AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **91(2)** (2025), Pages 282–300

About kings and dominating sets in tournaments

Ayman El Zein*

Lebanese University, KALMA Laboratory
Beirut, Lebanon

Abstract

In a digraph, a king is a vertex that can reach any other vertex by a directed path of length at most 2, and a dominating set is a set of vertices containing an in-neighbor of any vertex outside it. The domination number of a digraph is the cardinality of a smallest dominating set. E. Szekeres and G. Szekeres [*Math. Gaz.* 49 (1965), 290–293] proved that the domination number of an n-tournament is at most $\log_2(n) - \log_2(\log_2(n)) + 2$. In this paper the concept of kings is used in order to improve this bound. Moreover, we show that for every two integers $k \geq 3$ and $s \geq 2$, and for all $n \geq 3s + k - 2$, there exists an n-tournament T which has exactly k kings and $\delta^-(T) = s$. Furthermore, we characterize the tournaments with exactly three kings. We also treat the new concept of king degree introduced by El Sahili [Seminars on Graph Theory, 2021]; that is, in a digraph, the king degree of a vertex x, denoted by $k^+(x)$, is the number of vertices that can be reached from x by a directed path of length at most 2. The main result in this context is the characterization of the set of integers that can be the set of king degrees of the vertices of a tournament.

1 Introduction

A tournament is an orientation of a complete graph. An n-tournament is a tournament on n vertices. Let u and v be two vertices. A directed path starting from u and ending at v is said to be a uv-directed path. A tournament is said to be strong if there is a directed path between every two distinct vertices; otherwise, it is said to be reducible. A transmitter is a vertex dominating all other vertices. Consider a tournament T and a vertex u in T. The closed out-neighborhood of x, denoted by $N^+[x]$, is the set $N^+(x) \cup \{x\}$. As an abbreviation, we write $T^+(u)$ and $T^-(u)$ instead of $T[N^+(u)]$ and $T[N^-(u)]$. Also, we write $u \to v$ when $(u, v) \in E(T)$ and $A \to B$ when $u \to v$ for all $u \in A$ and $v \in B$. If x is a vertex and X is a

* Also at: University Savoie Mont Blanc, LAMA Laboratory, Chambéry, France.

ISSN: 2202-3518

tournament, we write $N^+_X(x)$ instead of $N^+(x) \cap V(X)$, and $d^+_X(x)$ denotes the cardinality of $N^+_X(x)$. Consider two disjoint tournaments T_1 and T_2. The tournament $T = T_1 \to T_2$ is the tournament such that $V(T_1)$ and $V(T_2)$ form a partition of $V(T)$ and $E(T) = E(T_1) \cup E(T_2) \cup \{(u, v); u \in T_1, v \in T_2\}$.

Let T be a tournament and consider two vertices u and v of T. The distance from u to v in T, denoted by $d^+_T(u, v)$, is the length of the shortest uv-directed path, if it exists. The second out-neighborhood of a vertex u is the set $N^{++}(u)$ containing every vertex v such that $d^+_T(u, v) = 2$. A vertex u in T is said to be a king if $d^+_T(u, v) \leq 2$ for all $v \in T$. Analogously, u is said to be a serf if $d^+_T(v, u) \leq 2$ for all $v \in T$. Now, $K(T)$ is the subtournament induced by the kings of T and we designate by $k(T)$ the order of $K(T)$. For an integer $i \geq 1$, we write $K^i(T) = \underbrace{K(K(\ldots K(T)\ldots))}_{i \text{ times}}$ and for $i = 0$, we write $K^i(T) = T$. We denote by $\xi(T)$ the first positive integer such that $K^{\xi(T)}(T) = K^{\xi(T)-1}(T)$. Here, T is said to be an all-kings tournament if $\xi(T) = 1$.

In a seminar on graph theory [2], El Sahili introduced the definition of the king degree $k^+(u)$ of a vertex u, in a digraph D, to be the number of vertices v such that $d^+_D(u, v) \leq 2$. The subdigraph induced by the vertices v such that $d^+_D(u, v) \leq 2$ is said to be the kingdom of u and is denoted by $K^+(u)$.

A set of vertices $M \subseteq V(T)$ is said to be a dominating set of T if it contains an in-neighbor of every vertex outside M. The domination number of T, denoted by $\gamma(T)$, is the cardinality of a smallest dominating set. A dominating set is said to be minimum if it has cardinality $\gamma(T)$.

Erdős [3] demonstrated that, for any tournament T of order n, $\gamma(T)$ is less than or equal to $\lceil \log_2 n \rceil$. Moreover, we can derive that $\gamma(T) \geq \log_2(n) - 2\log_2(\log_2(n)) - \log_2(\log_2(2+\epsilon))$ for $\epsilon > 0$, for sufficiently large n. Subsequently, Szekeres and Szekeres [12] improved the upper bound of the domination number to $\log_2(n) - \log_2(\log_2(n)) + 2$. In [7], Lu et al. gave a short direct proof of the same result. Furthermore, Erdős [3] proved the existence of tournaments with domination number greater than k for an arbitrary positive integer k using probabilistic methods. After that, Graham and Spencer wrote an article [4] describing a technique for constructing tournaments T with $\gamma(T) > k$ for every positive integer k. Reid et al. [11] proved that $\gamma(T)$ is at most 2 if the order of T is less than 7 and it is at most 3 if the order of T is less than 19. Duncan and Jacobson [1] established that for arbitrary positive integers k and m, where $k > 1$, there exists a tournament with domination number k having exactly m minimum dominating sets. In this article, we build a correlation between the dominating sets and the kings of a tournament. Indeed, for a tournament T, we show that each dominating set of $K(T)$ is likewise a dominating set of T. Moreover, any minimum dominating set of $K(T)$ is a minimum dominating set of T, which improves the upper bound of $\gamma(T)$ to $\log_2(k) - \log_2(\log_2(k)) + 2$, where $k = |K^{\xi(T)}(T)|$.

