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The 3-connected binary matroids with circumference 8, part II

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Abstract

This is the second paper in a sequence of three that describe the 3-connected binary matroids with circumference 8. A matroid M is said to be bent provided it has a maximum size circuit C such that M/C has a connected component with rank exceeding 1. In this paper, we describe the bent 3-connected binary matroids with circumference 8.

1 Introduction

We assume familiarity with matroid theory. The notation and terminology used in this article follow Oxley [9]. For a positive integer n, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. Let $\mathcal{SC}(M)$ be the family of series classes of a matroid M.

There are many sharp extremal results in matroid theory whose bounds depend on the circumference. When one of these bounds is used to prove a theorem, it may imply that a counter-example to it must have small circumference. It is likely that the knowledge of all matroids with small circumference may simplify the proof of such a result. This was the motivation to construct the 3-connected binary matroids with circumference at most 7 and large rank by Cordovil, Maia Jr. and Lemos [2]. In this paper, we continue to construct all 3-connected binary matroids with circumference 8 and large rank. We hope to apply our results together with the main result of Lemos and Oxley [8] to describe the 3-connected binary matroids with no odd circuit with size exceeding 7, extending the main result of Chun, Oxley and Wetzler [1].

Lemos and Oxley [7] establish that 6 is a sharp lower bound for the circumference of a 3-connected matroid with large rank. Cordovil and Lemos [3] constructed the 3-connected matroids with circumference 6 and large rank. These matroids can be described using a natural generalization of book for non-binary maroids.

A binary matroid M is said to be a *book* having *pages* $M_1, M_2, \ldots, M_n$, for $n \geq 2$, and r-*spine* T, for $r \geq 2$, provided:

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(i) $M_1, M_2, \ldots, M_n$ are binary matroids; and
(ii) $T = E(M_1) \cap E(M_2) \cap \cdots \cap E(M_n)$; and
(iii) $E(M_1) - T, E(M_2) - T, \ldots, E(M_n) - T$ are pairwise disjoint sets; and
(iv) $M_1|T = M_2|T = \cdots = M_n|T = K$ is isomorphic to $PG(r-1, 2)$; and
(v) $M = P_K(M_1, M_2, \ldots, M_n)$, that is, the circuit space of M is spanned by $\mathcal{C}(M_1) \cup \mathcal{C}(M_2) \cup \cdots \cup \mathcal{C}(M_n)$.

The main results of Cordovil, Maia Jr. and Lemos [2] can be stated using the concept of book proposed by Chun, Oxley and Wetzler [1] (see Lemos [4]). We need books having a 3-spine in [4].

For an integer k exceeding 3, we denote by Z_k the rank-k binary spike. There is just one element of Z_k belonging to k triangles. This element is called the *tip* of Z_k. All matroids obtained from Z_k by deleting an element other than the tip are isomorphic. When $k = 4$, such a matroid is isomorphic to S_8. The *tip* of S_8 is its unique element belonging to three triangles. Remember that a matroid M is said to be *bent* provided it has a maximum size circuit C such that M/C has a connected component with rank exceeding 1. Now, we state the main result of this paper:

Theorem 1.1 *Let M be a bent 3-connected binary matroid with circumference 8. If $r(M) \geq 14$, then there is a book M' with pages $M_1, M_2, \ldots, M_n$ and 2-spine T such that, for a fixed $e \in T$, M_i is isomorphic to a matroid belonging to $\{Z_4, S_8, F_7, M(K_4)\}$ and, when $r(M_i) = 4$, e is the tip of M_i, for each $i \in [n]$, and $M = M' \backslash T'$, for some $T' \subseteq T$. Moreover, $m = |\{i \in [n] : r(M_i) = 4\}| \geq 3$ and $m + n \geq 12$.*

In Theorem 1.1, the circumference of M' is 8 provided $m \geq 2$ and $n \geq 3$. We need $m \geq 3$ to guaranty that M is bent. The condition $m + n \geq 12$ follows from $r(M) \geq 14$.

Let M be a *book* having *pages* M_1, M_2, M_3 and 2-*spine* $T = \{e, f, g\}$. Assume that M_1, M_2 and M_3 are isomorphic to Z_4 having e, e and f as tips respectively. For $i \in \{1, 2, 3\}$, M_i has a circuit C_i such that $|C_i| = 4, |C_i \cap T| = 1$ and $C_i \cap T$ can be chosen to be any element of T other than the tip of M_i. We can choose C_1, C_2 and C_3 such that $g \in C_1, f \in C_2$ and $e \in C_3$. Note that $C = C_1 \triangle C_2 \triangle C_3 \triangle T$ is a 9-element circuit of M. This example justify the condition imposed on all pages with rank-4 in Theorem 1.1 to have the same tip.

Now, we describe the main result of Lemos [4]. Let M be an unbent 3-connected binary matroid having circumference 8. We say that M is *crossing* when M has an 8-element circuit C, sets X and Y contained in different rank-1 connected components of M/C such that $|X| = |Y| = 2$ and $M|(C \cup X \cup Y)$ is a subdivision of $M(K_4)$.

Theorem 1.2 *Let M be an unbent crossing 3-connected binary matroid with circumference 8. If $r(M) \geq 11$, then*

(i) M is a 3-connected rank-preserving restriction of M'', where M'' is a book with pages $M_1, M_2, \ldots, M_t$, for $t = r(M) - 3$, and 3-spine F such that M_i is isomorphic to $PG(3, 2)$, for every $i \in [t]$; or

(ii) $M = M''\backslash T'$, *where* $T' \subseteq T$ *and* M'' *is a book with pages* $M_1, M_2, \ldots, M_t$, *for* $t = r(M) - 5$, *and* 2*-spine* T *such that, for each* $i \in [t] - \{1\}$, M_i *is isomorphic to* $K(K_4)$ *or* F_7 *and* M_1 *is a* 3*-connected binary matroid satisfying:*

(A) M_1 *has a circuit* D *such that* $|D| = 6$ *and* $|D \cap T| = 2$*; and*

(B) the simplification of M_1/T *is isomorphic to* F_7^* *or* $AG(3, 2)$.

If M'' is the book described in Theorem 1.2(i), then M'' is internally 4-connected and $M''\backslash F$ is 4-connected. Both M'' and $M''\backslash F$ have circumference equal to 8. Note that $M''\backslash F$ has a rank-preserving restriction isomorphic to $M(K_{4,t})$.

Every matroid described in the conclusion of Theorem 1.1 is a bent 3-connected binary matroid with circumference 8. To restrict the matroids described in Theorem 1.2(i) so that they are contained in the class of unbent crossing 3-connected binary matroids with circumference 8 would produce a cumbersome statement. In Lemos [4], we state the condition. Moreover, we establish that any matroid described in Theorem 1.2(i) has circumference at most 8. We also prove that, when M'' satisfies Theorem 1.2(ii), the circumference of both M'' and $M''\backslash T$ are 8.

The next results about the circuit space of a binary matroid M are used without reference along this paper:

(i) A cycle of M is an union of pairwise disjoint circuits of M.
(ii) The symmetric difference of circuits of M is a cycle of M.
(iii) The circuit space of M is spanned by the circuits of M and it has dimension equal to $r^*(M)$.

2 Seymour's Arcs Theorem

A result of Seymour [10] that gives conditions to extend a k-separation of a restriction to the whole matroid will be fundamental in this paper. To state this result, we need to give more definitions. Let M be a matroid. For $F \subseteq E(M)$, an *F-arc* (see Section 3 of [10]) is a minimal non-empty subset A of $E(M) - F$ such that there exists a circuit C of M with $C - F = A$ and $C \cap F \neq \emptyset$. Such a circuit C is called an *F-fundamental for A*. Let A be an F-arc and $P \subseteq F$. Then $A \to P$ if there is an F-fundamental for A contained in $A \cup P$. Thus $A \not\to P$ denotes that there is no such F-fundamental. Note that A is an F-arc if and only if $A \in \mathcal{C}(M/F) - \mathcal{C}(M)$.

Theorem 2.1 ((3.8) of Seymour [10]) *Let* M *be a matroid on* S, *let* $Z \subseteq S$, *and let* (P_1, P_2) *be a partition of* Z. *Then either there is a* Z*-arc* A *such that* $A \not\to P_1, A \not\to P_2$, *or there is a partition* (X_1, X_2) *of* S *such that* $X_i \cap Z = P_i (i = 1, 2)$ *and*

$$r(X_1) + r(X_2) - r(S) = r(P_1) + r(P_2) - r(Z).$$

The proof of Theorem 2.1 can be adapted to establish that:

Theorem 2.2 *Let M be a matroid on S, let $Z \subseteq S$, and let (P_1, P_2) be a partition of Z. If $A \to P_1$ or $A \to P_2$, for each Z-arc A, and*

$$P_2' = \bigcup\{A : A \subseteq S - Z \text{ and } A \text{ is a } Z\text{-arc such that } A \to P_2\}$$

then, when $X_2 = P_2 \cup P_2'$ and $X_1 = S - X_2$,

$$r(X_1) + r(X_2) - r(S) = r(P_1) + r(P_2) - r(Z).$$

The next result is Lemma 2.2 of Lemos [4].

Lemma 2.3 *Let M be a connected matroid. Suppose that $M|F$ is connected, for $\emptyset \neq F \subsetneqq E(M)$. If $|A| \leq 2$, for every F-arc A, then every connected component of M/F has rank equal to 0 or 1.*

3 Basic results about theta sets and their arcs

We say that L is a *theta set* of a matroid M provided $L \subseteq E(M)$ and $M|L$ is a subdivision of $U_{1,3}$. When L_1, L_2 and L_3 are the series classes of $M|L$, $\{L_1, L_2, L_3\}$ is said to be *the canonical partition* of L in M. If $|L_1| = a, |L_2| = b$ and $|L_3| = c$, then L is said to be an *(a,b,c)-theta set* of M. The next two results are respectively Lemmas 2.3 and 2.4 of Lemos [4].

Lemma 3.1 *Let M be a matroid with circumference 8. If L is a theta set of M, then $|L| \leq 12$. Moreover, when $|L| \in \{11, 12\}$, L is an (a, b, c)-theta set of M, where $(a, b, c) \in \{(4, 4, 4), (4, 4, 3), (5, 3, 3)\}$.*

Lemma 3.2 *If M is a matroid with circumference 8, then the following statements are equivalent:*

(i) M is unbent.
(ii) Every theta set of M has at most 10 elements.

Now, we establish that Theorem 1.1 is equivalent to:

Theorem 3.3 *Let M be a 3-connected binary matroid with circumference 8. If $r(M) \geq 14$, then*

(i) $|L| \leq 10$, for every theta set L of M; or
(ii) there is a book M' with pages $M_1, M_2, \dots, M_n$ and 2-spine T such that, for a fixed $e \in T$, M_i is isomorphic to a matroid belonging to $\{Z_4, S_8, F_7, M(K_4)\}$ and, when $r(M_i) = 4$, e is the tip of M_i, for each $i \in [n]$, and $M = M' \backslash T'$, for some $T' \subseteq T$. Moreover, if $m = |\{i \in [n] : r(M_i) = 4\}|$, then $m \geq 3$ and $m + n \geq 12$.

Lemma 3.2 implies that Theorem 3.3(i) holds if and only if M is unbent. Hence Theorem 3.3(ii) is the description of bent 3-connected binary matroid with circumference 8. But it is just the conclusion of Theorem 1.1.

When Theorem 3.3(i) holds, we conclude that M/C has only rank-0 or rank-1 connected components, when C is a circuit of M such that $|C| = 8$. In [4, 5], we analyse how these small connected components attach to C to obtain the others 3-connected binary matroids having circumference 8.

Now, we fix some notation that we use along this section and in the next two. Let $\{L_1, L_2, L_3\}$ be the canonical partition of a theta set L of a binary matroid M. We assume that $|L_1| \geq |L_2| \geq |L_3|$. We define, for $i \in [3]$,

$$\begin{aligned}\mathcal{A} &= \{A \subseteq E(M) - L : A \text{ is an } L\text{-arc of } M\},\\ \mathcal{A}_i &= \{A \in \mathcal{A} : A \to L_i\} \text{ and}\\ \mathcal{A}' &= \mathcal{A} - (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3).\end{aligned}$$

(Note that $\mathcal{A}, \mathcal{A}_i$ and $\mathcal{A}'$ depend on M, L, L_1, L_2 and L_3. We do not emphasize these dependencies to avoid a cumbersome notation. Consequently, when we use any of these subsets of L-arcs of M, it is implicit that M is the binary matroid, L is the theta set and its canonical partition is $\{L_1, L_2, L_3\}$.)

Lemma 3.4 *If M is 3-connected, then $\mathcal{A}' \neq \emptyset$.*

Proof. Suppose that $\mathcal{A}' = \emptyset$. Therefore $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$. Consider the 2-separation $\{L_1, L_2 \cup L_3\}$ of $M|L$. As $A \to L_1$ or $A \to L_2 \cup L_3$, for every $A \in \mathcal{A}$, it follows, by Theorem 2.1, that M has a 2-separation $\{X, Y\}$ such that $X \cap L = L_1$ and $Y \cap L = L_2 \cup L_3$, a contradiction. Hence $\mathcal{A}' \neq \emptyset$. □

Lemma 3.5 *If $A \in \mathcal{A}'$, then the cosimplification of $M|(L \cup A)$ is isomorphic to $M(K_4)$ or F_7^*. Moreover, when $\mathrm{co}(M|(L \cup A)) \cong M(K_4)$, there is $i \in [3]$ such that L_i is a series class of $M|(L \cup A)$.*

Proof. Observe that $r^*(M|(L \cup A)) = 3$. If C is a circuit of M such that $A \subseteq C \subseteq L \cup A$, then the circuit space of $M|(L \cup A)$ is spanned by $L_1 \cup L_2$, $L_1 \cup L_3$ and C. To conclude the proof, we need to establish that any pair of circuits of $M|(L \cup A)$ meet, since $M|(L \cup A)$ is coloopless. Suppose that D and D' are circuits of $M|(L \cup A)$ such that $D \cap D' = \emptyset$. As A is a series class of $M|(L \cup A)$, we may assume that $A \cap D = \emptyset$, that is, $D = L_i \cup L_j$, for a 2-subset $\{i, j\}$ of [3]. Thus $A \subseteq D' \subseteq A \cup L_k$, where $\{i, j, k\} = [3]$. We arrive at a contradiction because $A \not\to L_k$. Therefore any two circuits of M meet. Consequently $M|(L \cup A)$ has 6 or 7 series classes. Moreover, when it has 6 series classes one must be L_i, for some $i \in [3]$. The result follows. □

Lemma 3.5 suggests the following partition for $\mathcal{A}'$:

$$\begin{aligned}\mathcal{A}'_F &= \{A \in \mathcal{A}' : \mathrm{co}(M|(L \cup A)) \cong F_7^*\},\\ \mathcal{A}'_K &= \{A \in \mathcal{A}' : \mathrm{co}(M|(L \cup A)) \cong M(K_4)\}.\end{aligned}$$

Now, we define a circuit C_A of M, for each $A \in \mathcal{A}$, such that $A \subseteq C_A \subseteq L \cup A$. We use this notation along this section and in the next two. We have three cases to deal with:

(i) When $A \in \mathcal{A}'_F$, let C be a circuit of M such that $A \subseteq C \subseteq L \cup A$. We use C_A to denote any circuit of M belonging to

$$\{C, C \triangle (L_1 \cup L_2), C \triangle (L_1 \cup L_3), C \triangle (L_2 \cup L_3)\}.$$

At a given proof, we may need to choose C_A conveniently.

(ii) When $A \in \mathcal{A}'_K$, let L_i be the series class of $M|(L \cup A)$, for $i \in [3]$. Let C be a circuit of M such that $A \subseteq C \subseteq (L - L_i) \cup A$. We use C_A to denote any circuit of M belonging to $\{C, C \triangle (L - L_i)\}$. Note that $M|(L \cup A)$ contains another two circuits containing A. These circuits are "too big" to be called C_A because they also contain L_i. We consider these circuits in Lemma 3.11.

(iii) When $A \in \mathcal{A}_i$, for some $i \in [3]$, let C_A be the unique circuit of M contained in $A \cup L_i$. If $\{i, j, k\} = [3]$, then $C = C_A \triangle (L_i \cup L_j)$ and $D = C_A \triangle (L_i \cup L_k)$ are the other circuits of $M|(L \cup A)$ that contain A. When, $C_A = A \cup L_i$, then $C = A \cup L_j$ and $D = A \cup L_k$. That is, $A \in \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3$. When we view A as an element of $\mathcal{A}_j$ or $\mathcal{A}_k$, C_A becomes C or D respectively. In this case C_A also depends on i. We do not need to emphasize this dependence because it will be clear from the context which C_A we are talking about in this very special case.

Lemma 3.6 *If $A \in \mathcal{A}_k$, for $k \in [3]$, then*

(i) $|A| \leq |C_A \cap L_k|$; or

(ii) M has a theta set L' such that $|L'| > |L|$.

Proof. Suppose that $|C_A \cap L_k| < |A|$. Observe that $r^*(M|(L \cup A)) = 3$. Hence $C_A, L_1 \cup L_2$ and $L_1 \cup L_3$ span the circuit space of $M|(L \cup A)$. Thus $C_A \cap L_k$ is a series class of $M|(L \cup A)$. If $L' = (L \cup A) - (C_A \cap L_k)$, then $r^*(M|L') = 2$ since $M|L' = [M|(L \cup A)] \backslash (C_A \cap L_k)$. Observe that L' is a theta set of M because $C_A \triangle (L_k \cup L_i), C_A \triangle (L_k \cup L_j)$ and $L_i \cup L_j$, for $\{i, j, k\} = [3]$, are pairwise different circuits of $M|L'$. Note that

$$|L'| = |L \cup A| - |C_A \cap L_k| = |L| + (|A| - |C_A \cap L_k|)$$

and so $|L'| > |L|$. We have (ii). □

Let A and A' be L-arcs of M. We say that $\{A, A'\}$ is *an apart pair of L-arcs of M* provided $A \cap A' = \emptyset$ and $(M/L)|(A \cup A') = [(M/L)|A] \oplus [(M/L)|A']$. Note that a pair of L-arcs $\{A, A'\}$ of M is apart if and only if A and A' are different series classes of $M|(L \cup A \cup A')$.

