not γ_2-2-critical. Hence, $n \geq 4$. If T is a star, then $T = K_{1,n-1}$. In this case, $\gamma_2(T) = n - 1$ (with the set of leaves in T as the unique γ_2-set of T) and $\gamma_2(T_e) = n$ for every edge $e \in E(T)$, implying that T is γ_2-1-critical, a contradiction. Thus, T is not a star. If $n = 4$, then since T is not a star, the tree T is a path, namely $T = P_4$, and the labeled tree $(T, S) \in \mathcal{F}_2$ where S is the labeling associated with the labeled base tree shown in Figure 7. This proves the base cases when $n \leq 4$. Let $n \geq 5$ and assume that if T' is γ_2-2-critical tree of order n' where $n' < n$, then $(T', S') \in \mathcal{F}_2$ for some labeling S'.

We now consider the γ_2-2-critical tree T of order $n \geq 5$ and diameter $\operatorname{diam}(T) \geq 3$. Among all longest paths in T (called a *diametrical path* in the literature), let $P\colon v_0 v_1 \ldots v_d$ be chosen so that

(1) $\deg_T(v_1)$ is a maximum, and
(2) subject to (1), the vertex v_2 has the minimum number of leaf neighbors.

We note that $d = \operatorname{diam}(T) \geq 3$. We now root the tree T at the vertex $r = v_d$. Necessary, v_1 is a support vertex of T and all children of v_1 are leaves. In particular, v_0 is a leaf in T. We proceed further by proving two claims.

Claim 1 *If $\deg_T(v_1) \geq 4$, then $(T, S) \in \mathcal{F}_2$ for some labeling S.*

Proof. Suppose that $\deg_T(v_1) \geq 4$. In this case, the vertex v_1 is a strong support vertex in T with at least three leaf neighbors. Let $T' = T - v_0$ and let T' have order n', and so $n' = n - 1$. By Lemma 4.4, the tree T' is a γ_2-2-critical tree. Applying the inductive hypothesis to the tree T' of order $n - 1$, the labeled tree $(T', S') \in \mathcal{F}_2$ for some labeling S'. Since v_1 is a strong support vertex in T', the vertex v_1 has status B in (T', S') by Observation 5.1(h). Moreover by Observation 5.1(a), every leaf has status A in (T', S'). Applying Operation $\mathcal{O}_5$ to the labeled tree (T', S') we add back the vertex v_0 and the edge $v_0 v_1$, and assign to v_0 the status A, thereby producing a labeled tree $(T, S) \in \mathcal{F}_2$. (□)

Claim 2 *If $\deg_T(v_1) = 3$, then $(T, S) \in \mathcal{F}_2$ for some labeling S.*

Proof. Suppose that $\deg_T(v_1) = 3$. Let u_0 be the child of v_1 different from v_0, and so $C(v_1) = \{u_0, v_0\}$ and $D[v_1] = C(v_1) \cup \{v_1\}$. Let T' be the tree obtained from T by deleting v_1 and its two children, that is, $T' = T - D[v]$. By Lemma 4.5, $\gamma_2(T) = \gamma_2(T') + 2$. Since the tree T is γ_2-2-critical, by Lemma 4.6 the tree T' is γ_2-2-critical. Applying the inductive hypothesis to the tree T' of order $n - 3$, the labeled tree $(T', S') \in \mathcal{F}_2$ for some labeling S'. Recall that v_2 is the parent of the vertex v_1, and v_3 is the parent if v_2 in the tree T.

Suppose, to the contrary, that $\operatorname{sta}(v_2) = A$ and the vertex v_2 has no neighbor of status B in the labeled tree (T', S'). From properties of labeled trees that belong to the family $\mathcal{F}_2$, we infer by Observation 5.1 that v_2 is a leaf in T' and has status A

and either its parent v_3 has status A or its parent v_3 has status X where $X \in \{Y, Z\}$ in the labeled tree (T', S'). In both cases, by Theorem 5.2(b) there exists a γ_2-set S' of T' that contains both v_2 and v_3. The set $S = (S' \setminus \{v_2\}) \cup \{u_0, v_0, v_1\}$ is a 2-dominating set of T, and so $\gamma_2(T') + 2 = \gamma_2(T) \leq |S| = |S'| + 2 = \gamma_2(T') + 2$. Consequently, we must have equality throughout this inequality chain, implying that S is a γ_2-set of T. We note that the set S contains the strong support vertex v_1. However by Lemma 4.2, $v_1 \in \mathcal{N}_2(T)$. This produces a contradiction.