In a charming explanation of a tournament model for dominance in chicken flocks, Maurer [8] established the twin terms king and serf. Landau [6] proved that any vertex with the greatest outdegree in a tournament is a king, and we can derive from [5] that a tournament T contains a minimum of three kings if $\delta^-(T) > 0$. The following theorem is proved by Maurer [8]:

Theorem 1.1 *[8] Let $n, k \in \mathbb{N}$ such that $n \geq k \geq 1$. There exists a tournament of order n with exactly k kings if and only if $(n, k) \notin \{(n, 2), (4, 4)\}$.*

Then Reid [9] provided an inductive proof of this result. In [8], Maurer asked about 4-tuples (n, k, s, b) for which there exists a tournament of order n, with exactly k kings, s serfs, and b vertices which are both kings and serfs. In [10], Reid characterized such 4-tuples. Maurer [8] also asked which tournaments can be the subtournament of kings of some tournament. He proved that such tournaments have a positive minimum indegree. For a non-trivial tournament T without transmitter, $m(T)$ represents the least order of a tournament W such that $T \subseteq W$ and $K(W) = T$. Reid [9] established a lower and upper bound of $m(T)$ and he requested an improvement of these bounds. Characterizing the n-tournaments included in an all-kings m-tournament is a comparable challenge. This is a property that every tournament possesses. Reid [9] determined, for a tournament T, the smallest order of an all-kings tournament that contains T. In [13], Yu et al. characterized, in a given tournament T, those arcs e in a way that the digraph $T - e$ that results from eliminating e has a king. And in [14], Yu characterized those pairs of arcs $\{e_1, e_2\}$ such that the digraph $T - \{e_1, e_2\}$ has a king.

Following the definition of "king degree", El Sahili [2] stated the subsequent two problems:

Problem 1 *Given a set of distinct integers $\{m_1, m_2, \ldots, m_t\}$, can we find a tournament on which the set of king degrees of its vertices is exactly $\{m_1, m_2, \ldots, m_t\}$?*

Problem 2 *Given a strong tournament T of order n, what is the lower bound on the number of vertices having a king degree at least $\frac{n}{2}$?*

In this article we give a construction for tournaments T with $\delta^-(T) = s$ and $k(T) = k$, where $s \geq 2$ and $k \geq 3$ are arbitrary. Also, we characterize the tournaments with exactly three kings.

Regarding Problem 2, we show that, in a strong n-tournament, the number of vertices having king degree at least $\frac{n}{2}$ is at least $\left\lfloor \frac{n}{2} + 3 \right\rfloor$. Concerning Problem 1, Theorem 1.1 is an answer in the case $t = 1$. In the last section of this paper, we settle completely Problem 1 in Theorem 4.13.

2 Minimum dominating set

In order to improve the upper bound of $\gamma(T)$, where T is a tournament, we first establish some results:

Lemma 2.1 *Let T be a tournament and M be a minimum dominating set of T. For any $x, y \in M$, we have $d_T^+(x, y) \leq 2$.*

Proof. Suppose that there exist $x, y \in M$ such that $d_T^+(x,y) \geq 3$; then $(y,x) \in E(T)$ and $N_{T-M}^+(x) \subseteq N_{T-M}^+(y)$, and thus $M - x$ is a dominating set with $|M - x| < |M|$, a contradiction. □

The previous lemma is useful in the proof of the theorem below:

Theorem 2.2 *Let T be a tournament. $K(T)$ contains a minimum dominating set of T.*

Proof. Denote by Γ the set of minimum dominating sets X of T such that $|X \cap K(T)| = \max\{|N \cap K(T)| : N \text{ is a minimum dominating set of } T\}$. Let $M \in \Gamma$ such that $\Sigma_{v \in M-K(T)} d_{T-M}^+(v)$ is maximal. Suppose that M is not contained in $K(T)$, i.e. there exists $x \in M$ which is not a king of T. By the previous lemma, $d_T^+(x,y) \leq 2$ for all $y \in M$. Thus, there exists $y \in N_{T-M}^-(x)$ such that $d_T^+(x,y) \geq 3$, and so $N_{T-M}^+(x) \subset N_{T-M}^+(y)$. Set $M' = (M \cup \{y\}) - x$; then M' is an element of Γ with $\Sigma_{v \in M'-K(T)} d_{T-M'}^+(v) > \Sigma_{v \in M-K(T)} d_{T-M}^+(v)$, a contradiction. □

Theorem 2.3 *Let T be a tournament. Any dominating set of $K(T)$ is a dominating set of T.*

Proof. Let M be a dominating set of $K(T)$. Set $X = \bigcap_{x \in M} N_{T-K(T)}^-(x)$ and suppose that $X \neq \emptyset$. Let $v \in K(X)$. For $y \in T - (K(T) \cup X)$, there exists $x \in M$ such that vxy is directed. For $y \in K(T)$, there exists $x \in M$ such that $x \to y$ and so vxy is directed. Hence $d_T^+(v,y) \leq 2$ for all $y \in T$; thus $v \in K(T)$, a contradiction. Therefore $X = \emptyset$, and thus for all $x \in T - M$ there exists $y \in M$ such that $y \to x$. □