Lemma 3.7 *Let $\{A, A'\}$ be an apart pair of L-arcs of M. If $A \in \mathcal{A}_k$, for some $k \in [3]$, and $A' \in \mathcal{A}'_F$, then $C_A \cap C_{A'} = \emptyset$ or $C_A \triangle C_{A'}$ is a circuit of M.*

Proof. Let $\{i, j\}$ be a 2-subset of [3] such that $\{i, j, k\} = [3]$. Consider

$$C = C_A \triangle C_{A'} = A \cup A' \cup [(C_A \triangle C_{A'}) \cap L_k] \cup [C_{A'} \cap (L_i \cup L_j)]. \tag{3.1}$$

If C is not a circuit of M, then, for $n \geq 2$, there are pairwise disjoint circuits $C_1, C_2, \ldots, C_n$ of $M|(L \cup A \cup A')$ such that $C = C_1 \cup C_2 \cup \cdots \cup C_n$. By (3.1),

$$C \cap L_i = C_{A'} \cap L_i \subsetneq L_i \text{ and } C \cap L_j = C_{A'} \cap L_j \subsetneq L_j, \tag{3.2}$$

and so $\mathcal{C}(M|L) \cap \{C_1, C_2, \ldots, C_n\} = \emptyset$. Hence each C_i contains a series class of $M|(L \cup A \cup A')$ avoiding L. These series classes are A and A'. Thus $n = 2$. We may assume that $A \subseteq C_1$ and $A' \subseteq C_2$. Therefore $C_1 = C_A$ because, by (3.2), $L_i \not\subseteq C_1$ and $L_j \not\subseteq C_1$. Consequently $C_A \cap C_{A'} = \emptyset$. □

Lemma 3.8 *Suppose that $A' \in \mathcal{A}'_F$ and $A \in \mathcal{A}_k$, for some $k \in [3]$. If $A \cup A'$ is a theta set of M/L, then $C_A \triangle C_{A'}$ is a circuit of M.*

Proof. Observe that $A \cap A' \neq \emptyset$ because $A \cup A'$ is a theta set of M/L. If $C_A \triangle C_{A'}$ is not a circuit of M, then, for $n \geq 2$, there are pairwise disjoint circuits $C_1, C_2, \ldots, C_n$ of $M|(L \cup A \cup A')$ such that $C = C_1 \cup C_2 \cup \cdots \cup C_n$. Note that $\mathcal{C}((M/L)|(A \cup A')) = \{A, A', A \triangle A'\}$, since $A \cup A'$ is a theta set of M/L. There is $i \in [n]$ such that $C_i - L = A \triangle A'$, say $i = 1$. As $(C_A \triangle C_{A'}) - (A \triangle A') \subseteq L$, it follows that $\{C_2, \ldots, C_n\} \subseteq \mathcal{C}(M|L)$. We arrive at a contradiction because $L_j - (C_A \triangle C_{A'}) = L_j - C_{A'} \neq \emptyset$ for every $j \in [3]$ such that $j \neq k$. □

Lemma 3.9 *Let $\{A, A'\}$ be a 2-subset of $\mathcal{A}_k$, for some $k \in [3]$. If $\{A, A'\}$ is an apart pair of L-arcs of M, then*

(i) $(C_A \triangle C_{A'}) \triangle (L_i \cup L_k)$ is a circuit of M, for $i \in [3]$ such that $i \neq k$; or

(ii) $C_A \cap L_k \subseteq C_{A'} \cap L_k$ or $C_{A'} \cap L_k \subseteq C_A \cap L_k$.

Proof. For $i \in [3]$ satisfying $i \neq k$, consider

$$C = (C_A \triangle C_{A'}) \triangle (L_i \cup L_k) = A \cup A' \cup [L_k - (C_A \triangle C_{A'})] \cup L_i. \tag{3.3}$$

If $(C_A \triangle C_{A'}) \cap L_k = \emptyset$, then $C_A \cap L_k = C_{A'} \cap L_k$ and (ii) follows. Assume that

$$(C_A \triangle C_{A'}) \cap L_k \neq \emptyset. \tag{3.4}$$

By (3.3) and (3.4),

$$\mathcal{C}(M|L) \cap \mathcal{C}(M|C) = \emptyset. \tag{3.5}$$

If C is a circuit of M, then (i) follows. Suppose that C is not a circuit of M. For $n \geq 2$, there are pairwise disjoint circuits $C_1, C_2, \ldots, C_n$ of $M|(L \cup A \cup A')$ such that

$C = C_1 \cup C_2 \cup \cdots \cup C_n$. By (3.5), $C_j - L \neq \emptyset$, for every $j \in [n]$. As A and A' are the unique series classes of $M|(L \cup A \cup A')$ avoiding L, it follows that $n = 2$, $A \subseteq C_1$ and $A' \subseteq C_2$, say. By (3.3), $C_1 \in \{C_A, C_A \triangle (L_i \cup L_k)\}$ and $C_2 \in \{C_{A'}, C_{A'} \triangle (L_i \cup L_k)\}$. As $L_i \subseteq [C_A \triangle (L_i \cup L_k)] \cap [C_{A'} \triangle (L_i \cup L_k)]$, it follows that $C_1 = C_A$ or $C_2 = C_{A'}$, say $C_1 = C_A$. By (3.3),

$$C_A = A \cup [C_A \cap L_k] \subseteq A \cup [L_k - (C_A \triangle C_{A'})]$$

and so $C_A \cap L_k \subseteq L_k - (C_A \triangle C_{A'})$. Thus $(C_A - C_{A'}) \cap L_k = \emptyset$, that is, $C_A \cap L_k \subseteq C_{A'} \cap L_k$. We have (ii). □

Lemma 3.10 *Suppose that M has circumference* 8 *and*

$$11 \leq |L| = \max\{|L'| : L' \text{ is a theta set of } M\}. \tag{3.6}$$

If $A \in \mathcal{A}'_F$, then

(i) $|A| \leq 2$*; and*

(ii) if $|A| = 2$, then $|L| = 12$ and every series class of $M|(L \cup A)$ has size 2.

Proof. By Lemma 3.1, L is a $(4,4,4)$- or $(4,4,3)$- or $(5,3,3)$-theta set of M. Replacing C_A by $C_A \triangle (L_2 \cup L_3)$, when necessary, we may assume

$$|C_A \cap L_3| \geq 2. \tag{3.7}$$

Replacing C_A by $C_A \triangle (L_1 \cup L_2)$, when necessary, we may suppose

$$|C_A \cap (L_1 \cup L_2)| \geq 4. \tag{3.8}$$

By (3.7) and (3.8),

$$8 \geq |C_A| = |A| + |C_A \cap (L_1 \cup L_2)| + |C_A \cap L_3| \geq |A| + 6. \tag{3.9}$$

Hence $|A| \leq 2$ and so (i) follows. Suppose that $|A| = 2$. We must have equality in (3.7), (3.8) and (3.9). Now, we establish that $|L_3| = 4$. If $|L_3| \neq 4$, then $|L_3| = 3$. By the equality in (3.7), we have that $|L_3 - C_A| = 1$. Remember that $L_1 \cup L_2, L_1 \cup L_3$ and C_A span the circuit space of $M|(L \cup A)$. Therefore

$$\mathcal{SC}(M|(L \cup A)) = \{A, L_1 \cap C_A, L_1 - C_A, L_2 \cap C_A, L_2 - C_A, L_3 \cap C_A, L_3 - C_A\}.$$

As $L_3 - C_A \in \mathcal{SC}(M|(L \cup A)), |L_3 - C_A| = 1$ and $r^*([M|(L \cup A)] \backslash (L_3 - C_A)) = 2$, it follows that $L' = (L \cup A) - (L_3 - C_A)$ is a theta set of M. We arrive at a contradiction to (3.6) since

$$|L'| = |L \cup A| - |L_3 - C_A| = |L| + |A| - |L_3 - C_A| > |L|.$$

Hence $|L_3| = 4$ and so L is a $(4,4,4)$-theta set of M. In particular, $|L| = 12$. By the equality of (3.7), we have that $|C_A \cap L_3| = |L_3 - C_A| = 2$. We conclude that any element of $\mathcal{SC}(M|(L \cup A))$ has size 2, since, by symmetry, any L_i can be chosen to be L_3. We have (ii). □

Lemma 3.11 *Suppose that M has circumference 8 and that $|L| \geq 11$. If $A \in \mathcal{A}'_K$, then*

(i) $|A| = 1$*; and*

(ii) there is $i \in [3]$ such that L_i is a series class of $M|(L \cup A)$ and $|L_i| = 3$; and

(iii) there are circuits D_1 and D_2 of $M|(L \cup A)$ such that $|D_1| = |D_2| = 8, A \cup L_i = D_1 \cap D_2, D_1 \triangle D_2 = L - L_i$ and $D_1, D_2, C_A, C_A \triangle (L - L_i)$ are the unique circuits of $M|(L \cup A)$ containing A; and

(iv) for $A' \in \mathcal{A}_k$, where $k \in [3]$ and $k \neq i$, set $S_1 = (C_A \cap L_k) - C_{A'}, S_2 = (C_A \cap L_k) \cap C_{A'}, S_3 = (L_k - C_A) \cap C_{A'}, S_4 = (L_k - C_A) - C_{A'}$. If $S_2 \neq \emptyset$ and $S_3 \neq \emptyset$, then

(a) $S_1 = S_4 = \emptyset$ *and* $C_{A'} = A' \cup L_k$*; or*

(b) $S_1 = \emptyset$ *and* $|S_3| + |A'| \leq |S_2|$*; or*

(c) $S_4 = \emptyset$ *and* $|S_2| + |A'| \leq |S_3|$.

Proof. By Lemma 3.1, L is a $(4, 4, 4)$- or $(4, 4, 3)$- or $(5, 3, 3)$-theta set of M. By Lemma 3.5, there is $i \in [3]$ such that L_i is a series class of $M|(L \cup A)$. If $\{i, j, k\} = [3]$, then, by definition, $C_A \subseteq A \cup (L_j \cup L_k)$. Consider the circuits of $M|(L \cup A)$:

$$D_1 = C_A \triangle (L_i \cup L_k) \text{ and } D_2 = C_A \triangle (L_i \cup L_j).$$

Note that $D_1 \cap D_2 = A \cup L_i$ and $D_1 \triangle D_2 = L_j \cup L_k$. Hence

$$\begin{aligned} 16 \geq |D_1| + |D_2| = 2|D_1 \cap D_2| + |D_1 \triangle D_2| = 2|A \cup L_i| + |L_j \cup L_k| = \\ = |L| + |L_i| + 2|A| \geq 14 + 2|A| \geq 16. \end{aligned} \tag{3.10}$$

We must have equality along (3.10). Thus $|A| = 1, |L_i| = 3, |L| = 11$ and $|D_1| = |D_2| = 8$. Therefore (i), (ii) and (iii) follows.

Now, we establish (iv). Observe that

$$L_1 \cup L_2, L_1 \cup L_3, C_A \text{ and } C_{A'} \text{ span the circuit space of } M|(L \cup A \cup A') \tag{3.11}$$

because $r^*(M|(L \cup A \cup A')) = 4$. In Figure 1, H is a graph such that $E(H) = \mathcal{SC}(M|(L \cup A \cup A'))$. (To simplify the figure, we set $S_5 = C_A \cap L_j$ and $S_6 = L_j - C_A$.) Let G be a subdivision of H such that each edge S is replaced by a path of length $|S|$ whose edges are labelled by the elements of S. Note that the cycles of G whose edge sets are displayed in (3.11) span the circuit space of $M(G)$. Therefore $M(G) = M|(L \cup A \cup A')$. By (iii), $S_1 \cup S_2 \subseteq D_2$ and $S_3 \cup S_4 \subseteq D_1$. Now, we show that:

$$D_2 \triangle C_{A'} \text{ is not a circuit of } M \text{ or } |S_3| + |A'| \leq |S_2|. \tag{3.12}$$

Suppose that $D_2 \triangle C_{A'}$ is a circuit of M. Hence

$$\begin{aligned} 8 \geq |D_2 \triangle C_{A'}| = |D_2 - C_{A'}| + |C_{A'} - D_2| = |D_2 - S_2| + |C_{A'} - S_2| = \\ = [|D_2| - |S_2|] + [|A'| + |S_3|] = 8 + (|S_3| + |A'| - |S_2|). \end{aligned}$$

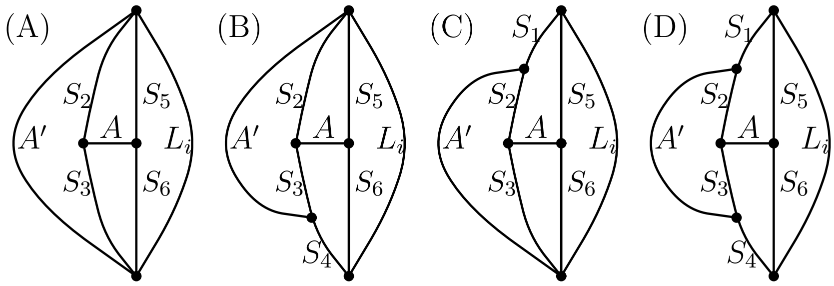

Figure 1: The graph H. In (A), when $S_1 = S_4 = \emptyset$. In (B), when $S_1 = \emptyset \neq S_4$. In (C), when $S_1 \neq \emptyset = S_4$. In (D), when $S_1 \neq \emptyset \neq S_4$.

Thus $|S_2| \geq |S_3| + |A'|$ and (3.12) holds. Now, we prove that

$$S_4 = \emptyset \text{ or } |S_3| + |A'| \leq |S_2|. \tag{3.13}$$

As $M|(L \cup A \cup A') = M(G)$, it follows that $D_2 \triangle C_{A'}$ is not a circuit of M if and only if $S_4 = \emptyset$. Thus (3.13) follows from (3.12). By symmetry, when we repeat this argument with D_1 in the place of D_2, we obtain that

$$S_1 = \emptyset \text{ or } |S_2| + |A'| \leq |S_3|. \tag{3.14}$$

When $S_1 = S_4 = \emptyset$, $C_{A'} = A' \cup S_2 \cup S_3 = A' \cup L_k$ and (iv)(a) follows. Assume that $S_1 \neq \emptyset$ or $S_4 \neq \emptyset$. By (3.13) and (3.14), $S_1 = \emptyset$ or $S_4 = \emptyset$. If $S_1 = \emptyset$, then, by (3.13), we have (iv)(b). If $S_4 = \emptyset$, then, by (3.14), we have (iv)(c). □

The proof of the next result is very simple. It gives a condition for a cycle to be a circuit in a binary matroid. We state it because it covers a situation that occurs many times in this paper.

Lemma 3.12 *Suppose that C is a cycle of a binary matroid N. If $C - F$ is an F-arc of N and $C \cap F$ is independent in N, for $F \subseteq E(N)$, then C is a circuit of N.*

Proof. Note that $C \neq \emptyset$ because $C - F$ is an F-arc of N. If C is not a circuit of N, then there are pairwise disjoint circuits $C_1, C_2, \dots, C_n$ of N, for $n \geq 2$, such that $C = C_1 \cup C_2 \cup \cdots \cup C_n$. As $A = C - F$ is contained in a series class of $M|(F \cup C)$, then $A \subseteq C_i$, for some $i \in [n]$, say $i = 1$. Thus $C_2 \subseteq C \cap F$, a contradiction to hypothesis. □

4 Only $(5, 3, 3)$-theta sets

In this section, we establish the next result:

Theorem 4.1 *Let M be a 3-connected binary matroid having circumference 8. If $r(M) \geq 14$ and L is a theta set of M satisfying $|L| \in \{11, 12\}$, then L is a $(5,3,3)$-theta set of M. Moreover, for each L-arc A of M, the matroid $M|(L \cup A)$ is graphic.*

We divide the proof of Theorem 4.1 into a sequence of lemmas. We set

$$\begin{aligned}\mathcal{L}_1 &= \{L' \subseteq E(M) : L' \text{ is a } (4,4,4)\text{-theta set of } M\},\\ \mathcal{L}_2 &= \{L' \subseteq E(M) : L' \text{ is a } (4,4,3)\text{-theta set of } M\} \text{ and}\\ \mathcal{L}_3 &= \{L' \subseteq E(M) : L' \text{ is a } (5,3,3)\text{-theta set of } M\}.\end{aligned}$$

By Lemma 3.1, $L \in \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$. Choose L and $u \in [3]$ such that $L \in \mathcal{L}_u$ and u is a small as possible. (If $u = 1$, then L is a $(4,4,4)$-theta set of M. If $u = 2$, then L is a $(4,4,3)$-theta set of M and, by the choice of L and u, M does not have a theta set with 12 elements. If $u = 3$, then L is a $(5,3,3)$-theta set of M and, by the choice of L and u, every theta set of M with more than 10 elements is a $(5,3,3)$-theta set.) Let $\{L_1, L_2, L_3\}$ be the canonical partition of L. We assume that $|L_1| \geq |L_2| \geq |L_3|$. For $i \in [3]$, define $\mathcal{A}, \mathcal{A}_i, \mathcal{A}', \mathcal{A}'_K$ and $\mathcal{A}'_F$ as we did in the previous section. For $A \in \mathcal{A}$, define C_A as we did in the previous section. Our goal is to show that $u = 3$ and $\mathcal{A}'_F = \emptyset$. By the next result, we need to establish only that $\mathcal{A}'_F = \emptyset$.

Lemma 4.2 *If $u \in \{1, 2\}$, then $\mathcal{A}'_F \neq \emptyset$.*

Proof. If $u = 1$, then, by Lemma 3.11(ii), $\mathcal{A}'_K = \emptyset$ because $|L_1| = |L_2| = |L_3| = 4$. Thus $\mathcal{A}'_F = \mathcal{A}'$. In this case, the result follows from Lemma 3.4. Assume that $u = 2$ and $\mathcal{A}'_F = \emptyset$. By Lemma 3.11(ii), when $A \in \mathcal{A}' = \mathcal{A}'_K$, L_3 is a series class of $M|(L \cup A)$, since $|L_1| = |L_2| = 4$. Hence $A \to L_1 \cup L_2$. Therefore, when $A' \in \mathcal{A}$, $A' \to L_1 \cup L_2$ or $A' \to L_3$ depending on $A' \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}'$ or $A' \in \mathcal{A}_3$ respectively. By Theorem 2.1, M has a 2-separation $\{X, Y\}$ such that $X \cap L = L_1 \cup L_2$ and $Y \cap L = L_3$, a contradiction. □

Lemma 4.3 *If $A \in \mathcal{A}$, then $|A| \leq 3$.*

Proof. Suppose that $|A| \geq 4$. By Lemmas 3.10(i) and 3.11(i), $A \in \mathcal{A}_k$, for some $k \in [3]$. By Lemma 3.6 and the choice of L and u, $4 \leq |A| \leq |C_A \cap L_k| \leq |L_k|$. Thus $|L_k| \in \{4, 5\}$. Reordering the L_i's, when necessary, we may assume that $k = 1$. If $|L_3| = 3$, then $L' = A \cup L_1 \cup L_2$ is a theta set of M such that $|L'| = |A| + |L_1| + |L_2| \geq 12 > |L|$. We arrive at a contradiction to the choice of L and u. Thus $|L_3| = |A| = 4$. In particular, $u = 1$. Set $L_4 = A$. As the circuit space of $M|(L \cup L_4)$ is spanned by $L_1 \cup L_2, L_1 \cup L_3$ and $L_1 \cup L_4$, it follows that $M|(L \cup L_4)$ is a subdivision of $U_{1,4}$ having L_1, L_2, L_3 and L_4 as series classes. By Theorem 2.1, there is an $(L \cup L_4)$-arc A' of M such that $A' \not\to L_1 \cup L_2$ and $A' \not\to L_3 \cup L_4$. Choose a circuit C of M such that $A' \subseteq C \subseteq A' \cup L \cup L_4$ and $l = |J|$ is minimum, where $J = \{i \in [4] : C \cap L_i \neq \emptyset\}$. Observe that $J \cap \{1, 2\} \neq \emptyset$, otherwise $A' \to L_3 \cup L_4$. Similarly, $J \cap \{3, 4\} \neq \emptyset$. Assume that $l = 2$, say $J = \{2, 3\}$. Note that A' is an L-arc of M such that $A' \in \mathcal{A}'_K$. We arrive at a contradiction to Lemma 3.11(ii). Therefore $l \geq 3$. If $i \in J$

and $L_i \subseteq C$, then, by Lemma 3.12, for $j \in J - \{i\}$, $C \triangle (L_i \cup L_j)$ is a circuit of M contrary to the choice of C. Hence, for every $i \in J$, $\emptyset \neq C \cap L_i \subsetneqq L_i$. To simplify the notation, we may suppose that $\{1, 2, 3\} \subseteq J$. By Lemma 3.12, we can replace C by $C \triangle (L_1 \cup L_2)$, when necessary, to assume that

$$|C \cap (L_1 \cup L_2)| \geq 4. \tag{4.1}$$

By Lemma 3.12, $D = C \triangle (L_3 \cup L_4)$ is a circuit of M such that

$$16 \geq |C| + |D| = 2|C \cap (L_1 \cup L_2)| + |L_3 \cup L_4| + 2|A'|.$$

We have a contradiction by (4.1). □

Lemma 4.4 *If $A \in \mathcal{A}'_F$, then we can choose C_A such that*

$$v_A \in \{(4, 2, 1), (3, 3, 1), (3, 2, 2), (2, 3, 2), (2, 2, 3), (2, 2, 2)\}, \tag{4.2}$$

where $v_A = (|C_A \cap L_1|, |C_A \cap L_2|, |C_A \cap L_3|)$.