Hence, $\text{sta}(v_2) = A$ and the vertex v_2 has at least one neighbor with status B or $\text{sta}(v_2) \in \{B, X\}$ where $X \in \{Y, Z\}$ in the labeled tree (T', S'). In this case, applying Operation $\mathcal{O}_4$ to (T', S') we add back the path $u_0v_1v_0$ and the edge v_1v_2, and assign to v_1 the status B and to each of u_0 and v_0 the status A, thereby producing a labeled tree $(T, S) \in \mathcal{F}_2$. (□)

By Claims 1 and 2, we may assume that $\deg_T(v_1) = 2$, for otherwise $(T, S) \in \mathcal{F}_2$ for some labeling S. By our choice of the path P, we infer that if $Q\colon u_0u_1 \dots u_d$ is a diametrical path in T, then $\deg_T(u_1) = \deg_T(u_{d-1}) = 2$. Recall that $d = \text{diam}(T) \geq 3$. If $d = 3$, then $\deg_T(v_1) = \deg_T(v_{d-1}) = 2$, and so $T = P_4$, contradicting the fact that $n \geq 5$. Hence, $d \geq 4$.

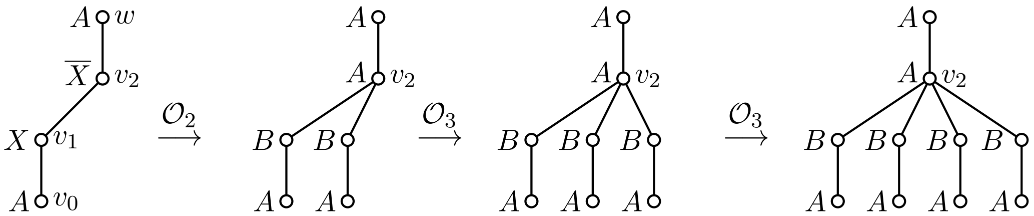
Claim 3 *If $d = 4$, then $(T, S) \in \mathcal{F}_2$ for some labeling S.*

Proof. Suppose that $d = 4$. Thus the path P is given by $v_0v_1v_2v_3v_4$. By our earlier observations, every neighbor of v_2 is either a leaf or a support vertex of degree 2. Let S be a γ_2-set of T. By Observation 4.1, the set S contains all leaves in T. In particular, $v_0 \in S$. If $v_1 \in S$, then we can simply replace the vertex v_1 in S with the vertex v_2. Hence, we may assume that $v_1 \notin S$, implying that $v_2 \in S$. By Lemma 4.2, we therefore infer that v_2 is not a strong support vertex in a T, and so either v_2 has exactly one leaf neighbor or every neighbor of v_2 is a support vertex of degree 2.

We show that v_2 has exactly one leaf neighbor. Suppose, to the contrary, that v_2 has no leaf neighbor, and so every neighbor of v_2 is a support vertex of degree 2. Thus in this case, T is obtained from a star $K_{1,k}$ where $k \geq 2$ by subdividing every edge exactly once. The resulting tree T has order $n = 2k + 1$ and satisfies $\gamma_2(T) = k + 1$. Further the set $L(T) \cup \{v_2\}$ consisting all k leaves of T together with the central vertex v_2 of T is the unique γ_2-set of T. However subdividing any edge of T increases the 2-domination number, contradicting the fact that T is a γ_2-2-critical tree.