As a corollary of the above, we reach our main result regarding the domination number:

Corollary 2.4 *Let T be a tournament; then $\gamma(T) \leq \log_2(k) - \log_2(\log_2(k)) + 2$, where $k = |K^{\xi(T)}(T)|$.*

Proof. If $\xi(T) = 0$, then $k = n$ and the result is the same as in [12]. Suppose now that $\xi(T) \geq 1$. By Theorem 2.3, any dominating set of $K^i(T)$ is a dominating set of $K^{i-1}(T)$ for all $i \in \{1, \ldots, \xi(T)\}$. Let M be a dominating set of $K^{\xi(Y)}(T)$; then M is a dominating set of $K^i(T)$ for all $i \in \{0, \ldots, \xi(T)\}$, and in particular, M is a dominating set of $K^0(T) = T$. Since M is a dominating set of $K^{\xi(T)}(T)$, by [12], $|M| \leq \log_2(k) - \log_2(\log_2(k)) + 2$. The result follows. □

3 Kings in tournaments

Notice that in any tournament T, for a vertex x, a king of the subtournament induced by the in-neighbors of x is a king of T. In particular, every king with positive in-degree is dominated by a king. From this fact, if $\delta^-(T) \geq 1$, then T has at least three kings. Recall that a tournament T has a unique king if and only if $\delta^-(T) = 0$. Given

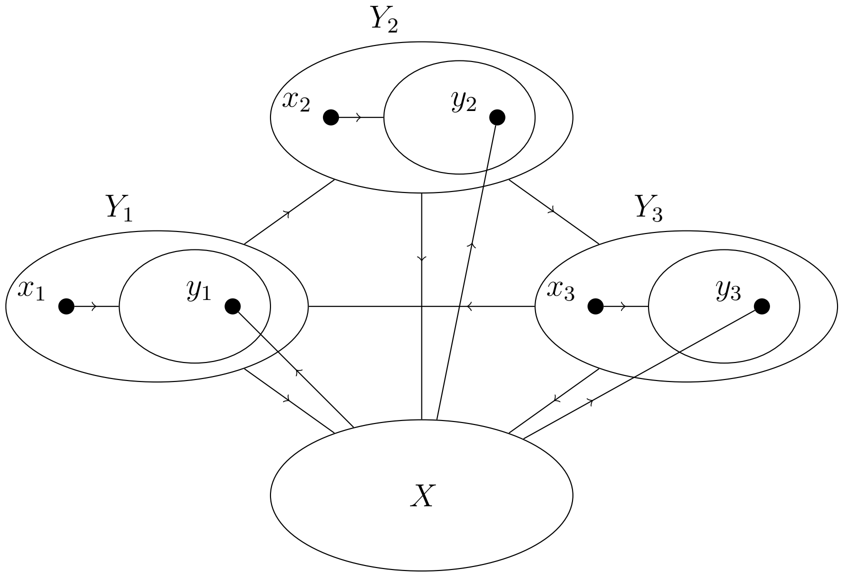


Figure 1: Tournament T for $k \neq 5$

these results, one might expect that there is a relation between $\delta^-(T)$ and $k(T)$ such as $k(T) \geq 2\delta^-(T)+1$ or $k(T) \geq \delta^-(T)+2$, where T is a tournament. The following result about the existence of tournaments having both an arbitrary number of kings and minimu in-degree confirms that such a relation does not exist.

Theorem 3.1 *Let $k, s \in \mathbb{N}$ where $k \geq 3$ and $s \geq 2$. For all $n \geq 3s+k-2$, there exists an n-tournament T such that $\delta^-(T) = s$ and $k(T) = k$.*

Proof. Let $n \geq 3s+k-2$ be an integer; we distinguish two cases:

Case 1: If $k \neq 5$, let X be an $(n-3s)$-tournament such that $k(X) = k-3$ (this tournament exists by Theorem 1.1), and let Y_1, Y_2 and Y_3 be three s-tournaments, with x_1, x_2 and x_3 being three transmitters of Y_1, Y_2, and Y_3 respectively. Consider the tournament T on n vertices represented in Figure 1. Note that no vertex in Y_i can reach x_i by a directed path of length at most 2 and $K_T^+(x) \cap X = K_X^+(x)$ for every vertex $x \in X$; thus one can easily obtain $V(K(T)) = V(K((X)) \cup \{x_1, x_2, x_3\}$. Hence T is an n-tournament such that $\delta^-(T) = s$ and $k(T) = k$.

Case 2: If $k = 5$, let X be an $(n-3s-2)$-tournament, let Y_1 and Y_2 be s-tournaments with x_1 and x_2 being transmitters of Y_1 and Y_2 respectively; and let Y_3 be an $(s+2)$-tournament with x_3, x_4, and x_5 forming a circuit triangle dominating any other vertex in Y_3. Consider the tournament T on n vertices represented in Figure 2. Clearly, the kings of T are x_1, x_2, x_3, x_4, and x_5. Moreover, x_2 has the minimum in-degree, which is s. The result follows. □

Is the bound of n sharp or can it be improved? Indeed, if $k \notin \{5, 7\}$, we can replace X by an all-kings $(k-3)$-tournament; this result remains true for $n = 3s+k-3$ and if $k = 5$, then $T' - X'$ gives the result. However, in order to characterize the tournaments with exactly three kings, we need to define a special form of tournament