Proof. We can choose C_A satisfying:

(i) $|C_A \cap L_1| \geq 2$. (If $|C_A \cap L_1| = 1$, then replace C_A by $C_A \triangle (L_1 \cup L_2)$.)

(ii) $|C_A \cap L_2| \geq 2$ and, when $|L_2| = 4$, $|C_A \cap (L_2 \cup L_3)| \geq 4$. (If $|C_A \cap L_2| = 1$ or, when $|L_2| = 4$, $|C_A \cap (L_2 \cup L_3)| \leq 3$, then replace C_A by $C_A \triangle (L_2 \cup L_3)$.)

(iii) $|C_A \cap L_3| \geq 2$, when $|L_1 - C_A| \geq 2$. (If $|C_A \cap L_3| = 1$ and $|L_1 - C_A| \geq 2$, then replace C_A by $C_A \triangle (L_1 \cup L_3)$.)

Now, we show that v_A satisfies (4.2). By (i), we have three cases to deal with $|C_A \cap L_1| \in \{2, 3, 4\}$.

If $|C_A \cap L_1| = 4$, then, by (ii), $|C_A \cap L_2| = 2$ because $|C_A \cap L_3| \geq 1$ and $|C_A| \leq 8$. Thus $v_A = (4, 2, 1)$.

If $|C_A \cap L_1| = 3$, then, by (ii), we have two subcases to deal with $|C_A \cap L_2| \in \{2, 3\}$. If $|C_A \cap L_2| = 3$, then $|C_A \cap L_3| = 1$ because $|C_A| \leq 8$. In this subcase, $v_A = (3, 3, 1)$. Assume that $|C_A \cap L_2| = 2$. If $|C_A \cap L_3| \geq 2$, then $v_A = (3, 2, 2)$. If $|C_A \cap L_3| = 1$, then $|C_A \cap (L_2 \cup L_3)| = 3$ and, by (ii), $|L_2| = 3$. Thus $|L_1| = 5$ and $|L_1 - C_A| \geq 2$, a contradiction to (iii).

If $|C_A \cap L_1| = 2$, then, by (ii), $|C_A \cap L_2| \in \{2, 3\}$. As $|L_1 - C_A| \geq 2$, it follows, by (iii), that $|C_A \cap L_3| \in \{2, 3\}$. Therefore $v_A \in \{(2, 3, 2), (2, 2, 3), (2, 2, 2)\}$ because $|C_A| \leq 8$. □

We improve Lemma 4.3 in a special case. Lemma 4.5 is central in this proof because it implies that every connected component of M/L has rank equal to 0 or 1, when $\mathcal{A}'_F \neq \emptyset$.

Lemma 4.5 *Suppose that $\mathcal{A}'_F \neq \emptyset$. If $A \in \mathcal{A}$, then $|A| \leq 2$.*

Proof. Consider $\mathcal{A}_T = \{A \in A : |A| \geq 3\}$. Suppose that $\mathcal{A}_T \neq \emptyset$. By Lemma 4.3, $|A| = 3$, for every $A \in \mathcal{A}_T$. By Lemmas 3.10(i) and 3.11(i), when $A \in \mathcal{A}_T$, there is $k \in [3]$ such that $A \in \mathcal{A}_k$. That is, $\mathcal{A}_T \subseteq \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$. When $A \in \mathcal{A}_T \cap \mathcal{A}_k$, for $k \in [3]$, by Lemma 3.6, we have that $|C_A \cap L_k| \geq |A| \geq 3$.

Claim 1. If $A' \in \mathcal{A}'_F$, then $|A'| = 1$.

Assume that $|A'| \geq 2$, for some $A' \in \mathcal{A}'_F$. By Lemmas 3.10 and 3.11(i), $|A'| = 2$ and $u = 1$. By Lemma 3.10(ii), $|C_{A'} \cap L_k| = 2$, for every $k \in [3]$. Choose $A \in \mathcal{A}_T$, say $A \in \mathcal{A}_T \cap \mathcal{A}_k$, for $k \in [3]$. As $|C_A \cap L_k| \geq 3, |C_{A'} \cap L_k| = 2$ and $|L_k| = 4$, it follows that $(C_A \cap C_{A'}) \cap L_k \neq \emptyset$. First, we show that

$$\{A, A'\} \text{ is not an apart pair of } L\text{-arcs of } M. \tag{4.3}$$

If (4.3) fails, then, by Lemma 3.7, $C = C_A \bigtriangleup C_{A'}$ is a circuit of M. Therefore

$$8 \geq |C| = |A| + |A'| + |C_{A'} \cap (L_i \cup L_j)| + |(C_A \bigtriangleup C_{A'}) \cap L_k|,$$

where $\{i, j, k\} = [3]$. We arrive at a contradiction because $|A| = 3, |A'| = 2$ and, by Lemma 3.10(ii), $|C_{A'} \cap (L_i \cup L_j)| = 4$. Therefore (4.3) follows.

In M/L, A' is contained in a parallel class P and A is a triangle. By (4.3), $(M/L)|(A \cup A')$ is connected. As M/L is binary, it follows that $A \cap P \neq \emptyset$. Next, we show that

$$A \cap A' = \emptyset. \tag{4.4}$$

Assume that $A \cap A' \neq \emptyset$, say $a \in A \cap A'$ and $A' = \{a, a'\}$. Hence $A \cup A'$ is a theta set of M/L. Choose $C_{A'}$ such that

$$|(C_A \bigtriangleup C_{A'}) \cap L_k| \text{ is maximum.} \tag{4.5}$$

By Lemma 3.8, $C = C_A \bigtriangleup C_{A'}$ is a circuit of M. Thus

$$8 \geq |C| = |A \bigtriangleup A'| + |C_{A'} \cap (L_i \cup L_j)| + |(C_A \bigtriangleup C_{A'}) \cap L_k|,$$

where $\{i, j, k\} = [3]$. By Lemma 3.10(ii), $|C_{A'} \cap (L_i \cup L_j)| = 4$. Hence $1 \geq |(C_A \bigtriangleup C_{A'}) \cap L_k|$ because $|A \bigtriangleup A'| = 3$. Therefore $C_{A'} \cap L_k \subseteq C_A \cap L_k$. Note that $D = C_{A'} \bigtriangleup (L_i \cup L_k)$ is contrary to the choice of $C_{A'}$ done in (4.5), since

$$(D \bigtriangleup C_A) \cap L_k = (C_{A'} \cap L_k) \cup (L_k - C_A)$$

has at least two elements. Thus (4.4) follows.

By (4.4), we may assume that $\alpha \in P \cap A$ and $A' = \{a, a'\}$ with $\alpha \notin A'$. In M/L, $A_1 = (A - \alpha) \cup a$ and $A_2 = (A - \alpha) \cup a'$ are triangles. As $A_1 \bigtriangleup A_2 = A'$, it follows that A_1 or A_2 is not a triangle of M, say A_1. Hence $A_1 \in \mathcal{A}_T$; a contradiction to (4.4) applied to A_1. Thus Claim 1 follows.

When $A' \in \mathcal{A}'_F$, we can choose $C_{A'}$ as in Lemma 4.4. Now, we show that:

Claim 2. If $A' \in \mathcal{A}'_F$ and $A \in \mathcal{A}_T \cap \mathcal{A}_k$, for some $k \in [3]$, then $C_A \cap C_{A'} = \emptyset$ or $C_A \cap L_k \subseteq C_{A'} \cap L_k \subsetneqq L_k$.

Assume that $C_A \cap C_{A'} \neq \emptyset$. By Claim 1, $|A'| = 1$. Thus $\{A, A'\}$ is an apart pair of L-arcs of M. By Lemma 3.7, $C = C_A \triangle C_{A'}$ is a circuit of M. Thus

$$8 \geq |C| = |A| + |A'| + |(C_A \triangle C_{A'}) \cap L_k| + |C_{A'} \cap (L_i \cup L_j)|,$$

where $\{i, j, k\} = [3]$. As $|A| + |A'| = 4$ and $|C_{A'} \cap (L_i \cup L_j)| \geq 3$, it follows that $|(C_A \triangle C_{A'}) \cap L_k| \leq 1$. Note that Claim 2 follows unless $|(C_A \triangle C_{A'}) \cap L_k| = 1$ and $|(C_A \cap L_k) - (C_{A'} \cap L_k)| = 1$. We may assume this is the case. Hence $|C_{A'} \cap (L_i \cup L_j)| = 3$. By Lemma 4.4, $\{i, j\} = \{2, 3\}, k = 1$ and $(|C_{A'} \cap L_1|, |C_{A'} \cap L_2|, |C_{A'} \cap L_3|) = (4, 2, 1)$. Thus $C_A = A \cup L_1$. Applying Lemma 3.7 to C_A and $C_{A'} \triangle (L_1 \cup L_3)$, we conclude that

$$C_A \triangle [C_{A'} \triangle (L_1 \cup L_3)] = A \cup A' \cup (C_{A'} \cap L_1) \cup (C_{A'} \cap L_2) \cup (L_3 - C_{A'})$$

is a circuit of M with 12 elements; a contradiction and Claim 2 follows.

Claim 3. If $A \in \mathcal{A}_T$, then $|C_A| = 6$.

By Claim 2, when $A \in \mathcal{A}_k$, for $k \in [3]$, then $C_A \cap L_k \subsetneqq L_k$. By Lemma 3.6, $|C_A \cap L_k| \in \{3, 4\}$. If $|C_A \cap L_k| = 3$, then Claim 3 follows. Assume that $|C_A \cap L_k| = 4$. Therefore $k = 1, |L_1| = 5, u = 3$ and $L' = A \cup L_1 \cup L_2$ is a $(4, 4, 3)$-theta set of M with canonical partition $\{C_A \cap L_1, (L_1 - C_A) \cup L_2, A\}$. We have a contradiction to the choice of L and u. Hence Claim 3 follows.

Claim 4. If $\{A, A'''\} \subseteq \mathcal{A}_T \cap \mathcal{A}_k$ and $C_A \cap L_k \neq C_{A'''} \cap L_k$, for some $k \in [3]$, then $k = 1, |L_1| = 5, u = 3$ and $(C_A \cup C_{A'''}) \cap L_1 = C_{A'} \cap L_1$, for every $A' \in \mathcal{A}'_F$.

By Claim 2, $C_A \cap L_k \subseteq X \in \{L_k \cap C_{A'}, L_k - C_{A'}\}$ and $C_{A'''} \cap L_k \subseteq Y \in \{L_k \cap C_{A'}, L_k - C_{A'}\}$, where $A' \in \mathcal{A}'_F$. As $\{L_k \cap C_{A'}, L_k - C_{A'}\}$ contains a unique element with cardinality exceeding 2, it follows that $X = Y$. By Claim 3, $|C_A \cap L_k| = |C_{A'''} \cap L_k| = 3$. As $C_A \cap L_k \neq C_{A'''} \cap L_k$, it follows that $|(C_A \cap L_k) \cup (C_{A'''} \cap L_k)| \geq 4$. Hence $X = (C_A \cup C_{A'''}) \cap L_k$ has 4 elements and so $k = 1; |L_1| = 5; u = 3$; and $C_{A'} \cap L_1 = (C_A \cup C_{A'''}) \cap L_1$.

Claim 5. Suppose that $A \in \mathcal{A}_T \cap \mathcal{A}_k$, for $k \in [3]$. There is an L-arc A'' of M such that

$$A'' \not\rightarrow C_A \cap L_k \text{ and } A'' \not\rightarrow L - C_A. \tag{4.6}$$

Moreover, when A'' satisfies (4.6),

(i) $A'' \in \mathcal{A}_k$;

(ii) $\{A, A''\}$ is not an apart pair of L-arcs of M. In particular, $|A''| \geq 2$;

(iii) $k = 1$ and there is $A''' \in \mathcal{A}_T \cap \mathcal{A}_1$ such that $C_A \cap L_1 \neq C_{A'''} \cap L_1$ and $(C_A \cup C_{A''}) \cap L_1 = (C_A \cup C_{A'''}) \cap L_1 = C_{A'} \cap L_1$, for every $A' \in \mathcal{A}'_F$. Moreover, $A'' \rightarrow C_{A'} \cap L_1$.

Observe that A'' exists, by Theorem 2.1, since $\{C_A \cap L_k, L - C_A\}$ is a 2-separation for $M|L$. Now, we prove (i).

First, suppose that $A'' \in \mathcal{A}'$. By Claim 2 and (4.6), $A'' \notin \mathcal{A}'_F$ and so $A'' \in \mathcal{A}'_K$. Set $S_1 = (C_{A''} \cap L_k) - C_A, S_2 = (C_{A''} \cap L_k) \cap C_A, S_3 = (L_k - C_{A''}) \cap C_A$ and $S_4 = (L_k - C_{A''}) - C_A$. By (4.6), $S_2 \neq \emptyset$ and $S_3 \neq \emptyset$. By Claim 3, $3 = |C_A \cap L_k| = |S_2| + |S_3|$ and so $\{|S_2|, |S_3|\} = \{1, 2\}$. As $|A| = 3$ and $\{|S_2|, |S_3|\} = \{1, 2\}$, it follows that Lemma 3.11(iv)(a) holds. That is, $S_1 = S_4 = \emptyset$ and $C_A = A \cup L_k$, a contradiction to Claim 2. Thus $A'' \notin \mathcal{A}'$. There is $l \in [3]$ such that $A'' \in \mathcal{A}_l$. By (4.6), $l = k$ and (i) follows.

Next, we establish (ii). Assume that $\{A, A''\}$ is an apart pair of L-arcs of M. By (4.6) and (i), all the sets $S_1 = (C_A - C_{A''}) \cap L_k, S_2 = (C_{A''} - C_A) \cap L_k$ and $S_3 = (C_A \cap C_{A''}) \cap L_k$ are non-empty. Choose $i \in [3], i \neq k$, such that $|L_i \cup L_k| = 8$. (We can do this because, by Claims 2 and 3, $|L_k| \geq 4$.) Note that A and A'' are series classes of $N = M|(A \cup A'' \cup L_k \cup L_i)$. As $r^*(N) = 3$ and $\mathcal{SC}(N) = \{A, A'', S_1, S_2, S_3, S_4\}$, where $S_4 = (L_i \cup L_k) - (S_1 \cup S_2 \cup S_3)$, it follows that N is isomorphic to a subdivision of $M(K_4)$. Thus

$$C = C_A \triangle C_{A''} \triangle (L_i \cup L_k) = A \cup A'' \cup S_3 \cup S_4$$

is a circuit of M. Hence

$$8 \geq |C| = |A| + |A''| + |S_3| + |S_4| \geq 6 + |A''| + |S_3| + |S_4 - L_i|.$$

So $|A''| = |S_3| = 1$ and $|S_4| = |L_i| = 3$. As $|L_k \cup L_i| = 8$, it follows that $|L_k| = 5$, that is, $k = 1$ and $u = 3$. By Claim 3, $|S_1| + |S_3| = |C_A \cap L_1| = 3$ and so $|S_1| = 2$. Therefore $|S_2| = |L_1| - (|S_1| + |S_3|) = 2$, say $S_2 = \{\alpha, \beta\}$. By Claim 2, we may assume that $C_{A'} \cap L_1 = L_1 - \beta$, for some $A' \in \mathcal{A}'_F$. (If $C_{A'} \cap L_1 = S_2$, then $[C_{A'} \triangle (L_1 \cup L_i)] \triangle C_{A''}$ is a circuit of M, by Lemma 3.7, having at least 9 elements, a contradiction.) By Claim 1, $|A'| = 1$ and so $A' \cap (A \cup A'') = \emptyset$. Note that $L' = A \cup S_3 \cup A'' \cup L_2 \cup L_3$ is a $(5, 3, 3)$-theta set of M having canonical partition $\{L'_1 = A \cup S_3 \cup A'', L_2, L_3\}$. If $D = C_{A'} \triangle (L_1 \cup L_3)$, then D is a circuit of M such that $D - L' = A' \cup \beta$. As β is a coloop of $M|(L' \cup \beta)$, it follows that $A' \cup \beta$ is an L'-arc of M. Observe that $D \cap L'_1 = \emptyset, \emptyset \neq D \cap L_2 = C_{A'} \cap L_2 \subsetneq L_2$ and $\emptyset \neq D \cap L_3 = L_3 - C_{A'} \subsetneq L_3$. Hence $M|(L' \cup A' \cup \beta)$ is isomorphic to a subdivision of $M(K_4)$; a contradiction to Lemma 3.11(i) applied to L' because $A' \cup \beta$ is an L'-arc of M satisfying $|A' \cup \beta| = 2$. Thus (ii) holds. Now, we establish (iii).