Therefore, v_2 has exactly one leaf neighbor, say w. Thus, T is obtained from a star $K_{1,k}$ where $k \geq 2$ by subdividing every edge exactly once, and then adding a new vertex w and adding the edge v_2w. Starting with the labeled base tree (T_1, S_1) given by the path $v_0v_1v_2w$ where v_0 and w have status A, v_1 has status X and v_2 has status $\overline{X}$, we apply Operation $\mathcal{O}_2$ to (T', S') by adding the path v_3v_4 and the edge v_2v_3, and changing the status of v_1 and v_2 to B and A, respectively, and letting v_3 and v_4 have status B and A, respectively, to produce the labeled tree (T_2, S_2). If $k = 2$, then we let $(T, S) = (T_2, S_2)$. Otherwise if $k \geq 3$, then by $k-2$ applications of Operation $\mathcal{O}_3$ with v_2 as the link vertex we produce the labeled tree $(T, S) = (T_k, S_k)$ where S is the labeling that labels all support vertices different from v_2 the label B

and labels all other vertices the label A. Thus, $(T, S) \in \mathcal{F}_2$ for some labeling S. In the special case when $k = 4$, the construction of the labeled tree $(T, S) = (T_4, S_4)$ is illustrated in Figure 17. (□)



(a) (T_1, S_1) (b) (T_2, S_2) (c) (T_3, S_3) (c) (T_4, S_4)

Figure 17: Construction a tree $(T, S) = (T_4, S_4)$ in the proof of Claim 3

By Claim 3, we may assume that $d = \text{diam}(T) \geq 5$, for otherwise the desired result follows. By our earlier observations, every neighbor of v_2 is either a leaf or a support vertex of degree 2. We show next that no neighbor of v_2 is a leaf.

Claim 4 *Every child of v_2 is a support vertex in T of degree 2.*

Proof. Suppose, to the contrary, that the vertex v_2 has a neighbor, say u_2, that is leaf. Let S be a γ_2-set of T. By Observation 4.1, the set S contains all leaves in T. In particular, $v_0 \in S$. If $v_1 \in S$, then we can replace the vertex v_1 in S with the vertex v_2. Hence, we may assume that $v_1 \notin S$, implying that $v_2 \in S$. By our choice of the path P, we note that $\deg_T(v_{d-1}) = 2$. Symmetrical arguments show that we can choose the set S so that $v_{d-1} \notin S$, implying that $\{v_{d-2}, v_d\} \subset S$. Moreover by our choice of the path P, the vertex v_{d-2} has at least as many leaf neighbors as does the vertex v_2, implying that the vertex v_{d-2} has at least one leaf neighbor, say u_{d-2}. By assumption, $d \geq 5$, and so v_2 and v_{d-2} are distinct vertices. By our earlier observations, $\{u_2, v_2, u_{d-2}, v_{d-2}\} \subset S$. Hence letting $e = u_2v_2$ and $f = u_{d-2}v_{d-2}$, the set S is a 2-dominating set of $T_{e,f}$, implying that $\gamma_2(T_{e,f}) \leq |S| = \gamma_2(T)$, contradicting the fact that T is a γ_2-2-critical tree. Hence, the vertex v_2 has no leaf neighbor, implying by our earlier observations that every child of v_2 is a support vertex in T of degree 2. (□)

Suppose that v_2 has k children, and so $k = \deg_T(v_2) - 1 \geq 1$. Let $C(v_2) = \{v_{1,1}, \ldots, v_{k,1}\}$ be the set of k children of v_2. Further, let $v_{i,0}$ denote the (unique) child of $v_{i,1}$ for $i \in [k]$. We note that $v_{i,0}$ is a leaf for all $i \in [k]$. Renaming vertices if necessary, we may assume that $v_0 = v_{1,0}$ and $v_1 = v_{1,1}$. Let $K = \{v_{1,0}, \ldots, v_{k,0}\}$, and let T' be obtained from T by deleting all $2k$ descendants of v_2. Thus, $T' = T - D(v_2)$, where $D(v_2) = C(v_2) \cup K$. We note that the vertex v_2 is a leaf in the tree T'. Let T' have order n', and so $n' = n - 2k$.

Claim 5 $\gamma_2(T) = \gamma_2(T') + k$.