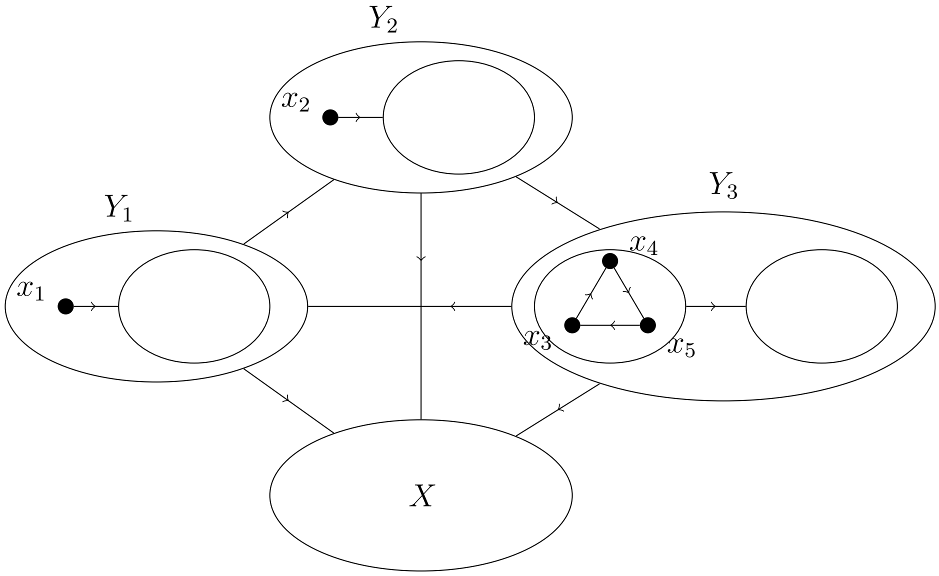


Figure 2: Tournament T for $k = 5$

T_3, represented in Figure 3, where X_1, X_2, X_3, Y_1, Y_2, Y_3 are each tournaments. Note that X_i and Y_i are allowed to be null. We have to mention first that x_1, x_2 and x_3 are the only kings of T. The aim now is to show that any tournament T such that $k(T) = 3$ is of the form T_3.

The following lemma is useful in the proof of the results in the rest of the article:

Lemma 3.2 *Let T be a tournament and let x and y be two vertices of T. If $k^+(x) \geq k^+(y)$, then $d^+(x, y) \leq 2$.*

Proof. If $d^+(x, y) \geq 3$ then $N^+(x) \subseteq N^+(y)$ and so $k^+(x) \leq k^+(y) - 1$, a contradiction. So $d^+(x, y) \leq 2$. $\square$

Theorem 3.3 *Let T be a tournament. T has exactly three kings if and only if T is of the form T_3.*

Proof. The sufficient condition follows from the fact that x_1, x_2, and x_3 are the sole kings of a tournament of the form T_3. Indeed, for any $i \in \{1, 2, 3\}$, any vertex in X_i cannot reach x_1 by a directed path of length at most 2. Moreover, any vertex in Y_1 (respectively, Y_2, Y_3) cannot reach x_2 (respectively, x_3, x_1) by a directed path of length at most 2. Assume now that a tournament T has exactly three kings, namely x_1, x_2 and x_3. Note that each x_i has a king as an in-neighbor. Therefore x_1, x_2, x_3 induced a directed cycle. Since $K(T^-(x_1)) \subseteq K(T)$, it follows that $k(T^-(x_1)) = 1$. Thus $K(T^-(x_1)) = \{x_3\}$ and $x_3 \to (T^-(x_1) - x_3)$. Similarly, $K(T^-(x_2)) = \{x_1\}$, $K(T^-(x_3)) = \{x_2\}$, $x_1 \to (T^-(x_2) - x_1)$, and $x_2 \to (T^-(x_3) - x_2)$. Set $X_1 = T^-(x_2) - x_1$, $X_2 = T^-(x_3) - x_2$, and $X_3 = T^-(x_1) - x_3$. We are going to prove that $X_1 \to X_2$. Indeed, suppose that there exists $x \in X_2$ such that $d^+_{X_1}(x) \geq 1$, and suppose that x is chosen with $k^+_{X_2}(x)$ being maximal. Clearly, $d^+(x, x_2) = 2$. By Lemma 3.2, we