By (i) and (ii), $A'' \in \mathcal{A}_k$ and $|A''| \in \{2, 3\}$. If $|A''| = 3$, then, by (4.6), $C_A \cap L_k \neq C_{A''} \cap L_k$. Therefore (iii) follows from Claim 4 applied to $A''' = A''$. Assume that $|A''| = 2$. By (ii), $(M/L)|(A \cup A'')$ is connected. In M/L, A is a triangle, A'' is contained in a parallel class P and $P \cap A \neq \emptyset$, say $\alpha \in P \cap A$. (Remember that M/L is binary.) Set $A'' = \{a, a'\}$. If $\alpha = a$, then $A''' = A \triangle A'' = (A - \alpha) \cup a'$ is an L-arc of M because, by (4.6), $(C_A - C_{A''}) \cap L_k \neq \emptyset \neq (C_{A''} - C_A) \cap L_k$ and so $C_A \triangle C_{A''} = A''' \cup [(C_A \triangle C_{A''}) \cap L_k] = C_{A'''}$. Thus $A''' \in \mathcal{A}_T \cap \mathcal{A}_k$. Observe that (iii) follows from Claim 2 since $(C_A \cup C_{A''}) \cap L_k = (C_A \cup C_{A'''}) \cap L_k$. We may assume that $\alpha \notin A''$. Set $A_1 = (A - \alpha) \cup a$ and $A_2 = (A - \alpha) \cup a'$. Note that A_1 and A_2

are triangles of M/L. For $i \in [2]$, A_i is a triangle of M or A_i is an L-arc of M. We define

$$C_i = \begin{cases} A_i, & \text{if } A_i \in \mathcal{C}(M); \\ C_{A_i}, & \text{if } A_i \in \mathcal{A}_T. \end{cases}$$

As $A_1 \triangle A_2 = A''$, it follows that $A_1 \notin \mathcal{C}(M)$ or $A_2 \notin \mathcal{C}(M)$, say $A_1 \notin \mathcal{C}(M)$. There is $j \in [3]$ such that $A_1 \in \mathcal{A}_T \cap \mathcal{A}_j$. By Claim 2, $C_{A_1} \cap L_j \subsetneq L_j$. Thus

$$C_A \triangle C_{A_1} = \begin{cases} \{a, \alpha\} \cup (C_{A_1} \cap L_j) \cup (C_A \cap L_k), & \text{when } j \neq k \\ \{a, \alpha\} \cup [(C_{A_1} \cap L_k) \triangle (C_A \cap L_k)], & \text{when } j = k \end{cases}$$

is a cycle of M. Hence $\{a, \alpha\} \in \mathcal{A}$ since $\{a\} \notin \mathcal{A}$ and $\{\alpha\} \notin \mathcal{A}$. By Lemma 3.12, $C_A \triangle C_{A_1}$ is a circuit of M and so $C_A \triangle C_{A_1} = C_{\{a,\alpha\}}$. By Lemma 3.11(i), $j = k$. Now, we prove that $C_{A_1} \cap L_k \neq C_A \cap L_k$. If $C_{A_1} \cap L_k = C_A \cap L_k$, then $C_A \triangle C_{A_1} = A \triangle A_1 = \{\alpha, a\}$, a contradiction. Hence $C_{A_1} \cap L_k \neq C_A \cap L_k$. By Claim 4, $k = 1, u = 3, |L_1| = 5$ and

$$(C_A \cup C_{A_1}) \cap L_1 = C_{A'} \cap L_1, \text{ for every } A' \in \mathcal{A}'_F. \tag{4.7}$$

We have two possibilities for C_2. If $C_2 = A_2$, then

$$C_{A_1} \triangle C_2 = (A_1 \triangle A_2) \cup (C_{A_1} \cap L_1) = \{a, a'\} \cup (C_{A_1} \cap L_1).$$

Thus $C_{A''} = \{a, a'\} \cup (C_{A_1} \cap L_1)$. Combining this with (4.7), we have (iii) taking $A''' = A_1$ in Claim 4. Assume that $C_2 = C_{A_2}$. By (4.7) applied to A_2, we get

$$(C_A \cup C_{A_2}) \cap L_1 = C_{A'} \cap L_1, \text{ for every } A' \in \mathcal{A}'_F. \tag{4.8}$$

By (4.7) and (4.8), we have that $(C_{A_1} \cap L_1) - C_A = (C_{A_2} \cap L_1) - C_A$. Hence

$$C_{A_1} \triangle C_{A_2} = \{a, a'\} \cup [(C_{A_1} \cap L_1) \triangle (C_{A_2} \cap L_1)] \subseteq A'' \cup (C_A \cap L_1).$$

We arrive at a contradiction to (4.6) because, by Lemma 3.12, $C_{A''} = C_{A_1} \triangle C_{A_2}$. With this contradiction we conclude the proof of Claim 5.

Now, we finish the proof of Lemma 4.5 by arriving at a contradiction. By Claim 5(iii), there are $A_1, A_2 \in \mathcal{A}_T \cap \mathcal{A}_1$, with $C_{A_1} \cap L_1 \neq C_{A_2} \cap L_1$, such that

$$(C_{A_1} \cup C_{A_2}) \cap L_1 = C_{A'} \cap L_1, \text{ for any } A' \in \mathcal{A}'_F. \tag{4.9}$$

Moreover, $u = 3$ and $|L_1| = 5$. By Theorem 2.1, there is an L-arc A'' of M such that

$$A'' \not\to C_{A'} \cap L_1 \text{ and } A'' \not\to L - C_{A'}, \text{ for } A' \in \mathcal{A}'_F. \tag{4.10}$$

We establish that A'' cannot satisfy (4.6). If A'' satisfies (4.6), for some $A \in \mathcal{A}_T$, then, by Claim 5(iii), $A'' \to C_{A'} \cap L_1$, a contradiction to (4.10). Thus A'' cannot satisfy (4.6), for every $A \in \mathcal{A}_T$. That is, $A'' \to C_{A_i} \cap L_1$ or $A'' \to L - C_{A_i}$, where $i \in [2]$. As $C_{A_i} \cap L_1 \subseteq C_{A'} \cap L_1$, it follows that $A'' \to L - C_{A_i}$. There is a circuit C_i of M such that

$$A'' \subseteq C_i \subseteq (L - C_{A_i}) \cup A''. \tag{4.11}$$

By (4.9) and Claim 3, there are elements a_1 and a_2 of M such that $\{a_1\} = (C_{A_1} - C_{A_2}) \cap (C_{A'} \cap L_1)$ and $\{a_2\} = (C_{A_2} - C_{A_1}) \cap (C_{A'} \cap L_1)$. By (4.10) and (4.11), $a_{3-i} \in C_i$. By (4.11), $\{a_1, a_2\} \subseteq C_1 \triangle C_2 \subseteq L - (C_{A_1} \cap C_{A_2})$; a contradiction because L_1 is a series class of $M|L$, $C_1 \triangle C_2$ is a cycle of $M|L$ and $\emptyset \neq (C_1 \triangle C_2) \cap L_1 \subseteq L_1 - (C_{A_1} \cap C_{A_2}) \subsetneqq L_1$. Therefore $\mathcal{A}_T = \emptyset$. □

Our goal is to prove Theorem 4.1, that is, $u = 3$ and $\mathcal{A}'_F = \emptyset$. By Lemma 4.2, it is enough to establish that $\mathcal{A}'_F = \emptyset$. Assume that $\mathcal{A}'_F \neq \emptyset$. By Lemma 4.5, $|A| \leq 2$, for every $A \in \mathcal{A}$. By Lemma 2.3, each connected component of M/L has rank 0 or 1. Let $H_1, H_2, \dots, H_n$ be the rank-1 connected components of M/L. Hence $r(M) = r(L) + r(M/L) = (|L| - 2) + n$. Thus

$$n = r(M) + 2 - |L| \geq 4. \tag{4.12}$$

For $i \in [n]$, choose $a_i \in E(H_i)$. If $B = (L - \{\alpha, \beta\}) \cup \{a_1, a_2, \dots, a_n\}$, where $\alpha \in L_1$ and $\beta \in L_2$, then B is a basis of M. As $\mathrm{cl}_M(B - a_i) = E(M) - E(H_i)$, it follows that $E(H_i)$ is a cocircuit of M, for every $i \in [n]$. Therefore $r(E(H_i)) \geq 3$ because M is 3-connected. For $i \in [n]$, there is an independent set I_i of M such that $|I_i| = 3$ and $I_i \subseteq E(H_i)$. We use $\binom{X}{2}$ to denote the family of 2-subsets of a set X. Note that each element of $\binom{E(H_i)}{2}$ is an L-arc of M.

Lemma 4.6 *If $A' \in \binom{I_i}{2} \cap \mathcal{A}'$ and $A \in \binom{I_j}{2} \cap \mathcal{A}_k$, for $k \in [3]$ and $\{i, j\}$ a 2-subset of $[n]$, then $C_A \cap L_k \in \{C_{A'} \cap L_k, L_k - C_{A'}\}$.*

Proof. By Lemmas 3.10 and 3.11(i), $A' \in \mathcal{A}'_F$, $u = 1$ and every element of $\mathcal{SC}(M|(L \cup A'))$ has size 2. By Lemma 3.6, $2 = |A| \leq |C_A \cap L_k|$. If $C_A \cap C_{A'} = \emptyset$, then $C_A \cap L_k \subseteq L_k - C_{A'}$. Thus $C_A \cap L_k = L_k - C_{A'}$ since $|L_k - C_{A'}| = 2$ and $|C_A \cap L_k| \geq 2$. Assume that $C_A \cap C_{A'} \neq \emptyset$. As $\{A, A'\}$ is an apart pair of L-arcs of M, it follows, by Lemma 3.7, that $C_A \triangle C_{A'}$ is a circuit of M. Thus

$$8 \geq |C_A \triangle C_{A'}| = |A| + |A'| + |(C_A \triangle C_{A'}) \cap L_k| + |C_{A'} \cap (L - L_k)|.$$

Therefore $|(C_A \triangle C_{A'}) \cap L_k| = 0$ because $|A| = |A'| = 2$ and $|C_{A'} \cap (L - L_k)| = 4$. Hence $C_A \cap L_k = C_{A'} \cap L_k$. □

Lemma 4.7 *If $A_1 \in \binom{I_i}{2} \cap \mathcal{A}_j$ and $A_2 \in \binom{I_i}{2} \cap \mathcal{A}_k$, $A_1 \neq A_2$, for $i \in [n]$ and $\{j, k\} \subseteq [3]$, then $\binom{I_i}{2} \subseteq \mathcal{A}_j$ or $\binom{I_i}{2} \subseteq \mathcal{A}_k$.*

Proof. There are $X_j \subseteq L_j$ and $X_k \subseteq L_k$ such that $C_{A_1} = A_1 \cup X_j$ and $C_{A_2} = A_2 \cup X_k$. By Lemma 3.6, $|X_j| \geq 2$ and $|X_k| \geq 2$. If $A_3 = A_1 \triangle A_2$, then $\binom{I_i}{2} = \{A_1, A_2, A_3\}$. First, we show that

$$j = k \text{ or } X_j = L_j \text{ or } X_k = L_k. \tag{4.13}$$

Assume that $j \neq k$ and $X_j \subsetneqq L_j$ and $X_k \subsetneqq L_k$. By Lemma 3.12,

$$C_{A_3} = C_{A_1} \triangle C_{A_2} = (A_1 \triangle A_2) \cup (X_j \cup X_k) = A_3 \cup (X_j \cup X_k).$$

In this case, $M|(L \cup A_3)$ is a subdivison of $M(K_4)$. We arrive at a contradiction to Lemma 3.11(i), since $|A_3| = 2$. Thus (4.13) follows. If $j = k$, then $A_3 \to L_j$ and $\binom{I_i}{2} \subseteq \mathcal{A}_j$ since, by Lemma 3.12, $C_{A_3} = A_3 \cup (X_j \triangle X_k)$. Suppose that $j \neq k$. By (4.13), $X_j = L_j$ or $X_k = L_k$, say $X_j = L_j$. Hence $C = C_{A_1} \triangle (L_j \cup L_k) = A_1 \cup L_k$ is a circuit of M and so $A_1 \to L_k$, that is, $A_1 \in \mathcal{A}_k$. Observe that

$$C_{A_3} = C \triangle C_{A_2} = (A_1 \cup L_k) \triangle (A_2 \cup X_k) = A_3 \cup (L_k - A_k).$$

Thus $A_3 \to L_k$ and $A_3 \in \mathcal{A}_k$. Therefore $\binom{I_i}{2} \subseteq \mathcal{A}_k$. □

Lemma 4.8 *Exactly one of the following statements holds:*

(a) $\binom{I_i}{2} \cap \mathcal{A}' \neq \emptyset$, *for every* $i \in [n]$; *or*

(b) $\binom{I_i}{2} \cap \mathcal{A}' = \emptyset$, *for every* $i \in [n]$.

Proof. Assume this result fails. There is a 2-subset $\{i, j\}$ of $[n]$ such that $\binom{I_i}{2} \cap \mathcal{A}' \neq \emptyset$ and $\binom{I_j}{2} \cap \mathcal{A}' = \emptyset$. Suppose that $\binom{I_j}{2} = \{A_1, A_2, A_3\}$ and $A' \in \binom{I_i}{2} \cap \mathcal{A}'$. By Lemma 4.7, there is $k \in [3]$ such that $\binom{I_j}{2} \subseteq \mathcal{A}_k$. By Lemma 4.6, $\{C_{A_1} \cap L_k, C_{A_2} \cap L_k, C_{A_3} \cap L_k\} \subseteq \{C_{A'} \cap L_k, L_k - C_{A'}\}$. We have a contradiction because, by Lemma 3.12,

$$C_{A_3} = C_{A_1} \triangle C_{A_2} = (A_1 \triangle A_2) \cup [(C_{A_1} \cap L_k) \triangle (C_{A_2} \cap L_k)]$$

and so $C_{A_3} = A_3$ or $C_{A_3} = A_3 \cup L_k$. □

Lemma 4.9 *If Lemma 4.8(a) holds, then*

$$\binom{I_1}{2} \cup \binom{I_2}{2} \cup \cdots \cup \binom{I_n}{2} \subseteq \mathcal{A}'. \tag{4.14}$$

Proof. Assume that (4.14) fails. First, we show that, for $i \in [n], k \in [3]$ and $A \in \binom{I_i}{2} \cap \mathcal{A}_k$,

(a) $\binom{I_i}{2} - \{A\} \subseteq \mathcal{A}'$; and

(b) $|C_A| = 4$; and

(c) for $j \in [n], j \neq i$, there is $A_j \in \binom{I_j}{2} - \mathcal{A}'$, say $A_j \in \mathcal{A}_{k_j}$, for some $k_j \in [3]$. Moreover, k_j is unique and $k_j \neq k$.

Note that (a) is a consequence of Lemma 4.7. By Lemma 4.7 and hypothesis, there are A'_j and A''_j in $\binom{I_j}{2} \cap \mathcal{A}'$, with $A'_j \neq A''_j$. By Lemma 4.6,

$$C_A \cap L_k \in \{C_{A'_j} \cap L_k, L_k - C_{A'_j}\} \cap \{C_{A''_j} \cap L_k, L_k - C_{A''_j}\}.$$

We can choose $C_{A'_j}$ and $C_{A''_j}$ such that $\alpha \notin C_{A'_j} \cup C_{A''_j}$, for $\alpha \in L_k - C_A$. Thus $C_A \cap L_k = C_{A'_j} \cap L_k = C_{A''_j} \cap L_k$. Therefore $|C_A| = |A| + |C_A \cap L_k| = |A| + |C_{A'_j} \cap L_k| = 4$. We have (b). By Lemma 3.12, $C_{A_j} = C_{A'_j} \triangle C_{A''_j}$. As $C_{A'_j} \triangle C_{A''_j} \subseteq (A'_j \triangle A''_j) \cup (L - L_k)$, it follows, by Lemma 3.11(i), that $A_j = A'_j \triangle A''_j \notin \mathcal{A}'$. Thus $A_j \in \mathcal{A}_{k_j}$, for some $k_j \in [3]$ with $k_j \neq k$. By (b) applied to A_j, we conclude that $|C_{A_j}| = 4$ and so k_j is unique. We define $k_i = k$. Note that k_i is unique by (b). By (c), $k_i \neq k_j$. Hence $k_1, k_2, \dots, k_n$ are pairwise different and so $n \leq 3$; a contradiction to (4.12). Therefore A does not exist and so (4.14) holds. □

Lemma 4.10 *Item (b) of Lemma 4.8 holds. In particular,* $|A'| = 1$*, for every* $A' \in \mathcal{A}'_F$.

Proof. Suppose that Lemma 4.8(a) holds. By Lemma 4.9, we have that

$$\mathcal{A}'' = \binom{I_1}{2} \cup \binom{I_2}{2} \cup \cdots \cup \binom{I_n}{2} \subseteq \mathcal{A}'.$$

For $A \in \mathcal{A}'', \alpha \in L_1$ and $\beta \in L_2$, choose C_A such that $C_A \subseteq A \cup (L_1 - \alpha) \cup (L_2 - \beta) \cup L_3$, say $C_A = A \cup X_A \cup Y_A \cup Z_A$, where $X_A \subseteq L_1 - \alpha, Y_A \subseteq L_2 - \beta$ and $Z_A \subseteq L_3$. (There is a unique C_A satisfying these conditions.) By Lemma 3.10(ii), $|X_A| = |Y_A| = |Z_A| = 2$. For $i \in [n]$, set $\binom{I_i}{2} = \{A_i, A'_i, A''_i\}$. First, we show that

$$\{X_{A_i}, X_{A'_i}, X_{A''_i}\} = \binom{L_1 - \alpha}{2}, \tag{4.15}$$

$$\{Y_{A_i}, Y_{A'_i}, Y_{A''_i}\} = \binom{L_2 - \beta}{2}, \tag{4.16}$$

$$\{Z_{A_i}, Z_{A'_i}, Z_{A''_i}\} = \binom{L_3 - \gamma_i}{2}, \tag{4.17}$$

for some $\gamma_i \in L_3$. By Lemma 3.12,

$$C_{A''_i} = C_{A_i} \triangle C_{A'_i} = (A_i \triangle A'_i) \cup [X_{A_i} \triangle X_{A'_i}] \cup [Y_{A_i} \triangle Y_{A'_i}] \cup [Z_{A_i} \triangle Z_{A'_i}].$$

As $X_{A_i}, X_{A'_i}$ and $X_{A''_i} = X_{A_i} \triangle X_{A'_i}$ are 2-subsets of $L_1 - \alpha$ and $|L_1 - \alpha| = 3$, it follows that (4.15) holds. A similar argument hods for (4.16). Observe that $Z_{A_i} \triangle Z_{A'_i} \triangle Z_{A''_i} = \emptyset$. Thus the 2-subsets $Z_{A_i}, Z_{A'_i}$ and $Z_{A''_i}$ of L_3 avoids an element γ_i of L_3. We have (4.17).