Proof. Let S be a γ_2-set of T. By Observation 4.1, $v_{i,0} \in \mathcal{A}_2(T)$, and so $v_{i,0} \in S$ for all $i \in [k]$. By our earlier observations, $\deg_T(v_{i,1}) = 2$ for all $i \in [k]$. If $v_{i,1} \in S$ for some $i \in [k]$, then we can replace $v_{i,1}$ in S by the vertex v_2. Hence we can choose the set S so that $S \cap C(v_2) = \emptyset$, implying that $v_2 \in S$. The set $S \setminus K$ is a 2-dominating set of T', and so $\gamma_2(T') \le |S| - |K| = \gamma_2(T) - k$. Conversely, let S' be a γ_2-set of T'. By our earlier observations, the vertex v_2 is a leaf in T', and so $v_2 \in S'$. The set S' can be extended to a 2-dominating set of T by adding to it the set K, and so $\gamma_2(T) \le |S'| + |K| = \gamma_2(T') + k$. Consequently, $\gamma_2(T) = \gamma_2(T') + k$. (□)

Claim 6 *The tree T' is γ_2-2-critical.*

Proof. Let $\{e_1, e_2\} \subset E(T')$, and let S' be a γ_2-set of T'_{e_1,e_2}. We show that $\gamma_2(T') < \gamma_2(T'_{e_1,e_2})$. Suppose, to the contrary, that $\gamma_2(T') \ge \gamma_2(T'_{e_1,e_2})$, implying that $\gamma_2(T') = \gamma_2(T'_{e_1,e_2})$ since subdividing edges cannot decrease the 2-domination number. By Claim 5, $\gamma_2(T) = \gamma_2(T')+k$. Since the vertex v_2 is a leaf in T'_{e_1,e_2}, we note that $v_2 \in S'$. The set S' can therefore be extended to a 2-dominating set of T by adding to it the set K, and so $\gamma_2(T_{e_1,e_2}) \le |S'|+|K| = \gamma_2(T'_{e_1,e_2})+k = \gamma_2(T')+k = \gamma_2(T)$. Hence we must have equality throughout this inequality chain. In particular, $\gamma_2(T_{e_1,e_2}) = \gamma_2(T)$. This contradicts the fact that T is a γ_2-2-critical tree. Hence, $\gamma_2(T') < \gamma_2(T'_{e_1,e_2})$.

It remains for us to show that there exists an edge $e' \in E(T')$ such that $\gamma_2(T') = \gamma_2(T'_e)$. Since T is a γ_2-2-critical tree, there exists an edge $e \in E(T)$ such that $\gamma_2(T) = \gamma_2(T_e)$. Suppose that $e \notin E(T')$. Renaming vertices if necessary, we may assume that $e = v_0v_1$ or $e = v_1v_2$. By symmetry, we may assume that $e = v_0v_1$ (noting that $T_{v_0v_1} \cong T_{v_1v_2}$). If $k \ge 2$, then by our earlier observations we may choose the set S_e so that $\{v_1, v_2\} \cup K \subset S_e$, implying that $|S_e| \ge \gamma_2(T) + 1$, a contradiction. Hence, $k = 1$, implying that $\deg_T(v_2) = 2$. However in this case, the tree T_e is isomorphic to the tree T_f where $f = v_2v_3$. Therefore we may choose the edge e so that $e \in E(T')$, where recall that $\gamma_2(T) = \gamma_2(T_e)$.

Let S_e be a γ_2-set of T'. By Theorem 5.2(e), we can choose the set S_e so that $v_2 \in S_e$. Thus the set $S_e \setminus K$ is a 2-dominating set of T'_e, and so $\gamma_2(T'_e) \le |S_e| - |K| = \gamma_2(T_e) - k = \gamma_2(T) - k = \gamma_2(T')$. Since subdividing an edge cannot decrease the 2-domination number, we infer that $\gamma_2(T'_e) = \gamma_2(T')$. Thus there exists an edge in T' which when subdivided does not change the 2-domination number. As observed earlier, if e_1 and e_2 are two arbitrary distinct edge of T', then $\gamma_2(T') < \gamma_2(T'_{e_1,e_2})$. These observations imply that the tree T' is γ_2-2-critical. (□)

By Claim 6, the tree T' is γ_2-2-critical. Applying the inductive hypothesis to the tree T', the labeled tree $(T', S') \in \mathcal{F}_2$ for some labeling S'. By Observation 5.1(a), $L(T') \subseteq S_A(T')$. In particular, the leaf v_2 in the labeled tree (T', S') has status A. We now consider the neighbor of v_2 in T', namely the vertex v_3.