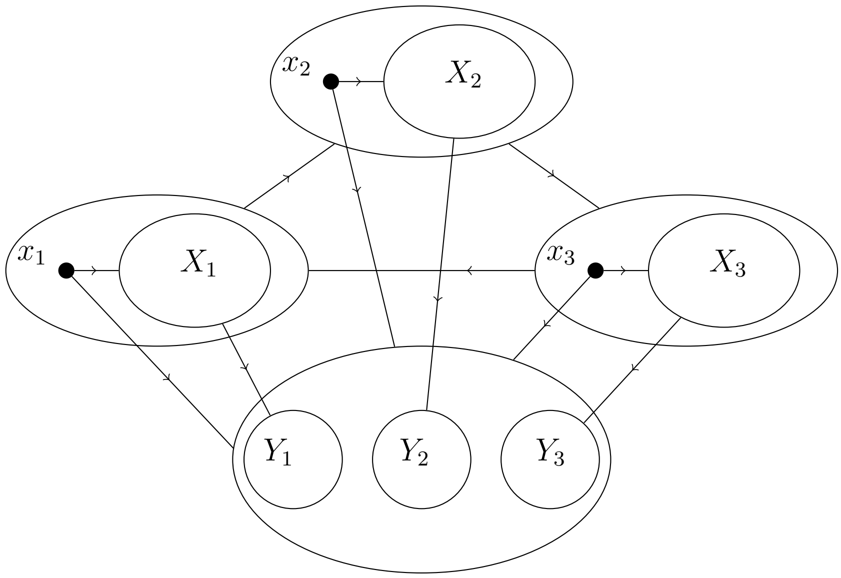


Figure 3: Tournament of the form T_3

have $d^+(x,y) \le 2$ for every $y \in X_2$ such that $k^+_{X_2}(x) \ge k^+_{X_2}(y)$. Let $y \in X_2$ such that $k^+_{X_2}(y) > k^+_{X_2}(x)$; then $X_1 \to y$ and so $d^+(x,y) \le 2$. Furthermore, let $y \in T - X_2$ such that $y \neq x_2$; then $y \in N^+(x_3)$ and so $d^+(x,y) \le 2$. Thus $x \in K(T)$, a contradiction. Likewise, $X_2 \to X_3$ and $X_3 \to X_1$. Set $A = T - \bigcup_{i=1}^{3}(X_i \cup \{x_i\})$. We claim that there exists $1 \le i \le 3$ such that $N^+(x) \cap X_i = \emptyset$ for every $x \in A$. Suppose to the contrary that there exists $x \in A$ such that $N^+(x) \cap X_i \neq \emptyset$ for all $1 \le i \le 3$ and suppose that x is chosen such that $k^+_A(x)$ is maximal. As above, $d^+(x,y) \le 2$ for every $y \in A$. For $y \in T - A$, depending on the relations between the vertices in $T - A$, one can easily check that $d^+(x,y) \le 2$. Hence $x \in K(T)$, a contradiction. Set $Y_i = \{x \in A; X_i \to x\}$ for all $1 \le i \le 3$. Thus T is of the form T_3. □

As we find a characterization for the tournaments having exactly three kings, can we characterize the tournaments having an arbitrary number of kings? Similarly, given a tournament T, can we characterize the tournaments W such that $K(W) = T$?

4 King degree

This section is structured as follows: the first subsection contains a result concerning Problem 2 and some useful lemmas in the treatment of Problem 1. The second subsection represents the main result of the section, a complete solution of Problem 1, which is a characterization of the set of integers which can be the set of king degrees of the vertices of a tournament.

4.1 Preliminary study

The first remark about the number of vertices with king degree at most s is the following:

Lemma 4.1 *Let T be a tournament; we have $|\{v \in T : k^+(v) \leq s\}| \leq s$. Moreover, if $|\{v \in T : k^+(v) \leq s\}| = s$, then $x \to y$ for all $x, y \in T$ such that $k^+(x) > s$ and $k^+(y) \leq s$.*

Proof. Set $X = \{v \in T : k^+(v) \leq s\}$ and let $x \in K(T[X])$. Clearly, $K^+_{T[X]}(x) \subseteq K^+_T(x)$ and so $|X| \leq s$. Suppose that $|X| = s$, and to show a contradiction suppose there exist y and $x(y) \in T$ satisfying $k^+(x(y)) > s$, $k^+(y) \leq s$ and $y \to x(y)$. Let y_0 be such a vertex with maximum king degree. For all $v \in T - y_0$ such that $k^+(v) \leq s$, we have $v \in K^+(y_0)$. In fact, if $k^+(v) \leq k^+(y_0)$, then by Lemma 3.2, $v \in K^+(y_0)$, and if $k^+(v) > k^+(y_0)$, the path $y_0x(y_0)v$ is directed, so $v \in K^+(y_0)$. Since $x(y_0) \in K^+(y_0)$, we obtain $k^+(y_0) > s$, a contradiction. □

As the equality may occur in the case of reducible tournaments, this number decreases at least by 2 in the case of a strong tournament.

Theorem 4.2 *Let T be a strong tournament of order n and $3 \leq s \leq n - 1$ be an integer. Then $|\{v \in T : k^+(v) \leq s\}| \leq s - 2$.*

Proof. Set $X = \{v \in T : k^+(v) \leq s\}$ and $Y = \bigcup_{v \in X} \bigcup_{u \in N^+_{T-X}(v)} N^+_{T-X}[u]$. If $|Y| = 1$ then $(T - Y) \to Y$, contradicting the fact that T is strong. So $|Y| \geq 2$. Let $x \in X$ such that $|K^+(x) \cap Y| \geq 2$ and suppose that it is chosen of maximal king degree. For any $y \in X$ such that $k^+(y) > k^+(x)$, there exists $z \in N^+_{T-X}(x)$ such that $z \to y$. Hence $|X| + 2 \leq k^+(x) \leq s$. The result follows. □

As a consequence, one can easily notice the following remark:

Remark 4.3 *Let T be a tournament. We have $|\{v \in T : k^+(v) = 3\}| \neq 2$.*

Proof. To show a contradiction, suppose $|\{v \in T : k^+(v) = 3\}| = 2$ and let x and y be the two vertices with king degree 3 such that $x \to y$. If $d^+(x) = 2$, then $N^{++}(x) = \emptyset$, and so $k^+(y) < 3$, a contradiction. Otherwise, $d^+(x) = 1$. Let z be the unique out-neighbor of y. As $k^+(y) = 3$, z is dominated by all vertices other than x, y, and z. Then, $k^+(z) = 3$, a contradiction. □

As a corollary of the previous theorem, a lower bound on the number of vertices having a king degree at least the half of the order of the tournament results:

Corollary 4.4 *Let T be a strong tournament of order n. We have*

$$|\{v \in T : k^+(v) \geq \frac{n}{2}\}| \geq \left\lfloor \frac{n}{2} + 3 \right\rfloor.$$