For $A \in \mathcal{A}''$, we set $\mathcal{X}_A = \{X_A, Y_A, Z_A\}$. Now, we prove that

$$\mathcal{X}_A \cap \mathcal{X}_{A'} \neq \emptyset, \text{ when } \{i, j\} \in \binom{[n]}{2}, A \in \binom{I_i}{2} \text{ and } A' \in \binom{I_j}{2}. \tag{4.18}$$

Suppose that (4.18) fails. Consider

$$C = C_A \triangle C_{A'} = A \cup A' \cup [X_A \triangle X_{A'}] \cup [Y_A \triangle Y_{A'}] \cup [Z_A \triangle Z_{A'}].$$

Observe that $|C| \geq 10$ becuase: (a) $|A| = |A'| = 2$; (b) X_A and $X_{A'}$ are different 2-subsets of $L_1 - \alpha$ and so $|X_A \triangle X_{A'}| = 2$; and (c) similarly, $|Y_A \triangle Y_{A'}| = 2$ and $|Z_A \triangle Z_{A'}| \in \{2, 4\}$. Thus C is not a circuit of M. As $C \cap L \subseteq L - \{\alpha, \beta\}$ is independent in M and A and A' are the unique series classes of $M|(L \cup A \cup A')$ avoiding L, it follows that $C_1 \cup C_2 = C \subseteq (L - \{\alpha, \beta\}) \cup A \cup A'$, where C_1 and C_2 are disjoint circuits of M with $A \subseteq C_1$ and $A' \subseteq C_2$. Hence $C_1 = C_A$ and $C_2 = C_{A'}$, a contradiction. Therefore (4.18) holds. Now, we refine (4.18) to

$$|\mathcal{X}_A \cap \mathcal{X}_{A'}| = 1, \text{ when } \{i, j\} \in \binom{[n]}{2}, A \in \binom{I_i}{2} \text{ and } A' \in \binom{I_j}{2}. \tag{4.19}$$

Assume that (4.19) fails. By (4.18), we have that $|\mathcal{X}_A \cap \mathcal{X}_{A'}| \geq 2$, say $X_A = X_{A'}$ and $Y_A = Y_{A'}$. By (4.15), (4.16) and (4.17),

$$\mathcal{X}_{A_j}, \mathcal{X}_{A'_j} \text{ and } \mathcal{X}_{A''_j} \text{ are pairwise disjoint.} \tag{4.20}$$

Assume that $A' = A''_j$. By (4.18), we have that

$$|\mathcal{X}_A \cap \mathcal{X}_{A_j}| \geq 1 \text{ and } |\mathcal{X}_A \cap \mathcal{X}_{A'_j}| \geq 1. \tag{4.21}$$

By (4.20), $\mathcal{X}_A \cap \mathcal{X}_{A_j} \subseteq \mathcal{X}_A - \mathcal{X}_{A'} \subseteq \{Z_A\}$ and $\mathcal{X}_A \cap \mathcal{X}_{A'_j} \subseteq \mathcal{X}_A - \mathcal{X}_{A'} \subseteq \{Z_A\}$. By (4.21), $Z_A = Z_{A_j}$ and $Z_A = Z_{A'_j}$, a contradiction to (4.20). We have (4.19).

By (4.15), (4.16) and (4.17), we may ssume that

$$\{\mathcal{X}_{A_1}, \mathcal{X}_{A'_1}, \mathcal{X}_{A''_1}\} = \{\{X_1, Y_1, Z_1\}, \{X_2, Y_2, Z_2\}, \{X_3, Y_3, Z_3\}\}, \text{ where}$$

$$\binom{L_1 - \alpha}{2} = \{X_1, X_2, X_3\}, \binom{L_1 - \beta}{2} = \{Y_1, Y_2, Y_3\}, \binom{L_3 - \gamma_1}{2} = \{Z_1, Z_2, Z_3\}.$$

By (4.19) applied nine times when $i = 1$ and $j \in [n] - \{1\}$, we obtain that

$$\{\mathcal{X}_{A_j}, \mathcal{X}_{A'_j}, \mathcal{X}_{A''_j}\} = \{\{X_{i_1}, Y_{i_2}, Z_{i_3}\}, \{X_{j_1}, Y_{j_2}, Z_{j_3}\}, \{X_{k_1}, Y_{k_2}, Z_{k_3}\}\}$$

with $\{i_1, i_2, i_3\} = \{j_1, j_2, j_3\} = \{k_1, k_2, k_3\} = [3]$. By (4.15), (4.16) and (4.17) applied to the index j, we have that $\{i_1, j_1, k_1\} = \{i_2, j_2, k_2\} = \{i_3, j_3, k_3\} = [3]$. Therefore there are two possibilities for $\{\mathcal{X}_{A_j}, \mathcal{X}_{A'_j}, \mathcal{X}_{A''_j}\}$:

$$\{\{X_1, Y_2, Z_3\}, \{X_2, Y_3, Z_1\}, \{X_3, Y_1, Z_2\}\} \text{ or } \{\{X_1, Y_3, Z_2\}, \{X_2, Y_1, Z_3\}, \{X_3, Y_2, Z_1\}\}.$$

Hence $n \leq 3$, a contradiction to (4.12). □

By Lemma 4.10, $|A'| = 1$, for every $A' \in \mathcal{A}'_F$. Fix $C_{A'}$ satisfying Lemma 4.4. Remember that our goal is to prove that $\mathcal{A}'_F = \emptyset$. We are assuming that $\mathcal{A}'_F \neq \emptyset$.

In the next lemma, item (iii) refines item (i). We state item (i) because it is a step towards the proof of item (iii).

Lemma 4.11 *Suppose that $A \in \mathcal{A}'_F$. If $A' \in \mathcal{A}_i$, for $i \in [3]$, and $|A'| = 2$, then*

(i) $C_A \cap L_i \subseteq C_{A'} \cap L_i$ or $C_{A'} \cap L_i \subseteq C_A \cap L_i$ or $C_A \cap C_{A'} = \emptyset$;

(ii) $L_i \not\subseteq C_{A'}$; and

(iii) $C_{A'} \cap L_i \subseteq C_A \cap L_i$ *or* $C_A \cap C_{A'} = \emptyset$.

Proof. Suppose that $C_A \cap C_{A'} \neq \emptyset$. As $|A| = 1$, it follows that $\{A, A'\}$ is an apart pair of L-arcs of M. By Lemma 3.7, $C_A \triangle C_{A'}$ is a circuit of M and

$$8 \geq |C_A \triangle C_{A'}| = |A| + |A'| + |C_A \cap (L_j \cup L_k)| + |(C_A \triangle C_{A'}) \cap L_i|,$$

where $\{i, j, k\} = [3]$. Thus

$$5 \geq |C_A \cap (L_j \cup L_k)| + |(C_A \triangle C_{A'}) \cap L_i|. \tag{4.22}$$

Now, we begin the proof of (i) to (iii).

(i) Assume that (i) fails. Hence $(C_A - C_{A'}) \cap L_i \neq \emptyset \neq (C_{A'} - C_A) \cap L_i$ and so

$$|(C_A \triangle C_{A'}) \cap L_i| \geq 2. \tag{4.23}$$

Replacing (4.23) into (4.22), we get $3 \geq |C_A \cap (L_j \cup L_k)|$. Therefore $\{|C_A \cap L_j|, |C_A \cap L_k|\} = \{1, 2\}$. Hence $v_A = (4, 2, 1), i = 1$ and $|L_1| = 5$. Moreover, we have equality in (4.23). In particular, $|(C_A - C_{A'}) \cap L_1| = |(C_{A'} - C_A) \cap L_1| = 1$. Thus $|C_A \cap L_1| = |C_{A'} \cap L_1| = 4$. There is $\{a, b\} \in \binom{L_1}{2}$ such that $C_A \cap L_1 = L_1 - b$ and $C_{A'} \cap L_1 = L_1 - a$. Observe that $D = C_A \triangle (L_1 \cup L_3)$ is a circuit of M such that $D \cap L_1 = \{b\}$. By Lemma 3.7, $D \triangle C_{A'}$ is a circuit of M. Hence

$$8 \geq |D \triangle C_{A'}| = |A| + |A'| + |L_1 - \{a, b\}| + |C_A \cap L_2| + |L_3 - C_A|,$$

a contradiction because $|D \triangle C_{A'}| = 10$. Therefore (i) holds.

(ii) Assume that (ii) fails. Hence $C_{A'} = A' \cup L_i$. First, we show that

$$|L_i - C_A| = 1. \tag{4.24}$$

If $|L_i - C_A| \geq 2$, then, by (4.22), $|C_A \cap (L_j \cup L_k)| = 3$ and $|L_i - C_A| = 2$, since $(C_A \triangle C_{A'}) \cap L_i = L_i - C_A$. That is, $\{j, k\} = \{2, 3\}$ and $i = 1$, a contradiction because $v_A = (4, 2, 1)$ and $|L_1 - C_A| = 1$. Thus (4.24) follows. For $j \in [3], j \neq i$, we can view $A' \in \mathcal{A}_j$ since

$$D = C_{A'} \triangle (L_i \cup L_j) = (A' \cup L_i) \triangle (L_i \cup L_j) = A' \cup L_j$$

is a circuit of M. By (4.24) applied to j, we obtain $|L_j - C_A| = 1$. Thus $|L \cap C_A| = |L| - 3 \geq 8$. With this contradiction, we conclude the proof of (ii).

(iii) Assume that (iii) fails. By (i), $C_A \cap L_i \subsetneqq C_{A'} \cap L_i$. By (ii), $C_{A'} \cap L_i \subsetneqq L_i$. In resume,

$$C_A \cap L_i \subsetneqq C_{A'} \cap L_i \subsetneqq L_i. \tag{4.25}$$

Remember that $C_A \triangle C_{A'} = [C_A - (C_A \cap L_i)] \cup A' \cup [(C_{A'} - C_A) \cap L_i]$ is a circuit of M. Thus

$$8 \geq |C_A| + |A' \cup [(C_{A'} - C_A) \cap L_i]| - |C_A \cap L_i|. \tag{4.26}$$

As $|C_A| \in \{7, 8\}$, it follows that $|C_A \cap L_i| \geq 2$ because $|A' \cup [(C_{A'} - C_A) \cap L_i]| \geq 3$. By (4.25), $|L_i| \geq 4$. Next, we show that $|L_i| = 4$. If $|L_i| = 5$, then $i = 1$ and, by (4.25), $v_A \in \{(3, 2, 2), (2, 2, 2)\}$. Consider $D = C_A \triangle (L_1 \cup L_3)$. By Lemma 3.7, $D \triangle C_{A'}$ is a circuit of M. As

$$D \triangle C_{A'} = A \cup A' \cup [(L_1 - C_{A'}) \cup (C_A \cap L_1)] \cup (C_A \cap L_2) \cup (L_3 - C_A),$$

it follows that $|D| \geq 1 + 2 + [1 + 2] + 2 + 1 = 9$, a contradiction. Therefore $|L_i| = 4$. By (4.25), $|C_{A'} \cap L_i| = 3$ and $|C_A \cap L_i| = 2$. Moreover, $u \in \{1, 2\}$. As $|A' \cup [(C_{A'} - C_A) \cap L_i] \geq 3$, it follows, by (4.26), that $|C_A| = 7$. Thus $v_A = (2, 2, 2)$. If $j \in \{1, 2\}$ and $j \neq i$, consider $D = C_A \triangle (L_i \cup L_j)$. Observe that $(|D \cap L_1|, |D \cap L_2|, |D \cap L_3|) = (2, 2, 2)$. Thus D can be taken to be C_A; a contradiction because (i) is not satisfied in this case. Therefore (iii) follows. □

By Lemmas 4.7 and 4.10, for $i \in [n]$, there is $k_i \in [3]$ such that

$$\binom{I_i}{2} \subseteq \mathcal{A}_{k_i}.$$

Now, we prove:

Lemma 4.12 *Suppose that $A \in \mathcal{A}'_F$. If $i \in [n]$, then $C_A \cap L_{k_i}$ or $L_{k_i} - C_A$ contains*

$$X_i = \bigcup \left\{ C_{A'} \cap L_{k_i} : A' \in \binom{I_i}{2} \right\}.$$

Proof. Assume this result fails. By Lemma 4.11(iii), there are different 2-subsets A_i and A'_i of I_i such that $C_{A_i} \cap L_{k_i} \subseteq C_A \cap L_k$ and $C_{A'_i} \cap L_{k_i} \subseteq L_{k_i} - C_A$. As $A''_i = A_i \triangle A'_i$ is the third 2-subset of I_i, it follows, by Lemma 3.12, that

$$C_{A''_i} = C_{A_i} \triangle C_{A'_i} = (A_i \triangle A'_i) \cup (C_{A_i} \cap L_{k_i}) \cup (C_{A'_i} \cap L_{k_i}).$$

We arrive at a contradiction to Lemma 4.11(iii) applied to A''_i and A. □

Lemma 4.13 *If $i \in [n]$, then $|X_i| \geq 3$.*

Proof. By Lemma 3.6, when $|X_i| \leq 2$, then $|X_i| = 2$ and $C_{A'} = A' \cup X_i$, for every $A' \in \binom{I_i}{2}$, a contradiction because $C_{A_1} \triangle C_{A_2} = A_3$, when $\binom{I_i}{2} = \{A_1, A_2, A_3\}$. □

Lemma 4.14 *If $\{i, j\}$ is a 2-subset of $[n]$ such that $k_i = k_j$, then, for $A_i \in \binom{I_i}{2}$ and $A_j \in \binom{I_j}{2}$, $C_{A_i} \cap C_{A_j} = \emptyset$ or $C_{A_i} \cap L_{k_i} \subseteq C_{A_j} \cap L_{k_i}$ or $C_{A_j} \cap L_{k_i} \subseteq C_{A_i} \cap L_{k_i}$.*

Proof. Suppose that $C_{A_i} \cap C_{A_j} \neq \emptyset$, say $a \in C_{A_i} \cap C_{A_j}$. By Lemmas 4.12 and 4.13, $|L_{k_i}| \geq 4$. We can choose $k \in [3]$ such that $k \neq k_i$ and $L_k \cup L_{k_i}$ is a circuit of M with 8 elements. Consider the following cycle of M:

$$D = (C_{A_i} \triangle C_{A_j}) \triangle (L_k \cup L_{k_i}) = A_i \cup A_j \cup L_k \cup [L_{k_i} - (C_{A_i} \triangle C_{A_j})].$$

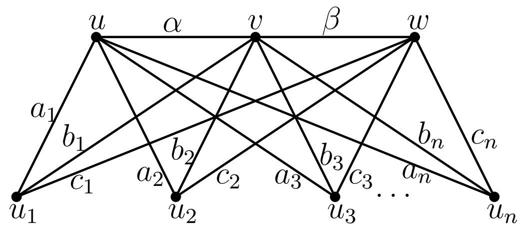

Figure 2: The graph G isomorphic to $K''_{3,n}$.

There is $b \in L_{k_i} - (C_{A_i} \cup C_{A_j})$ because, by Lemmas 4.12 and 4.13, X_i and X_j are contained in the unique set in $\{C_A \cap L_{k_i}, L_{k_i} - C_A\}$, where $A \in \mathcal{A}'_F$, with at least 3 elements. (Choose b in the other set.) As $\{a, b\}$ is a 2-subset of $L_{k_i} - (C_{A_i} \triangle C_{A_j})$, it follows that $|D| \geq 9$ and D is not a circuit of M. The result follows from Lemma 3.9(ii) applied to $\{A_i, A_j\}$. □

Lemma 4.15 *If $A \in \mathcal{A}'_F$, then $|L_1| = 5, u = 3, k_i = 1$, for every $i \in [n]$, and there is a partition $\{Y_1, Y_2\}$ of $C_A \cap L_1$ such that we can label the elements of Y_1, Y_2, I_k, for $k \in [n]$, by respectively $\{\alpha, \alpha'\}$, $\{\beta, \beta'\}$, $\{a_k, b_k, c_k\}$ such that $M|(Y_1 \cup Y_2 \cup I_1 \cup I_2 \cup \cdots \cup I_n)/\{\alpha', \beta'\} = M(G)$, where G is the graph such that $v(G) = \{u, v, w, u_1, u_2, \dots, u_n\}$ and $E(G) = \{\alpha, \beta, a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n, c_1, c_2, \dots, c_n\}$ with the incidences $\alpha = uv, \beta = vw, a_k = u_k u, b_k = u_k v, c_k = u_k w$, for $k \in [n]$ (see Figure 2). Moreover, $\{\alpha, \alpha'\}$ and $\{\beta, \beta'\}$ are series classes of $M|(Y_1 \cup Y_2 \cup I_1 \cup I_2 \cup \cdots \cup I_n)$.*

Proof. There is a 2-subset $\{i, j\}$ of $[n]$ such that $k_i = k_j$ because, by (4.12), $n \geq 4$ and $\{k_1, k_2, \dots, k_n\} \subseteq [3]$. First, we establish the existence of $X \in \{C_A \cap L_{k_i}, L_{k_i} - C_A\}$ such that

$$X_i \cup X_j \subseteq X \text{ and } 3 \leq \min\{|X_i|, |X_j|\} \text{ and } |X| \leq 4. \tag{4.27}$$

By Lemma 4.12, there is $X \in \{C_A \cap L_{k_i}, L_{k_i} - C_A\}$ such that $X_i \subseteq X$. By Lemma 4.13, $|X| \geq |X_i| \geq 3$. So $|L_{k_i} - X| \leq 2$. By Lemmas 4.12 and 4.13, $X_j \subseteq X$, since $|X_j| \geq 3$. Moreover, $|X| \leq 4$. Hence (4.27) follows. Set

$$\{Y_1, Y_2, Y_3\} = \{C_{A'} \cap L_{k_i} : A' \in \tbinom{I_i}{2}\} \text{ and } \{Z_1, Z_2, Z_3\} = \{C_{A'} \cap L_{k_i} : A' \in \tbinom{I_j}{2}\}.$$

Next, we prove that

$$X_i = X_j = X, |X| = 4, k_i = 1, |L_1| = 5 \text{ and } u = 3. \tag{4.28}$$

First, we show that $|X_i| = |X_j| = 4$. Assume that $|X_i| = 3$ or $|X_j| = 3$, say $|X_i| = 3$. In this case,

$$\{Y_1, Y_2, Y_3\} = \binom{X_i}{2}.$$

As $|X_i| \geq 3, |X_j| \geq 3$ and $4 \geq |X_i \cup X_j|$, it follows that $|X_i \cap X_j| \geq 2$. Choose a 2-subset $\{a, b\}$ of $X_i \cap X_j$. Thus $\{a, b\} = Y_r$, for some $r \in [3]$, say $r = 1$, since

$\{Y_1, Y_2, Y_3\} = \binom{X_i}{2}$. As $Z_1 \triangle Z_2 \triangle Z_3 = \emptyset$, it follows that $a \in Z_s \cap Z_t$, for a 2-subset $\{s, t\}$ of $[3]$, say $s = 1, t = 2$. By Lemma 4.14, $Y_1 \subseteq Z_1$ and $Y_1 \subseteq Z_2$ because $|Y_1| = 2$ and $\min\{|Z_1|, |Z_2|\} \geq 2$. Thus $Z_3 \cap Y_1 = \emptyset$. As $|Y_1 \cup Z_3| \geq 4$, it follows that $Y_1 \cup Z_3 = X$ and $|Z_3| = 2$. Note that $|Y_2 \cap Y_1| = |Y_2 \cap Z_3| = 1$. This is a contradiction to Lemma 4.14 because $Y_2 \cap Z_3 \neq \emptyset$, $Z_3 \not\subseteq Y_2$ and $Y_2 \not\subseteq Z_3$. Therefore $|X_i| = |X_j| = |X| = 4$. By (4.27), $X_i = X_j = X = C_A \cap L_{k_i}$ and so $k_i = 1$. Moreover, $|L_1| = 5$ and $u = 3$. Thus (4.28) holds.