Claim 7 *If v_3 has status B in (T', S'), then $(T, S) \in \mathcal{F}_2$ for some labeling S.*

Proof. Suppose that the vertex v_3 has status B in the labeled tree (T', S'). Thus the vertex v_2 has status A in (T', S'), with its (unique) neighbor of status B. Applying Operation $\mathcal{O}_3$ to (T', S') we add back the deleted vertices v_0 and v_1 and the deleted edges v_0v_1 and v_1v_2, and assign to v_0 the status A and to v_1 the status B, thereby producing a labeled tree $(T, S) \in \mathcal{F}_2$. (□)

Claim 8 *If v_3 does not have status B in (T', S'), then $(T, S) \in \mathcal{F}_2$ for some labeling S.*

Proof. Suppose that the vertex v_3 does not have status B in the labeled tree (T', S'). Thus the vertex v_3 has status A or status X where $X \in \{Y, Z\}$ in (T', S'). In both cases, applying Operation $\mathcal{O}_1$ to (T', S') we add back the deleted vertices v_0 and v_1 and the deleted edges v_0v_1 and v_1v_2. Further we assign to v_0 the status A, to v_1 the status X, and to v_2 the status $\overline{X}$, thereby producing a labeled tree $(T, S) \in \mathcal{F}_2$. (□)

The proof of Theorem 2.1 now follows from Claims 7 and 8. □

Acknowledgments

The second author, Michael A. Henning, acknowledges support from the CoE-MaSS under graph number 2025-014-GRA-Structural Graph Theory. Opinions expressed and conclusions arrived at by the second author are those of the author and not necessarily to be attributed to the CoE-MaSS.

References

[1] M. Atapour, S. M. Sheikholeslami, A. Hansberg, L. Volkmann and A. Khodkar, 2-Domination subdivision number of graphs, *AKCE J. Graphs. Combin.* **5** (2008), 165–173.

[2] M. Atapour, S. M. Sheikholeslami and A. Khodkar, Trees whose 2-domination subdivision number is 2, *Opuscula Math.* **32** (2012), 423–437.

[3] M. Chellali, Bounds on the 2-domination number in cactus graphs, *Opuscula Math.* **26** (2006), 5–12.

[4] E. J. Cockayne, B. Gamble and B. Shepherd, An upper bound for the k-domination number of a graph, *J. Graph Theory.* **9** (1985), 533–534.

[5] M. Dettlaff, M. A. Henning, M. Lemańska, A. Roux and J. Topp, 2-Domination edge subdivision in trees, (submitted).

[6] O. Favaron, A. Hansberg and L. Volkmann, On k-domination and minimum degree in graphs, *J. Graph Theory.* **57** (2008), 33–40.

[7] J. F. Fink and M. S. Jacobson, n-Domination in graphs, in: "Graph Theory with Application to Algorithms and Computer Sciencei", John Wiley and Sons, New York, 282–300 (1985).

[8] J. Fujisawa, A. Hansberg, T. Kubo, A. Saito, M. Sugita and L. Volkmann, Independence and 2-domination in bipartite graphs, *Australas. J. Combin.* **57** (2008), 265–268.

[9] A. Hansberg and L. Volkmann, Multiple domination, *Topics in Domination in Graphs,* 151–203. Series: Developments in Mathematics, Vol. 64, Springer, Cham., 2020.

[10] T. W. Haynes, S. T. Hedetniemi and M. A. Henning (eds), *Topics in Domination in Graphs*; Series: Developments in Mathematics, Vol. 64, Springer, Cham., 2020, viii + 545 pp.

[11] T. W. Haynes, S. T. Hedetniemi and M. A. Henning (eds), *Structures of Domination in Graphs*; Series: Developments in Mathematics, Vol. 66, Springer, Cham., 2021, viii + 536 pp.

[12] T. W. Haynes, S. T. Hedetniemi and M. A. Henning, *Domination in Graphs: Core Concepts*; Series: Springer Monographs in Mathematics, Springer, Cham., 2023, xx + 644 pp.

[13] M. A. Henning and A. Yeo, *Total Domination in Graphs*; Series: Springer Monographs in Mathematics, Springer, Cham, 2013. xiv + 178 pp.

(Received 26 Sep 2024; revised 29 May 2025)