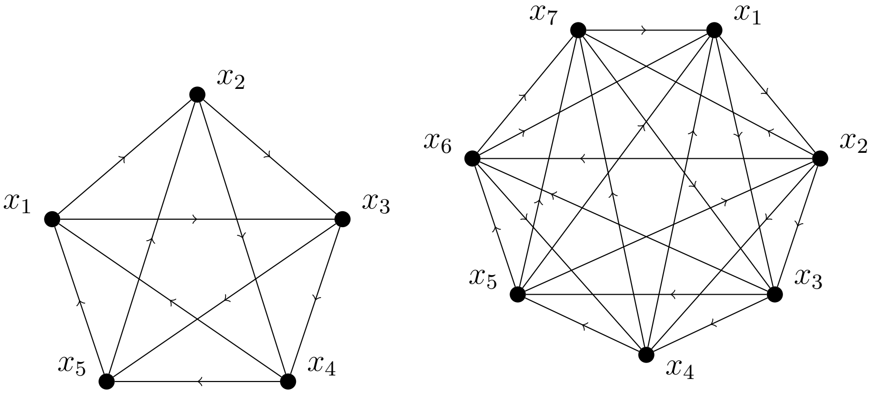


Figure 4: All-kings n-tournaments T with $n \in \{5, 7\}$ and $\delta^+(T) = 2$

Proof. We have

$$\begin{aligned} \left|\{v \in T : k^+(v) \geq \tfrac{n}{2}\}\right| &= n - |\{v \in T; k^+(v) \leq \lceil \tfrac{n}{2} - 1 \rceil\}| \\ &\geq n - \lceil \tfrac{n}{2} - 3 \rceil \\ &\geq \lfloor \tfrac{n}{2} + 3 \rfloor . \end{aligned}$$

□

In the rest of this section, we establish some lemmas that are practical in the treatment of Problem 1.

Lemma 4.5 *Let T_1 and T_2 be two tournaments and let x be a vertex of T_1. Let T be a tournament such that $V(T) = V(T_1) \cup V(T_2)$ and $E(T) = E(T_1) \cup E(T_2) \cup \{(v, y) : v \in T_2, y \in N^+[x]\} \cup \{(y, v) : y \in N^-(x), v \in T_2\}$. Then $k_T^+(x) = k_{T_1}^+(x)$, $k_T^+(v) = k_{T_2}^+(v) + k_{T_1}^+(x)$ for all $v \in T_2$, and $k_T^+(u) = k_{T_1}^+(u) + |T_2|$ for all $u \in T_1 - x$ such that $k_{T_1}^+(u) \geq k_{T_1}^+(x)$.*

Proof. It is clear that $N_T^+(x) = N_{T_1}^+(x)$ and $N_T^{++}(x) = N_{T_1}^{++}(x)$, so $k_T^+(x) = k_{T_1}^+(x)$. Let $v \in T_2$; we have $N_T^+(v) = N_{T_2}^+(v) \cup N_{T_1}^+(x) \cup \{x\}$ and $N_T^{++}(v) = N_{T_2}^{++}(v) \cup N_{T_1}^{++}(x)$, so $k_T^+(v) = k_{T_2}^+(v) + k_{T_1}^+(x)$. Let $u \in T_1 - x$; then obviously $K_T^+(u) \cap T_1 = K_{T_1}^+(u)$. Moreover, if $k_{T_1}^+(u) \geq k_{T_1}^+(x)$, by Lemma 3.2, $d_{T_1}^+(u, x) \leq 2$, i.e. there exists $y \in N^+[u]$ which dominates x. By the definition of T, y dominates every vertex in T_2, so then $d_T^+(u, v) \leq 2$ for all $v \in V(T_2)$. Thus $k_T^+(u) = k_{T_1}^+(u) + |T_2|$. □

From Figure 4 we can notice the following remark.

Remark 4.6 *There exists an all-kings tournament T of order $n \in \{5, 7\}$ with $\delta^+(T) = 2$.*

It can be simply observed for a transitive tournament T on n vertices that $\{k^+(v) : v \in T\} = \{n, n-1, \ldots, 1\}$. This idea will be used in proving the following lemma.

Lemma 4.7 *There exists a tournament T such that $\{k^+(v) : v \in T\} = \{n, n-1, \ldots, 5, 4\}$ containing two vertices x and y such that $|N^+[x] \cup N^+[y]| = 3$.*

Proof. Let T' be a transitive tournament of order $n-5$ and define the tournament T such that $V(T) = V(T') \cup \{x_1, x_2, x_3, x_4, x_5\}$ where T' is a subtournament of T dominating all other vertices, $N^+(x_1) = \{x_2, x_3\}$, $N^+(x_2) = \{x_3\}$, $N^+(x_3) = \{x_4, x_5\}$, and $N^+(x_4) = \{x_1, x_2, x_5\}$. Hence $\{k_T^+(v) : v \in T'\} = \{n, n-1, \ldots, 6\}$, $k_T^+(x_i) = 5$ for all $i \in \{1, 3, 4\}$ and $k_T^+(x_i) = 4$ for all $i \in \{2, 5\}$. We have $N^+[x_1] \cup N^+[x_2] = \{x_1, x_2, x_3\}$. $\square$

Lemma 4.8 *Let $t \geq 3$ be an integer and $m_1 > m_2 > \cdots > m_t > 2$ be t integers with $m_{t-1} - m_t = 2$ and $m_t \notin \{4, 6\}$. If there exists a tournament T_1 such that $\{k^+(v) : v \in T_1\} = \{m_1 - m_t + 2, m_2 - m_t + 2, \ldots, m_{t-1} - m_t + 2\}$, then there exists a tournament T such that $\{k^+(v) : v \in T\} = \{m_1, \ldots, m_t\}$.*