Now, we establish that

$$\text{if } Z \in \{Y_1, Y_2, Y_3, Z_1, Z_2, Z_3\}, \text{ then } |Z| \in \{2, 4\}. \tag{4.29}$$

Assume that (4.29) fails. There is $Z \in \{Y_1, Y_2, Y_3, Z_1, Z_2, Z_3\}$ such that $|Z| \notin \{2, 4\}$. As $2 \leq |Z| \leq 4$, it follows that $|Z| = 3$, say $|Y_1| = 3$. For each $r \in [3]$, $Y_1 \cap Z_r \neq \emptyset$ since $|Z_r| \geq 2, |Y_1| = 3$ and $|Z_r \cup Y_1| \leq |X| = 4$. By Lemma 4.14, $Z_r \subseteq Y_1$ or $Y_1 \subseteq Z_r$, for each $r \in [3]$. Hence there is a 2-subset $\{s, t\}$ of $[3]$ such that

(a) $Z_s \subseteq Y_1$ and $Z_t \subseteq Y_1$; or

(b) $Y_1 \subseteq Z_s$ and $Y_1 \subseteq Z_t$.

If (a) happens, then $X_j = Z_s \cup Z_t \subseteq Y_1 \subsetneq X$, a contradiction to (4.28). If (b) happens, then $Z_s \triangle Z_t \subseteq X_j - Y_1$ and so $|Z_s \triangle Z_t| \leq 1$, a contradiction because $Z_s \triangle Z_t \in \{Z_1, Z_2, Z_3\}$. Hence (4.29) follows.

In this paragraph, we show that

$$X \in \{Y_1, Y_2, Y_3\} \cap \{Z_1, Z_2, Z_3\}. \tag{4.30}$$

If (4.30) fails, then, by symmetry, we may assume that $X \notin \{Y_1, Y_2, Y_3\}$. By (4.29), $|Y_1| = |Y_2| = |Y_3| = 2$. Thus $X_i = Y_1 \cup Y_2$ has 3 elements because $Y_3 = Y_1 \triangle Y_2$, a contradiction to (4.28). We have (4.30).

Finally, we show that,

$$\text{when } Y_3 = Z_3 = X, \text{ then } \{Y_1, Y_2\} = \{Z_1, Z_2\} \text{ is a partition of } X. \tag{4.31}$$

By (4.30), we may assume that $Y_3 = Z_3 = X$. By (4.29), $|Y_1| = |Y_2| = |Z_1| = |Z_2| = 2$. As $X = Y_1 \cup Y_2 = Z_1 \cup Z_2$ and $|X| = 4$, it follows that $\{Y_1, Y_2\}$ and $\{Z_1, Z_2\}$ are partitions of X. By Lemma 4.14, $\{Y_1, Y_2\} = \{Z_1, Z_2\}$. Thus (4.31) follows.

By (4.28), L is a $(5, 3, 3)$-theta set of M. As $|L_2| = |L_3| = 3$, it follows, by Lemmas 4.12 and 4.13, that $k_1 = k_2 = \cdots = k_n = 1$. By (4.31), we can label the elements of I_k, for $k \in [n]$, by a_k, b_k, c_k such that $\{a_k, b_k\} \cup Y_1, \{b_k, c_k\} \cup Y_2$ and $\{a_k, c_k\} \cup Y_1 \cup Y_2$ are circuits of M. These circuits span the circuit space of $N = M|(Y_1 \cup Y_2 \cup I_1 \cup I_2 \cup \cdots \cup I_n)$. Note that Y_1 and Y_2 are series classes of N. Moreover, $\{a_k, b_k, \alpha\}, \{b_k, c_k, \beta\}, \{a_k, c_k, \alpha, \beta\}$, for $k \in [n]$, span the circuit space of both $N/\{\alpha', \beta'\}$ and $M(G)$. Therefore $N/\{\alpha', \beta'\} = M(G)$. □

Lemma 4.16 $\mathcal{A}'_K = \emptyset$.

Proof. Assume that $A' \in \mathcal{A}'_K$. By Lemma 3.11, $|A'| = 1$ and L_i, for some $i \in \{2, 3\}$ is a series class of $M|(L \cup A')$, say $i = 3$. Let Y_1, Y_2, a_1, b_1 and c_1 be as defined in Lemma 4.15. First, we show that $|C_{A'} \cap Y_j| \in \{0, 2\}$, for every $j \in [2]$. Assume that $|C_{A'} \cap Y_j| = 1$, say $j = 1$. By Lemma 3.11(iv) applied to A' and $\{a_1, b_1\}$, we have that (a) happens because $|C_{A'} \cap Y_1| = |Y_1 - C_{A'}| = 1$. Thus $C_{\{a_1,b_1\}} = \{a_1, b_1\} \cup L_1$, a contradiction. Therefore $\{|C_{A'} \cap Y_1|, |C_{A'} \cap Y_2|\} \subseteq \{0, 2\}$. Next, we establish that $|C_{A'} \cap (Y_1 \cup Y_2)| \in \{0, 4\}$. If $|C_{A'} \cap (Y_1 \cup Y_2)| = 2$, then, by Lemma 3.11(iv) applied to A' and $\{a_1, c_1\}$, we have that (a) occurs since $|C_{A'} \cap (Y_1 \cup Y_2)| = |(Y_1 \cup Y_2) - C_{A'}| = 2$. Thus $C_{\{a_1,c_1\}} = \{a_1, c_1\} \cup L_1$; a contradiction. Hence $|C_{A'} \cap (Y_1 \cup Y_2)| \in \{0, 4\}$. Replacing $C_{A'}$ by $C_{A'} \triangle (L_1 \cup L_2)$, when $|C_{A'} \cap (Y_1 \cup Y_2)| = 4$, we may assume that $|C_{A'} \cap (Y_1 \cup Y_2)| = 0$. Therefore $x \in C_{A'}$, where $\{x\} = L_1 - (Y_1 \cup Y_2)$. Consider the following circuit of $M|(L \cup A')$:

$$C = C_{A'} \triangle (L_1 \cup L_3) = A' \cup (L_1 - x) \cup (L_2 \cap C_{A'}) \cup L_3.$$

Hence $|C| = 8 + |L_2 \cap C_{A'}|$, a contradiction because $L_2 \cap C_{A'} \neq \emptyset$. □

By Lemma 4.10, $|A'| = 1$, for every $A' \in \mathcal{A}'_F$. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be pairwise different elements of M, with $m \geq 1$, such that $\mathcal{A}'_F = \{\{\alpha_1\}, \{\alpha_2\}, \ldots, \{\alpha_m\}\}$.

Lemma 4.17 *If $i \in [m]$, then there are 2-subsets W_i and W'_i of L_2 and L_3 respectively such that $\{x, \alpha_i\} \cup W_i \cup W'_i$ is a circuit of M, where $\{x\} = L_1 - (Y_1 \cup Y_2)$. ($Y_1$ and Y_2 are defined in Lemma 4.15.)*

Proof. If $A = \{\alpha_i\}$, then, by Lemmas 4.4 and 4.15, we can choose C_A such that $C_A = A \cup (Y_1 \cup Y_2) \cup W_i \cup d_i$, where W_i is a 2-subset of L_2 and $d_i \in L_3$. Consider the circuit $C = C_A \triangle (L_1 \cup L_3) = \{x, \alpha_i\} \cup W_i \cup W'_i$, where $W'_i = L_3 - d_i$. The result follows. □

Lemma 4.18 *If $\{i, j\}$ is a 2-subset of $[m]$, then*

- *(a) $W_i = W_j$ and $W'_i \neq W'_j$; or*
- *(b) $W_i \neq W_j$ and $W'_i = W'_j$.*

Proof. By Lemma 4.17,

$$C = (\{x, \alpha_i\} \cup W_i \cup W'_i) \triangle (\{x, \alpha_j\} \cup W_j \cup W'_j) = \{\alpha_i, \alpha_j\} \cup (W_i \triangle W_j) \cup (W'_i \triangle W'_j)$$

is a cycle of M. Observe that $\{|W_i \triangle W_j|, |W'_i \triangle W'_j|\} \subseteq \{0, 2\}$ because, by Lemma 4.15, $|L_2| = |L_3| = 3$. If $|W_i \triangle W_j| = |W'_i \triangle W'_j| = 0$, then $C = \{\alpha_i, \alpha_j\}$; a contradiction because M is 3-connected. Thus $2 \in \{|W_i \triangle W_j|, |W'_i \triangle W'_j|\}$. If 0 also belongs to this set, then (a) or (b) holds. Assume that 0 does not belong it, that is, $|W_i \triangle W_j| = |W'_i \triangle W'_j| = 2$. Consider the cycle D of M:

$$D = C \triangle (L_1 \cup L_3) = \{\alpha_i, \alpha_j\} \cup L_1 \cup (W_i \triangle W_j) \cup d,$$

where $\{d\} = L_3 - (W'_i \triangle W'_j)$. Note that $|D| = 10$. There are pairwise disjoint circuits $D_1, D_2, \dots, D_l$ of M, for $l \geq 2$, such that $D = D_1 \cup D_2 \cup \cdots \cup D_l$. Note that $\mathcal{C}(M|L) \cap \{D_1, D_2, \dots, D_l\} = \emptyset$. Thus $D_k \cap \{\alpha_i, \alpha_j\} \neq \emptyset$, for every $k \in [l]$. Hence $l = 2$. We may assume that $\alpha_i \in D_1$ and $\alpha_j \in D_2$. Observe that $d \in D_1$ or $d \in D_2$, say $d \in D_1$. Therefore $\alpha_j \in D_2 \subseteq L_1 \cup L_2 \cup \alpha_j$, a contradiction since $\{\alpha_j\} \in \mathcal{A}'_F$. □

Lemma 4.19 $m \leq 3$. *Moreover,*

(a) $W_1 = W_2 = \cdots = W_m$ *and* $W'_1, W'_2, \dots, W'_m$ *are pairwise different* 2*-subsets of* L_3*; or*

(b) $W'_1 = W'_2 = \cdots = W'_m$ *and* $W_1, W_2, \dots, W_m$ *are pairwise different* 2*-subsets of* L_2.

Proof. If $m = 1$, then the result follows. Assume that $m \geq 2$. By Lemma 4.18, permuting L_2 with L_3, when necessary, we may assume that

$$W_1 = W_2 \text{ and } W'_1 \neq W'_2. \tag{4.32}$$

Choose $j \in [m] - \{1, 2\}$. (If $m = 2$, then the result follows.) First, we show that $W_j = W_1 = W_2$. If $W_j \neq W_1 = W_2$, then, by Lemma 4.18, $W'_j = W'_1$ and $W'_j = W'_2$, a contradiction to (4.32). Thus $W_j = W_1 = W_2$. As j is any element of $[m] - \{1, 2\}$, it follows that $W_1 = W_2 = \cdots = W_m$. By Lemma 4.18, $W'_1, W'_2, \dots, W'_m$ are pairwise different 2-subsets of L_3. We have (a). Hence $m \leq 3$ since L_3 contains only 3 different 2-subsets. □

By symmetry, we may assume that Lemma 4.19(a) holds. By Theorem 2.1, M has an L-arc $\widetilde{A}$ such that $\widetilde{A} \not\to W_1$ and $\widetilde{A} \not\to L - W_1$. By Lemma 4.16, $\widetilde{A} \notin \mathcal{A}'_K$. Note that $\widetilde{A} \notin \mathcal{A}'_F$, since we are assuming that Lemma 4.19(a) holds. Thus $\widetilde{A} \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$. Hence $\widetilde{A} \in \mathcal{A}_2$. By Lemma 4.15, $|\widetilde{A}| = 1$ and $C_{\widetilde{A}} = \widetilde{A} \cup W$, where $W \subseteq L_2, |W| = 2$ and $|W \cap W_1| = 1$. Consider, for $A = \{\alpha_1\}$, the cycle of M:

$$C = C_A \triangle C_{\widetilde{A}} = A \cup \widetilde{A} \cup (L_1 - \{x\}) \cup (W \triangle W_1) \cup (C_A \cap L_3).$$

Note that $|C| = 9$. By Lemma 3.7, C is a circuit of M. With this contradiction we finish the proof of Theorem 4.1.

5 Proof of Theorem 3.3

We divide the proof of Theorem 3.3 into a sequence of lemmas. Assume that Theorem 3.3(i) fails, that is, M has a theta set L such that $|L| \geq 11$. By Theorem 4.1, L is a $(5, 3, 3)$-theta set. When $\{L_1, L_2, L_3\}$ is the canonical partition of L in M, we may assume that $|L_1| = 5$ and $|L_2| = |L_3| = 3$. For $i \in [3]$, define $\mathcal{A}, \mathcal{A}_i, \mathcal{A}', \mathcal{A}'_K$ and $\mathcal{A}'_F$ as we did in Section 2. By Theorem 4.1, $\mathcal{A}'_F = \emptyset$. By Lemma 3.4, $\emptyset \neq \mathcal{A}' = \mathcal{A}'_K$.

By Lemma 3.11(i), $|A| = 1$, for every $A \in \mathcal{A}'$. For $A \in \mathcal{A}$, define C_A as we did in Section 2. We fix more notation. For $i \in \{2, 3\}$, set

$$V = \{a \in E(M) - L : \{a\} \in \mathcal{A}'\} \text{ and}$$
$$V_i = \{a \in V : \{a\} \to L_1 \cup L_i\}.$$

By Lemma 3.11(ii), $\{V_2, V_3\}$ is a partition of V.

Lemma 5.1 *If $a \in V_i$, for $i \in \{2, 3\}$, then there is a partition $\{Z_a, W_a\}$ of L_1, with $|Z_a| = 2$ and $|W_a| = 3$, a partition $\{X_a, Y_a\}$ of L_i, with $|X_a| = 1$ and $|Y_a| = 2$, such that $a \cup X_a \cup Z_a$ and $a \cup Y_a \cup W_a$ are circuits of M. (These circuits are the two options for $C_{\{a\}}$.)*

For $a \in V$, we use the sets X_a, Y_a, Z_a and W_a and theirs properties establish in Lemma 5.1 along this section without referring to this lemma.

Proof. Replacing $C_{\{a\}}$ by $C_{\{a\}} \triangle (L_1 \cup L_i)$, when necessary, we may assume that $|C_{\{a\}} \cap L_1| > |L_1 - C_{\{a\}}|$. Set $W_a = C_{\{a\}} \cap L_1, Z_a = L_1 - C_{\{a\}}, Y_a = C_{\{a\}} \cap L_i$ and $X_a = L_i - C_{\{a\}}$. We have just chosen $C_{\{a\}}$ such that $|W_a| > |Z_a|$. Hence $|W_a| \geq 3$. If j satisfies $\{i, j\} = \{2, 3\}$, then, by Lemma 3.11(iii),

$$D_1 = C_{\{a\}} \triangle (L_1 \cup L_j) = a \cup Y_a \cup Z_a \cup L_j \text{ and } D_2 = C_{\{a\}} \triangle (L_i \cup L_j) = a \cup X_a \cup W_a \cup L_j$$

are circuits of M satisfying $|D_1| = |D_2| = 8$. Thus $|Y_a| + |Z_a| = |X_a| + |W_a| = 4$. As $\emptyset \notin \{X_a, Y_a, Z_a, W_a\}$ because $\{a\} \not\to L_1$ and $\{a\} \not\to L_i$, it follows that $3 \leq |W_a| = 4 - |X_a| \leq 3$. Therefore $|W_a| = 3$ and $|X_a| = 1$. Consequently $|Z_a| = |L_1| - |W_a| = 2$ and $|Y_a| = |L_i| - |X_a| = 2$. □

Lemma 5.2 *For $i \in \{2, 3\}$, $V_i \neq \emptyset$.*

Proof. By symmetry, we may assume $i = 3$. By Theorem 2.1, there is an L-arc A such that $A \not\to L_1 \cup L_2$ and $A \not\to L_3$ because $\{L_1 \cup L_2, L_3\}$ is a 2-separation for $M|L$. Thus $A \in \mathcal{A}' = \mathcal{A}'_K$. By Lemma 3.11(i)(ii), $|A| = 1$, say $A = \{a\}$, and $M|(L \cup A)$ is a subdivision of $M(K_4)$ with $A \to L_1 \cup L_2$ or $A \to L_1 \cup L_3$. Hence $A \to L_1 \cup L_3$ and so $a \in V_3$. □

Lemma 5.3 *If $A \in \mathcal{A}_i$ and $a \in V_i$, for $i \in \{2, 3\}$, satisfy $X_a \subseteq C_A$, then $C_A = A \cup L_i$.*

Proof. Suppose that $C_A = A \cup X$, for $X \subseteq L_i$. Take $C_{\{a\}} = a \cup X_a \cup Z_a$. If $X = L_i$, then the result follows. Assume that $X \subsetneq L_i$. By Lemma 3.6, $|A| \leq |X| \leq |L_i| - 1 \leq 2$. Thus $|X| = 2$ because every parallel class of M is trivial. By hypothesis, $X_a \subseteq X$. Set $S_2 = (C_{\{a\}} \cap L_i) \cap C_A = X_a$ and $S_3 = (L_i - C_{\{a\}}) \cap C_A = X - X_a$. As $|S_2| = |S_3| = 1$, it follows, by Lemma 3.11(iv), that Lemma 3.11(iv)(a) holds. Thus $C_A = A \cup L_i$, a contradiction. The result follows. □

Lemma 5.4 *Suppose that N is a binary matroid. Let $\{X, Y\}$ be an exact 2-separation for $N|(X \cup Y)$. Assume that $N|(X \cup Y)$ is connected. If A is an $(X \cup Y)$-arc of N such that $A \not\rightarrow X$ and $A \not\rightarrow Y$ and C is a circuit of M such that $A \subseteq C \subseteq A \cup X \cup Y$, then $C \cap X \neq \emptyset, X - C \neq \emptyset, Y - C \neq \emptyset$ and $C \cap Y \neq \emptyset$.*

Proof. As $A \not\rightarrow X$, it follows that $C \cap Y \neq \emptyset$. Similarly, $C \cap X \neq \emptyset$. The result follows provided $X - C \neq \emptyset$ and $Y - C \neq \emptyset$, say $X - C \neq \emptyset$. Assume that $X - C = \emptyset$. Hence $X \subsetneq C$ because $C \cap Y \neq \emptyset$. Thus X is independent in N. Let C' be a circuit of $N|(X \cup Y)$ such that $C' \cap X \neq \emptyset$ and $C' \cap Y \neq \emptyset$. As X is independent in N, it follows that $X \subseteq C'$. Consider the cycle $D = C \triangle C'$ of M. Note that $A \subseteq D \subseteq Y \cup A$. There is a circuit D' of N such that $A \subseteq D' \subseteq D$ because A is a series class of $N|(X \cup Y \cup A)$. Hence $A \rightarrow Y$, a contradiction. Therefore $X - C \neq \emptyset$. □

Lemma 5.5 *For $i \in \{2, 3\}$, there is a 2-subset $\{a, b\}$ of V_i such that $X_a \neq X_b$.*

Proof. By Lemma 5.2, there is $a \in V_i$. Observe that $\{Y_a, (L - Y_a) \cup a\}$ is a 2-separation for $M|(L \cup a)$. By Theorem 2.1, there is an $(L \cup a)$-arc A of M such that $A \not\rightarrow Y_a$ and $A \not\rightarrow (L - Y_a) \cup a$. As L spans a in M, it follows that A is an L-arc of M. Consider C_A. By Lemma 5.4, $C_A \cap Y_a \neq \emptyset$ and $Y_a - C_A \neq \emptyset$. Hence $|C_A \cap Y_a| = |Y_a - C_A| = 1$. We have two cases to deal with: $A \in \mathcal{A}'$ and $A \notin \mathcal{A}'$. If $A \in \mathcal{A}' = \mathcal{A}'_K$, say $A = \{b\}$, then $b \in V_i$ and

$$Y_b = \begin{cases} C_A \cap L_i, & \text{when } |C_A \cap L_i| = 2; \\ L_i - C_A, & \text{when } |C_A \cap L_i| = 1. \end{cases}$$

If $Y_b = C_A \cap L_i$, then $X_b = L_i - C_A \supseteq Y_a - C_A \neq \emptyset$ and so $X_b = Y_a - C_A$. In particular, $X_a \cap X_b = \emptyset$ because $X_a \cap Y_a = \emptyset$. Hence $X_a \neq X_b$. If $Y_b = L_i - C_A$, then $X_b = C_A \cap L_i \supseteq C_A \cap Y_a \neq \emptyset$ and so $X_b = Y_a \cap C_A$. Again, $X_a \cap X_b = \emptyset$ and so $X_a \neq X_b$. When $A \in \mathcal{A}'$, the result follows. Assume that $A \notin \mathcal{A}'$. Therefore $A \in \mathcal{A}_j$, for some $j \in [3]$. Note that $j = i$ since $|C_A \cap Y_a| = |Y_a - C_A| = 1$. If $C_A = A \cup X$, for $X \subsetneq L_i$, then, by Lemma 3.6, $|X| \geq 2$ and so $|X| = 2$. Thus $X_a \subseteq X \subseteq C_A$, a contradiction to Lemma 5.3. □

Lemma 5.6 $\mathcal{A}_2 \cup \mathcal{A}_3 \subseteq \mathcal{A}_1$.