Proof. There exists a vertex $x \in T_1$ such that $d^+(x) = 1$. Indeed, let $v \in T_1$ such that $k^+(v) = m_{t-1} - m_t + 2 = 4$. If $d^+(v) \geq 2$ then v has an out-neighbor of out-degree 1. Let T_2 be an all-kings tournament of order $m_t - 2$ and define a tournament T such that $V(T) = V(T_1) \cup V(T_2)$ and $E(T) = E(T_1) \cup E(T_2) \cup \{(w, u) : w \in T_1 - x, u \in T_2\} \cup \{(u, x) : u \in T_2\}$. We have $k_T^+(u) = m_t$ for all $u \in T_2$ and $k_T^+(w) = k_{T_1}^+(w) + m_t - 2$ for all $w \in T_1$. $\square$

Lemma 4.9 *Let $t \geq 3$ be an integer and $m_1 > m_2 > \cdots > m_{t-2} > 8$ be $t-2$ integers. If there exists a tournament T_1 such that $\{k^+(v) : v \in T_1\} = \{m_1 - 4, m_2 - 4, \ldots, m_{t-2} - 4, 4\}$, then there exists a tournament T such that $\{k^+(v) : v \in T\} = \{m_1, \ldots, m_{t-2}, 8, 4\}$.*

Proof. Consider the tournament T defined by $V(T) = V(T_1) \cup \{x_1, x_2, x_3, x_4\}$ and

$$\begin{aligned} E(T) &= E(T_1) \cup \{(v, x_1), (x_2, v), (v, x_3), (v, x_4) : v \in T_1\} \\ &\quad \cup \{(x_1, x_2), (x_1, x_3), (x_2, x_3), (x_3, x_4), (x_4, x_1), (x_4, x_2)\}. \end{aligned}$$

For any vertex $u \in T_1$, we have $k_T^+(u) = k_{T_1}^+(u) + 4$. On the other hand, x_1, x_2, and x_4 are kings in T, i.e. $k_T^+(x_i) = m_t$ for all $i \in \{1, 2, 4\}$. Finally, $k_T^+(x_3) = 4$. $\square$

4.2 The set of king degrees

The theorem below gives a characterization of the set of two integers that can be the set of the king degrees of the vertices of a reducible tournament.

Theorem 4.10 *Let $n, m \in \mathbb{N}$ be two integers such that $n > m$. There exists a reducible tournament T of order n such that $\{k^+(v) : v \in T\} = \{n, m\}$ if and only if $m \notin \{n-2, n-4, 2, 4\}$.*

Proof. Let T be a reducible tournament such that $\{k^+(v) : v \in T\} = \{n, m\}$. As T is reducible, $T = T_1 \to T_2$, so for all $x \in T_1$, $y \in T_2$, we have $k^+(x) > k^+(y)$. Thus $k^+(x) = n$ and $k^+(y) = m$, and so $K(T) = T_1$ where T_1 and T_2 are all-kings tournaments. Consequently, $n - m, m \notin \{2, 4\}$ and therefore $m \notin \{n-2, n-4, 2, 4\}$.

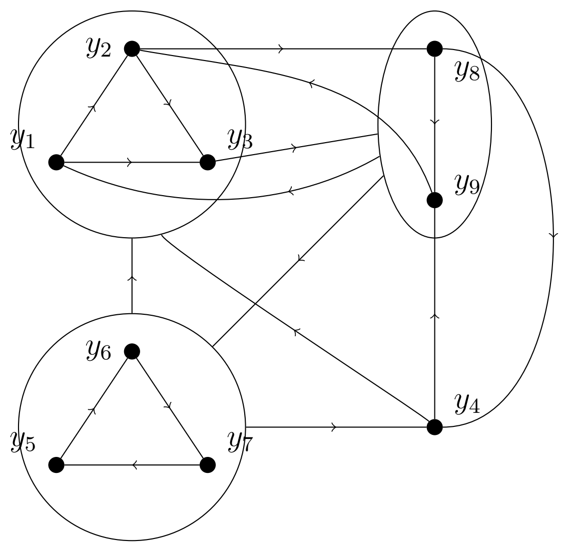


Figure 5: Tournament T with $\{k^+(v); v \in T\} = \{9, 5\}$

Suppose that $m \notin \{n-2, n-4, 2, 4\}$, so $n-m, m \notin \{2, 4\}$. Let T_1 and T_2 be two all-kings tournaments of orders $n-m$ and m respectively. Then the tournament $T = T_1 \to T_2$ is the desired tournament. $\square$

The following theorem imposes a necessary condition on two integers n and m, for the existence of strong tournaments whose set of king degrees is $\{n, m\}$.

Theorem 4.11 *Let $n, m \in \mathbb{N}$ be two integers such that $\frac{n+1}{2} < m < n$. Then there exists a strong tournament T of order n such that $\{k^+(v) : v \in T\} = \{n, m\}$.*

Proof. First we will prove the statement for $m \notin \{\frac{n+3}{2}, \frac{n+5}{2}\}$. Let T_1 be a regular $(2(n-m)+1)$-tournament and T_2 be an all-kings $(2m-n-1)$-tournament. Let $x_0 \in T_1$ and define the tournament T such that $V(T) = V(T_1) \cup V(T_2)$ and $E(T) = E(T_1) \cup E(T_2) \cup \{(x, y) : x \in T_1 - x_0, y \in T_2\} \cup \{(y, x_0) : y \in T_2\}$. It is clear that $K(T) = T_1$ and for all $y \in T_2$, $K^+(y) = T - N^-_{T_1}(x_0)$, which implies that $k^+(y) = m$, and thus $\{k^+(v) : v \in T\} = \{n, m\}$.