Proof. If $A \in \mathcal{A}_i - \mathcal{A}_1$, for some $i \in \{2, 3\}$, then $C_A = A \cup X$, for some $X \subsetneq L_i$. By Lemma 3.6, $|X| = 2$. By Lemma 5.5, X contains X_a, for some $a \in V_i$, a contradiction to Lemma 5.3. □

Lemma 5.7 *If $\{a, a'\}$ is a 2-subset of V with $X_a \neq X_{a'}$, then $Z_a \cap Z_{a'} \neq \emptyset$.*

Proof. Suppose that $Z_a \cap Z_{a'} = \emptyset$. If $b \in L_1 - (Z_a \cup Z_{a'})$, then $\{\{b\}, Z_a, Z_{a'}\}$ is a partition of L_1. We deal with the two cases simultaneously: $\{|\{a, a'\} \cap V_2|, |\{a, a'\} \cap$

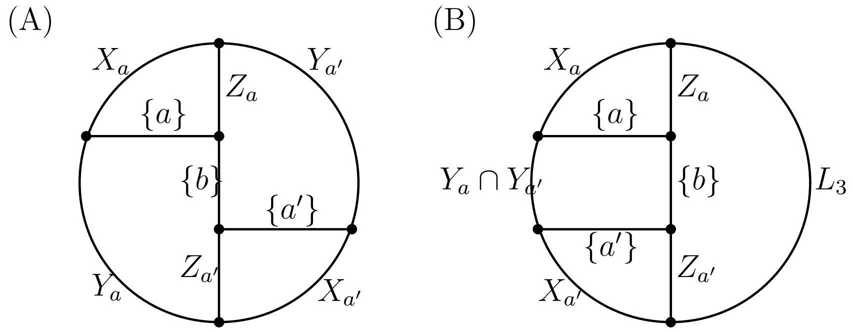

Figure 3: The two possibilities for the graph H. In (A), when $a \in V_2$ and $a' \in V_3$. In (B), when $\{a, a'\} \subseteq V_2$.

$V_3|\}$ can be equal to $\{1\}$ or $\{0, 2\}$, say $a \in V_2$ and $a' \in V_3$; or $\{a, a'\} \subseteq V_2$. In both cases the circuit space of $N = M|(L \cup \{a, a'\})$ is spanned by

$$L_1 \cup L_2, L_1 \cup L_3, a \cup X_a \cup Z_a, a' \cup X_{a'} \cup Z_{a'}. \tag{5.1}$$

In Figure 3, H is a graph such that $E(H) = \mathcal{SC}(N)$. Let G be a subdivision of H such that each edge S of H is replaced by a path of length $|S|$ whose edges are labeled by the elements of S. Note that the circuits displayed in (5.1) span the circuit space of $M(G)$. Therefore $M(G) = M|(L \cup \{a, a'\})$. When $a \in V_2$ and $a' \in V_3$,

$$(a \cup Y_a \cup W_a) \triangle (a' \cup Y_{a'} \cup W_{a'}) = \{a, a'\} \cup Z_a \cup Z_{a'} \cup Y_a \cup Y_{a'}$$

is a circuit of M with 10 elements, a contradiction. When $\{a, a'\} \subseteq V_2$,

$$(L_2 \cup L_3) \triangle (a \cup X_a \cup Z_a) \triangle (a' \cup X_{a'} \cup Z_{a'}) = \{a, a'\} \cup Z_a \cup Z_{a'} \cup (Y_a \cap Y_{a'}) \cup L_3$$

is a circuit of M with 10 elements, a contradiction. □

Lemma 5.8 *For $i \in \{2, 3\}$, there is a connected component N_i of M/L_1 such that $E(N_i) = L_i \cup V_i$, L_i is a triangle of N_i, each element of V_i is in parallel with some element of L_i and N_i has at least two non-trivial parallel classes.*

Proof. By symmetry, we may assume $i = 2$. The set of L-arcs of $M \backslash V_2$ is $\mathcal{A} - \mathcal{A}'_2 = \mathcal{A}'_3 \cup \mathcal{A}_1$, since, by Lemma 5.6, $\mathcal{A}_2 \cup \mathcal{A}_3 \subseteq \mathcal{A}_1$. Therefore $A \to L_1 \cup L_3$, for every $A \in \mathcal{A} - \mathcal{A}'_2$. By Theorem 2.1, $M \backslash V_2$ has a 2-separation $\{X, Y\}$ such that $X \cap L = L_2$ and $Y \cap L = L_1 \cup L_3$. Moreover, by Theorem 2.2, we may take

$$Y = (L_1 \cup L_3) \cup \left(\bigcup \{A : A \in \mathcal{A} - \mathcal{A}'_2\} \right)$$

and $X = L_2$. Thus L_2 is a series class of $M \backslash V_2$. As L_2 is a circuit of $(M \backslash V_2)/L_1$, it follows that L_2 is the ground set of a connected component H of $(M \backslash V_2)/L_1$. By Lemma 5.1, for each $a \in V_2$, $a \cup X_a \cup Z_a$ is a circuit of M. Therefore $a \cup X_a$ is circuit of M/L_1. As $|a \cup X_a| = 2$ and $X_a \subseteq L_2$, say $X_a = \{x_a\}$, it follows that, in M/L_1,

a is in parallel with $x_a \in L_2$. If N_2 is the connected component of M/L_1 satisfying $L_2 \subseteq E(N_2)$, then $V_2 \subseteq E(N_2)$ and $N_2 \backslash V_2 = H$. Thus $E(N_2) = L_2 \cup V_2$. Moreover, when $a \in V_2$, $a \cup x_a$ is contained in a non-trivial parallel class of N_2. By Lemma 5.5, N_2 has at least two non-trivial parallel classes. □

Now, we refine Lemma 5.8. Let C_3^2 be the unique connected matroid over $\{1, 2, 3, 4, 5, 6\}$ having $\{1, 2\}, \{3, 4\}$ and $\{5, 6\}$ as parallel classes.

Lemma 5.9 *If $\{a, a'\}$ is a 2-subset of V_i, for $i \in \{2, 3\}$, then $Z_a = Z_{a'}$. Moreover, N_i is isomorphic to C_3^2 or $C_3^2 \backslash 6$.*

Proof. First, we establish that

$$Z_a = Z_{a'}. \tag{5.2}$$

Assume that $i = 2$. Observe that $L' = (L - X_a) \cup a$ is a $(5, 3, 3)$-theta set of M with canonical partition $\{L_3 \cup Z_a, W_a, a \cup Y_a\}$. Applying Lemma 5.8 to L', we conclude that $M/(L_3 \cup Z_a)$ has rank-2 connected components H_2 and H_3 having W_a and $a \cup Y_a$ as triangles respectively. As $[M/(L_3 \cup Z_a)]/W_a = (M/L_1)/L_3$, it follows that $H_3 = N_2$ because $a \cup Y_a \subseteq L_2 \cup V_2 = E(N_2)$. By Lemma 5.8, there is $b \in L_2$ such that $\{a', b\}$ is a circuit of H_3. Let C be a circuit of M such that $C - (L_3 \cup Z_a) = \{a', b\}$. As $C \subseteq L \cup a'$ and $C \cap L_2 = \{b\}$, it follows that $C = \{a', b\} \cup Z_{a'}$ or $C = \{a', b\} \cup W_{a'} \cup L_3$. Hence $C = \{a', b\} \cup Z_{a'}$ because $|L_1 \cap (L_3 \cup Z_a)| = 2$ and $|W_{a'}| = 3$. Therefore $Z_{a'} \subseteq (L_3 \cup Z_a) \cap L_1 = Z_a$ and so $Z_a = Z_{a'}$. We have (5.2). Now, we show that every parallel class of N_2 has at most two elements. Assume that P is a parallel class of N_2 such that $|P| \geq 3$. Choose a 2-subset $\{a_1, a_2\}$ of $P - L_2$. Observe that $X_{a_1} = X_{a_2} = \{x\}$, where $x \in L_2 \cap P$. By (5.2), $Z_{a_1} = Z_{a_2}$. Hence

$$(a_1 \cup X_{a_1} \cup Z_{a_1}) \triangle (a_2 \cup X_{a_2} \cup Z_{a_2}) = \{a_1, a_2\}$$

is a circuit of M, a contradiction. Thus every parallel class of N_2 has at most two elements. The result follows from Lemma 5.8. □

Lemma 5.10 *If $A \in \mathcal{A}_1$ and $|A| = 3$, then*

(i) $C_A = A \cup L_1$; or

(ii) $C_A = A \cup X$, with $X \subseteq L_1$ and $|X| = 3$.

Moreover, when (ii) happens, $X = W_a = W_{a'}$, for every $a \in V_2$ and $a' \in V_3$.

Proof. There is $X \subseteq L_1$ such that $C_A = A \cup X$. If $X = L_1$, then (i) happens. Assume that $X \subsetneq L_1$. By Lemma 3.6, $|A| \leq |X| < |L_1| = 5$. Thus $X \in \{3, 4\}$. If $|X| = 4$, then $L_1 \cup L_2 \cup A$ is a $(4, 4, 3)$-theta set of M with canonical partition $\{X, (L_1 \cup L_2) - X, A\}$, a contradiction to Theorem 4.1. Hence $|X| = 3$. We have (ii). If $X = W_a = W_{a'}$, then the result follows. By symmetry, we may assume that $X \neq W_a$. Set $S_2 = [(a \cup X_a \cup Z_a) \cap L_1] \cap C_A$ and $S_3 = [L_1 - (a \cup X_a \cup Z_a)] \cap C_A$. Note that $S_2 = Z_a \cap X \neq \emptyset$ and $S_3 = W_a \cap X \neq \emptyset$. As $\{|S_2|, |S_3|\} = \{1, 2\}$, since $|S_2| + |S_3| = |X|$, it follows that Lemma 3.11(iv)(b)(c) cannot happen because $|A| = 3$. Hence Lemma 3.11(iv)(a) holds and $C_A = A \cup L_1$, a contradiction. □

Lemma 5.11 *If N is a connected component of M/L_1, then*

(i) $r(N) = 2$ *and* N *is isomorphic to* C_2^2 *or* $C_3^2\backslash 6$*; or*

(ii) $r(N) = 1$ *and* N *is isomorphic to* $U_{1,|E(N)|}$*; or*

(iii) $r(N) = 0$ *and* N *is isomorphic to* $U_{0,1}$*.*

Proof. If $N = N_2$ or $N = N_3$, then, by Lemma 5.9, (i) follows. Assume that $N \notin \{N_2, N_3\}$. If

$$\mathcal{A}'' = \{A \in \mathcal{A} : A \cap [E(N_2) \cup E(N_3)] = \emptyset\},$$

then, by Lemma 5.6, $\mathcal{A}'' \subseteq \mathcal{A}_1$. As every theta set of M has at most 11 elements, it follows that $|A| \leq 3$, for every $A \in \mathcal{A}''$. If $|A| = 3$, then, by Lemma 5.10, we have two possibilities:

(a) $C_A = A \cup L_1$. In this case, by Lemma 5.9, when applied to $L' = L_1 \cup L_2 \cup A$ whose canonical partition is $\{L_1, L_2, A\}$, M/L_1 has connected component N such that $A \subseteq E(N)$ and N is isomorphic to C_2^2 or $C_3^2\backslash 6$. We have (i).

(b) $C_A = A \cup W_a$, for $a \in V$. Observe that $L' = L_1 \cup L_2 \cup A$ is a theta set of M with canonical partition $\{L_2 \cup Z_a, W_a, A\}$. By Lemma 5.9 applied to L', $M/(L_2 \cup Z_2)$ has connected components N and N' such that $A \subseteq E(N), W_a \subseteq E(N')$ and N and N' are isomorphic to C_3^2 or $C_3^2\backslash 6$. If H is the connected component of M/L_1 that contains A, then $H \notin \{N_2, N_3\}$. Note that H is a connected component of $M/[L_1 \cup E(N_2)]$. Thus H is a connected component of $M/(L_1 \cup L_2)$ because $E(N_2) - L_2$ is a set of loops of this matroid. As $M/(L_1 \cup L_2) = M/(L_2 \cup Z_a)/W_a$ and $W_a \subseteq E(N')$, it follows that $H = N$. We have (i).

Now, let K be a connected component of M/L_1 such that every L-arc A of M satisfying $A \subseteq E(K)$ has cardinality at most 2. If $|A| = 1$, for some $A \in \mathcal{A}''$ such that $A \subseteq E(K)$, then $r(K) = 0$ and (iii) follows. Assume that $|A| = 2$, for every $A \in \mathcal{A}''$ such that $A \subseteq E(K)$. By Lemma 2.3, $r(K) = 1$ and (ii) follows. □

Lemma 5.12 *If* $A \in \mathcal{A}_1$ *and* $a \in V$*, then* $C_A \supseteq Z_a$ *or* $C_A \supseteq W_a$ *or* $W_a \supseteq C_A \cap L_1$*.*

Proof. Assume this result fails. If $C_A = A \cup X$, for $X \subseteq L_1$, then $|X \cap Z_a| = |Z_a - X| = 1$ and $W_a - X \neq \emptyset$. By Lemma 3.6, $|A| \leq |X|$ and so $|X| \geq 2$ because M is simple. Set $S_1 = Z_a - X, S_2 = Z_a \cap X, S_3 = W_a \cap X$ and $S_4 = W_a - X$. Note that $\emptyset \notin \{S_1, S_2, S_3, S_4\}$. We have a contradiction to Lemma 3.11(iv) applied to $C_a = a \cup X_a \cup Z_a$ and C_A. □

Lemma 5.13 *If* $a \in V_2$ *and* $a' \in V_3$*, then* $Z_a = Z_{a'}$*.*

Proof. Note that $L' = (L - X_a) \cup a$ is a theta-set of M with canonical partition $\{L_3 \cup Z_a, (L_2 - X_a) \cup a, W_a\}$. By Lemma 5.9 applied to L', there is a connected component N_1 of $M/(L_3 \cup Z_a)$ such that W_a is a circuit of N_1 and N_1 is isomorphic to C_3^2 or $C_3^2\backslash 6$. Let $\{b, b'\}$ and $\{x_b, x_{b'}\}$ be 2-subsets of $E(N_1) - W_a$ and W_a respectively such that $\{b, x_b\}$ and $\{b', x_{b'}\}$ are parallel classes of N_1. By Lemma 5.9 applied to L', there is a 2-subset X of $L_3 \cup Z_a$ such that $\{b, x_b\} \cup X$ and $\{b', x_{b'}\} \cup X$ are circuits of M. We have three possibilities for X:

(i) If $X \subseteq L_3$, then $b \cup X$ and $b' \cup X$ are cycles of M/L_1. As L_3 is a triangle of M/L_1, it follows that $b \cup X$ and $b' \cup X$ are also triangles of M/L_1. If $\{b''\} = L_3 - X$, then $\{b, b', b''\}$ is contained in a parallel class of N_3; a contradiction to Lemma 5.9.

(ii) If $|X \cap L_3| = 1$, say $b'' \in X \cap L_3$, then $\{b, b''\}$ and $\{b', b''\}$ are cycles of M/L_1. Similarly, we conclude that $\{b, b', b''\}$ is contained in a parallel class of N_3. Again, we arrive at a contradiction to Lemma 5.9. Thus we must have (iii).

(iii) If $X \cap L_3 = \emptyset$, then $X = Z_a$.

Similarly, we conclude that $X = Z_{a'}$. The result follows. □

By Lemmas 5.9 and 5.13, for a 2-subset $\{a, a'\}$ of V, we have $Z_a = Z_{a'}$ and so $W_a = L_1 - Z_a = L_1 - Z_{a'} = W_{a'}$. Therefore the connected component N_1 of $M/(L_3 \cup Z_a)$ that contains W_a does not depend on $a \in V_2$. By the proof of Lemma 5.13, N_1 is isomorphic to C_3^2 or $C_3^2\backslash 6$.

Lemma 5.14 *Let K be a connected component of M/L_1 such that $r(K) = 1$. If $Z_a = \{x, y\}$, for $a \in V_2$, then:*

(i) If I is a 2-subset of $E(K)$, then $I \in \mathcal{A}_1$ and $C_I \cap L_1 \in \{Z_a, W_a, L_1 - x, L_1 - y, L_1\}$.

(ii) If I is a 3-subset of $E(K)$ and $\binom{I}{2} = \{I_1, I_2, I_3\}$, then $\mathcal{X}_I = \{C_{I_1} \cap L_1, C_{I_2} \cap L_1, C_{I_3} \cap L_1\}$ is equal to $\{Z_a, W_a, L_1\}$ or $\{Z_a, L_1 - x, L_1 - y\}$.

(iii) $|E(K)| \in \{3, 4\}$ and, when $|E(K)| = 4$, $E(K)$ is a circuit-cocircuit of M.

(iv) If I and J are 3-subsets of $E(K)$, then $\mathcal{X}_I = \mathcal{X}_J$. (We denote this set by $\mathcal{X}_K$.)

Proof. (i) By Lemma 5.6, $I \in \mathcal{A}_1$. There is $X \subseteq L_1$ such that $C_I = I \cup X$. In $M/(L_3 \cup Z_a)$, $C_I - (L_3 \cup Z_a) = I \cup (X - Z_a)$ is a union of pairwise disjoint circuits. As $X - Za \subseteq W_a \subseteq E(N_1)$, where N_1 is the connected component of $M/(L_3 \cup Z_a)$ such that $W_a \subseteq E(N_1)$, it follows that $X - Z_a = \emptyset$ or $X - Z_a = W_a$ because W_a is a triangle of N_1. If $X - Z_a = \emptyset$, then $X = Z_a$, since $|X| \geq 2$, by Lemma 3.6. If $X - Z_a = W_a$, then $X \in \{W_a, L_1 - x, L_1 - y, L_1\}$.

(ii) If $L_1 - x$ and $L_1 - y$ do not belong to $\mathcal{X}_I$, then $\mathcal{X}_I = \{Z_a, W_a, L_1\}$ and (ii) follows. Assume that $L_1 - x \in \mathcal{X}_I$, say. Observe that W_a and L_1 do not belong to $\mathcal{X}_I$ otherwise $(L_1 - x) \triangle W_a = \{y\}$ or $(L_1 - x) \triangle L_1 = \{x\}$ belong to $\mathcal{X}_I$, because $C_{I_1} \triangle C_{I_2} = C_{I_3}$. (That is, $\mathcal{X}_I$ is closed under symmetric differences.) By (i), $\mathcal{X}_I = \{Z_a, L_1 - x, L_1 - y\}$.

(iii) Suppose that $I = \{a, b, c\}$ is a 3-subset of $E(K)$. By (iii), L_1 or $L_1 - x$ belongs to $\mathcal{X}_I$. Suppose that $C_{\{a,b\}} = \{a, b\} \cup L_1$, when $L_1 \in \mathcal{X}_I$, or $C_{\{a,b\}} = \{a, b\} \cup (L_1 - x)$, when $L_1 - x \in \mathcal{X}_I$. Assume that $d \in E(K) - I$. Set $J = \{a, b, d\}$. Note that $L_1 \in \mathcal{X}_J$, when $L_1 \in \mathcal{X}_I$, and $L_1 - x \in \mathcal{X}_J$, when $L_1 - x \in \mathcal{X}_I$. By (ii), $\mathcal{X}_I = \mathcal{X}_J$. There is a 2-subset $\{\alpha, \beta\}$ of J such that $L_1 \cap C_{\{\alpha,\beta\}} = L_1 \cap C_{\{a,c\}} \in \mathcal{X}_I = \mathcal{X}_J$. Hence

$$c \in C_{\{\alpha,\beta\}} \triangle C_{\{a,c\}} = \{\alpha, \beta\} \triangle \{a, c\} \subseteq I \cup J = I \cup d$$

is a cycle of M contained in $E(K)$. As M is 3-connected, it follows that $\{\alpha, \beta\} \cap \{a, c\} = \emptyset$. Therefore $I \cup d$ is a circuit of M. Assume that $d' \in E(K) - (I \cup d)$. Thus $I \cup d'$ is a circuit of M; a contradiction because $(I \cup d) \triangle (I \cup d') = \{d, d'\}$ is a cycle of M.

(iv) If $|E(K)| = 3$, then the result follows. Assume that $|E(K)| = 4$. By (iii), $E(K)$ is a circuit of M. If $\{X, Y\}$ is a partition of $E(K)$ with $|X| = |Y| = 2$, then $E(K) \triangle C_X \triangle C_Y = L_1 \cap (C_X \triangle C_Y)$ is a union of pairwise disjoint circuits of M. As $L_1 \cap (C_X \triangle C_Y)$ is independent in M, it follows that $L_1 \cap (C_X \triangle C_Y) = \emptyset$ and so $C_X \cap L_1 = C_Y \cap L_1$. The result follows from (ii). □

Lemma 5.15 *Suppose that $Z_a = \{x, y\}$, for $a \in V_2$. If K is a connected component of M/L_1 such that $r(K) = 1$ and $\mathcal{X}_K = \{Z_a, L_1 - x, L_1 - y\}$, then,*

(i) K is unique; and

(ii) M does not have an element z such that $(L_1 - x) \cup z$ or $(L_1 - y) \cup z$ is a circuit of M.

Proof. We prove (i) and (ii) simultaneously. Assume that (i) or (ii) fails. First, we choose $A'' \in \mathcal{A}_1$. We have two possibilities. When (ii) fails, say $(L_1 - x) \cup z$ is a circuit of M, set $A'' = \{z\}$ and $C_{A''} = z \cup (L_1 - x)$. When (i) fails, choose a 2-subset A'' of $E(K')$, where K' is a connected component of M/L_1 such that $K' \neq K$, $r(K') = 1$ and $\mathcal{X}_K = \mathcal{X}_{K'}$, satisfying $C_{A''} = A'' \cup (L_1 - x)$. Next, we choose a 2-subset A of $E(K)$ such that $C_A = A \cup (L_1 - y)$. Note that $\{A, A''\}$ is an apart pair of L-arcs of M. Hence A and A'' are series classes of $M|(L \cup A \cup A'')$. Consider

$$C = C_A \triangle C_{A''} \triangle (L_1 \cup L_3) = A \cup A'' \cup W_a \cup L_3.$$

As $|C| \geq 9$, it follows that C is the union of pairwise disjoint circuits $C_1, C_2, \dots, C_n$ of M, for $n \geq 2$. Observe that $\emptyset \notin \{C_1 - L, C_2 - L, \dots, C_n - L\}$ because $C \cap L = W_a \cup L_3$ is independent in M. Thus $n = 2$. Also, we may assume that $A \subseteq C_1$ and $A'' \subseteq C_2$. As L_3 is a series class of $M(A \cup A'' \cup L)$, it follows that

(a) $L_3 \subseteq C_1$ and $A'' \subseteq C_2 \subseteq A'' \cup W_a \subseteq A'' \cup L_1$; a contradiction because $C_{A''}$ is unique; or

(b) $L_3 \subseteq C_2$ and $A \subseteq C_1 \subseteq A \cup W_a \subseteq A \cup L_1$; a contradiction because C_A is unique.

The result follows. □

Let $N_2, N_3, N_4, \dots, N_l$ be the connected components of M/L_1 such that $r(N_i) = 2$, for every $i \in [l]$, $i \neq 1$. (In Lemma 5.8, we have defined N_2 and N_3 satisfying $L_2 \subseteq E(N_2)$ and $L_3 \subseteq E(N_3)$.) Let $K_{l+1}, K_{l+2}, \dots, K_{n-1}$ be the connected components of M/L_1 such that, for each $i \in [n-1] - [l]$, $r(K_i) = 1$ and $\mathcal{X}_{K_i} = \{Z_a, W_a, L_1\}$, where $a \in V$. By Lemmas 5.11 and 5.15, we have two cases:

(a) The other connected components of M/L_1 have rank equal to 0; or

(b) M/L_1 has a connected component K' such that $r(K') = 1$ and $\mathcal{X}_{K'} = \{Z_a, L_1 - x, L_1 - y\}$, where $Z_a = \{x, y\}$, and the other connected components of M/L_1 have rank equal to 0.

Remember that N_1 is the connected component of $M/(Z_3 \cup Z_a)$, for $a \in V_2$, such that $W_a \subseteq E(N_1)$.

We are going to decompose M. We may need to add elements α, β, γ to M to obtain a matroid M'.

(a) If $\alpha \cup Z_a$ is a triangle of M, for some $\alpha \in E(M)$, then set $M_\alpha = M$. If Z_a is not contained in a triangle of M, then let M_α be the binary matroid over $E(M) \cup \alpha$, where α is a new element, whose circuit space is spanned by $\mathcal{C}(M)$ and $Z_a \cup \alpha$.

(b) If $\beta \cup W_a$ is a circuit of M_α, for some $\beta \in E(M_\alpha)$, then set $M_\beta = M_\alpha$. If W_a is not contained in a 4-element circuit of M_α, then let M_β be the binary matroid over $E(M_\alpha) \cup \beta$, where β is a new element, whose circuit space is spanned by $\mathcal{C}(M_\alpha)$ and $W_a \cup \beta$.

(c) If $\{\alpha, \beta, \gamma\}$ is a triangle of M_β, for some $\gamma \in E(M_\beta)$, then set $M' = M_\beta$. If $\{\alpha, \beta\}$ is not contained in a triangle of M_β, then let M' be the binary matroid over $E(M_\beta) \cup \gamma$, where γ is a new element, whose circuit space is spanned by $\mathcal{C}(M_\beta)$ and $\{\alpha, \beta, \gamma\}$.

We can resume this construction as follows. Let M' be a binary matroid such that $E(M) \subseteq E(M')$, $M'|E(M) = M$, $r(M') = r(M)$, M' has a triangle $\{\alpha, \beta, \gamma\}$ such that $\alpha \cup Z_a$ and $\beta \cup W_a$ are circuits of M and $|E(M')|$ is minimum. (We denote M' by $M\langle L\rangle$ when we need to emphasize the dependence on L.)

Observe that M' is 3-connected and $E(M') = E(M) \cup \{\alpha, \beta, \gamma\}$. The next lemma is probability known and its proof is very simple.

Lemma 5.16 *For a matroid N and $X \subseteq E(N)$, let H be a connected component of N/X. If $Y \subseteq X$ and $(N/Y)|E(H) = (N/X)|E(H)$, then H is a connected component of N/Y.*

Proof. As $(N/Y)|E(H) = (N/X)|E(H)$, it follows that $E(H)$ is contained in a connected component of K of N/Y. If $E(K) = E(H)$, then $K = (N/Y)|E(H) = (N/X)|E(H) = H$ and the result follows. Assume that $E(K) - E(H) \neq \emptyset$. Let C be a circuit of N/Y such that $C \cap E(H) \neq \emptyset$ and $C - E(H) \neq \emptyset$. Hence $C - (X - Y)$ contains a circuit D of $(N/Y)/(X - Y) = N/X$ such that $D \cap E(H) \neq \emptyset$. Thus $D \subseteq E(H)$. As $(N/Y)|E(H) = (N/X)|E(H)$, it follows that D is a circuit of N/Y properly contained in C, a contradiction. □

Lemma 5.17 *If $i \in [l]$, then $M_i = M'|[E(N_i) \cup \{\alpha, \beta, \gamma\}]$ is*

(i) a 4-legged spike having tip α, when $|E(N_i)| = 6$; or

(ii) a matroid obtained from a 4-legged spike having tip α by deleting an element outside the leg $\{\alpha, \beta, \gamma\}$, when $|E(N_i)| = 5$.

Moreover, $N_1, N_2, N_3, \ldots, N_l$ are connected components of $M'/\{\alpha, \beta, \gamma\}$.

Proof. First, we show this result for $i = 2$; $i = 3$; and $i = 1$ (in this sequence). Then to a general i.

Assume that $i = 2$. Suppose that $L_2 = \{a_2, b_2, c_2\}$. By Lemma 5.9, we may assume that $\{a_2, a_2'\}$ and $\{b_2, b_2'\}$ are non-trivial parallel classes of N_2 and, when $N_2 \cong C_3^2$, let $\{c_2, c_2'\}$ be the third parallel class of it. By Lemma 5.9, $\{a_2, a_2'\} \cup Z_a$ is a circuit of M, for every $a \in V$. As $\alpha \cup Z_a$ is a circuit of M', it follows that $(\alpha \cup Z_a) \triangle (\{a_2, a_2'\} \cup Z_a) = \{a_2, a_2', \alpha\}$ is a triangle of M_2. Similarly, $\{b_2, b_2', \alpha\}$ and, when c_2' exists, $\{c_2, c_2', \alpha\}$ are triangles of M_2. Note that $\{\gamma, a_2, b_2, c_2\} = (L_1 \cup L_2) \triangle (L_1 \cup \gamma)$ is a circuit of M_2. As α, β, γ are loops of M'/L_1 and

$$(M'/L_1)|E(N_2) = (M'/[L_1 \cup \{\alpha, \beta, \gamma\}])|E(N_2) = (M'/\{\alpha, \beta, \gamma\})|E(N_2), \qquad (5.3)$$

it follows that the circuit space of M_2 is spanned by

$$\{\alpha, \beta, \gamma\}, \{a_2, a_2', \alpha\}, \{b_2, b_2', \alpha\}, \{\gamma, a_2, b_2, c_2\} \text{ and, when } c_2' \text{ exists, } \{c_2, c_2', \alpha\}$$

because $r(M_2) = r(\{\alpha, \beta, \gamma\}) + r(M_2/\{\alpha, \beta, \gamma\}) = 2 + r(N_2) = 4$. By (5.3) and Lemma 5.16, we conclude that N_2 is a connected component of $M'/\{\alpha, \beta, \gamma\}$. The result follows in this case.

Assume that $i = 3$. By Lemma 5.13, when $a' \in V_3$, then $Z_a = Z_{a'}$. We conclude the result in this case permuting L_2 with L_3 and using the previous case.

Assume that $i = 1$. Consider the theta set $L' = (L - X_a) \cup a$, for $a \in V_2$, whose canonical partition is $\{L_3 \cup Z_a, W_a, (L_2 - X_a) \cup a\}$. Note that N_1 and N_2 are the connected components of $M/(L_3 \cup Z_a)$. In M', the sets $\gamma \cup L_3, \beta \cup W_a, \alpha \cup Z_a$ are circuits. For the theta set L', the roles of β and γ commutes. But α have the same role. (Note that $M\langle L' \rangle$ is obtained from $M\langle L \rangle$ by permuting γ with β.) By the previous paragraph, applied to L' and N_1, we obtain the result in this case.

Now, assume that $i \geq 4$. By Lemma 5.10, we have two cases to deal with. There is an L-arc A of M such that $|A| = 3, A \subseteq E(N_i)$ and

(1) $C_A = A \cup L_1$. In this case $L' = L_1 \cup L_2 \cup A$ is a theta set of M with canonical partition $\{L_1, L_2, A\}$. The result follows in this case because A takes the place of L_3.

(2) $C_A = A \cup W_a$. In this case, $L' = L_1 \cup L_3 \cup A$ is a theta set of M whose canonical partition is $\{L_3 \cup Z_a, W_a, A\}$ and $L_3 \cup Z_a$ takes the place of L_1. The result follows from the case $i = 1$ because A takes the place of $(L_2 - X_a) \cup a$.

□

Lemma 5.18 *If $i \in [n-1] - [l]$, then $M_i = M'|[E(K_i) \cup \{\alpha, \beta, \gamma\}]$ is isomorphic to $M(K_4)$ or F_7. Moreover, K_i is a connected component of $M'/\{\alpha, \beta, \gamma\}$.*

Proof. Let I_i be a 3-subset of $E(K_i)$, say $I_i = \{a_i, b_i, c_i\}$. We may assume that

$$C_{\{a_i,b_i\}} = \{a_i, b_i\} \cup Z_a, C_{\{a_i,c_i\}} = \{a_i, c_i\} \cup W_a, C_{\{b_i,c_i\}} = \{b_i, c_i\} \cup L_1.$$

Thus $\{a_i, b_i, \alpha\}, \{a_i, c_i, \beta\}$ and $\{b_i, c_i, \gamma\}$ are triangles of M_i. If $|E(M_i)| = 6$, then these triangles spans the circuit space of M_i. That is, M_i is isomorphic to $M(K_4)$. If $|E(M_i)| = 7$, then those triangles together with $E(K_i)$ spans the circuit space of M_i. In this case, M_i is isomorphic to F_7. By Lemma 5.16, K_i is a connected component of $M'/\{\alpha, \beta, \gamma\}$ because $E(K_i)$ is contained in a parallel class of $M'/\{\alpha, \beta, \gamma\}$. □

Let $M_n = M' \backslash [E(N_1) \cup E(N_2) \cup \cdots \cup E(N_l) \cup E(K_{l+1}) \cup E(K_{l+2}) \cup \cdots \cup E(K_{n-1})]$. To finish the proof of Theorem 3.3, we need to analyse M_n. By Lemma 5.15, we have two cases to deal with:

Case 1. Each connected component of M/L_1 different from $N_2, N_3, \ldots, N_l$, K_{l+1}, $K_{l+2}, \ldots, K_{n-1}$ has rank 0.

Note that $Z_a \cup \{\alpha, \beta, \gamma\} \subseteq E(M_n)$, when $a \in V$. If $b \in E(M_n) - (Z_a \cup \{\alpha, \beta, \gamma\})$, then b is a loop of M/L_1. Assume that $b \cup X$, for $X \subseteq L_1$, is a circuit of M. As $(b \cup X) - (Z_a \cup L_3)$, for $a \in V_2$, is a cycle of $M/(Z_a \cup L_3)$ and $b \notin E(N_1)$, it follows that $X \cap W_a = \emptyset$ or $X \supseteq W_a$. If $X = Z_a$, then $\{\alpha, b\} = (b \cup Z_a) \triangle (\alpha \cup Z_a)$ is a circuit of M', a contradiction. If $X = W_a$ or $X = L_1$, then, similarly, $\{\beta, b\}$ or $\{\gamma, b\}$ is a circuit of M' respectively, a contradiction. Thus $X \in \{L_1 - x, L_1 - y\}$, where $Z_a = \{x, y\}$. (If $X = L_1 - x$, then $[b \cup (L_1 - x)] \triangle [\gamma \cup L_1] = \{b, \gamma, x\}$ is a triangle of M_n. Hence $\{b, \gamma, x\}$ or $\{b, \gamma, y\}$ is a triangle of M_n.) As Z_a is not a cocircuit of M, at least one of these two possibilities for b must exist. Thus M_n is isomorphic to $M(K_4)$ or F_7. Theorem 3.3 follows in this case with $m = l$.

Case 2. M/L_1 has a connected component K such that $r(K) = 1$ and $\mathcal{X}_K = \{Z_a, L_1 - x, L_1 - y\}$, where $Z_a = \{x, y\}$, for $a \in V$.

Suppose that $b \in E(M_n) - [E(K) \cup Z_a \cup \{\alpha, \beta, \gamma\}]$. Note that b is a loop of M/L_1. Similarly to the previous case, $b \cup X$ is a circuit of M, for some $X \in \{L_1 - x, L_1 - y\}$, a contradiction to Lemma 5.15(ii). Therefore $E(M_n) = E(K) \cup Z_a \cup \{\alpha, \beta, \gamma\}$. Let $\{a_n, b_n, c_n\}$ be a 3-subset of $E(K)$. Suppose that $\{a_n, b_n\} \cup Z_a$, $\{a_n, c_n\} \cup (L_1 - x)$, $\{b_n, c_n\} \cup (L_1 - y)$ are circuits of M. Note that $\{a_n, b_n, \alpha\}$, $\{a_n, c_n, \beta, y\}$, $\{b_n, c_n, \beta, x\}$, $\{\alpha, \beta, \gamma\}$, $\{x, y, \alpha\}$ are circuits of M. If $|E(K)| = 4$, say $d_n \in E(K) - \{a_n, b_n, c_n\}$, then the circuit space of M_n is spanned by

$$\{\alpha, \beta, \gamma\}, \{\alpha, x, y\}, \{\alpha, a_n, b_n\}, \{\alpha, c_n, d_n\} \text{ and } \{a_n, c_n, y, \beta\}.$$

and M_n is isomorphic to a 4-leged spike. (Note that $\{\alpha, c_n, d_n\} = \{\alpha, a_n, b_n\} \triangle E(K)$.) If $|E(K)| = 3$, then M_n is obtained from the 4-legged spike described above removing d_n. Therefore Theorem 3.3 also follows in this case with $m = l + 1$.

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