Now we will proceed for $m \in \{\frac{n+3}{2}, \frac{n+5}{2}\}$. Let T_1 be a regular $(2(n-m)+1)$-tournament and let T_2 be a $(2m-n-1)$-tournament whose vertices are all kings except for a vertex x of outdegree 0. Let (a, b) be an arc in T_1 and define $T = (V(T), E(T))$ such that $V(T) = V(T_1) \cup V(T_2)$ and

$$\begin{aligned} E(T) &= E(T_1) \cup E(T_2) \cup \{(y, a) : y \in K(T_2)\} \\ &\quad \cup \{(v, y) : v \in T_1 - a, y \in K(T_2)\} \cup \{(v, x) : v \in T_1 - b\} \cup \{(x, b)\}. \end{aligned}$$

We have $K(T) = T_1$ and, for all $v \in K(T_2)$, $K^+(v) = T - N^-_{T_1}(a)$ and $K^+(x) = T - N^-_{T_1}(b)$, which implies that $k^+(v) = m$ for all $v \in T_2$. The result follows. $\square$

When seeking the sets of integers that qualify as exceptions, one can begin by imagining tournaments structured as "all-kings" with specific orders $m_i - m_{i+1}$ and

m_t, representing the connected components of the whole tournament. The king degrees in these tournaments align with the set of m_i's, and the count of vertices with king degree at most m_i is precisely m_i. The core challenge arises when $m_i - m_{i+1}$ is 2 or 4 for some i, or when m_t takes on these values. A useful intuition here is to identify the last integer m_i causing the problem. We will suppose first that this m_i is m_1. This means $m_i - m_{i+1} \in \{2, 4\}$ if and only if $i = 1$. The problem boils down to the "capability" of altering the king degree of a vertex with king degree m_j, $j \geq 2$, without losing m_j from the set of king degrees. If such an alteration is feasible, it allows us to adjust the king degree of the vertex having a king degree between m_1 and m_2 to m_1, rectifying the issue. However, this adjustment is impossible when $m_j - m_{j+1} = 1$ for all $j > 1$, culminating in $m_t = 1$. This configuration leads to the first expected exceptions: sets containing integers from 1 to $t-1$, followed by $t+1$ or $t+3$. These exceptions will be part of $A_1(t)$ and $A_2(t)$.

Furthermore, as $m_j - m_{j+1} \notin \{2, 4\}$ for all $j > 1$, at least one of the subsequent differences between the m_j's must be at least 3. When exactly one such difference is 3, the structure of the tournament induced by the vertices having king degree no more than m_2 is akin to a transitive tournament, except that one vertex is replaced by a directed triangle. This structural limitation means we cannot adjust the king degree of a vertex while preserving representation for all integers. Hence, additional exceptions arise: sets containing integers from 1 to s, and then jumping to $s+3$ and up to $t+1$, and finally $t+3$ or $t+5$. Alternatively, sets may exclude the initial segment (1 to s) entirely, starting instead from 3 to $t+1$, followed by $t+3$ or $t+5$. These exceptions represent $A_3(t)$ and $A_4(t)$. Intriguingly, while deriving these anticipated exceptions and proving the result, additional exceptions emerge, cascading naturally from the expected ones. These intuitive sets form the foundation of our inductive proof on t.

To prove the main result, we are going to classify formally the sets of t positive integers $\{m_1, m_2, \ldots, m_t\}$ with $m_1 > m_2 > \cdots > m_t$ for which we cannot find a tournament T with $\{k^+(v) : v \in T\} = \{m_1, m_2, \ldots, m_t\}$ into the sets $A_i(t)$ as follows:

- $A_1(1) = \{\{2\}\}$, $A_2(1) = \{\{4\}\}$, $A_3(1) = \emptyset$, $A_4(1) = \emptyset$.

For $t \geq 2$,

- $A_1(t) = \{\{m_1, m_2, \ldots, m_s, m_s{-}2, m_s{-}3, \ldots, 1\} : 1 \leq s \leq t{-}1\} \cup \{\{m_1, m_2, \ldots, m_{t-1}, 2\}\}$. That is to say, the first part of $A_1(t)$ is a decreasing sequence of s integers $m_1, m_2, \ldots, m_s$ followed by $m_s - 2, m_s - 3, \ldots, 1$. This of course implies that $m_s = t - s + 2$.

- $A_2(t) = \{\{t{+}3, t{-}1, t{-}2, \ldots, 1\}\}$. That is, a sequence of t consecutive integers with exactly one jump of 4 at the beginning.

- $A_3(t) = \{\{t+3, t+1, t, \ldots, s, s-3, s-4, \ldots, 1\} : 4 \leq s \leq t+1\}$
  $\cup\, \{\{t+3, t+1, t, \ldots, 3\}\}$.

That is, a sequence of t consecutive integers with a jump of 2 at the beginning and another jump of 3 in the middle or at the end (i.e., ending with 3).

- $A_4(t) = \{\{t+5, t+1, t, \ldots, s, s-3, s-4, \ldots, 1\} : 4 \le s \le t+1\}$
$\cup \{\{t+5, t+1, t, \ldots, 3\}\}$.
That is, a sequence of t consecutive integers with a jump of 4 at the beginning and another jump of 3 in the middle or at the end (i.e., ending with 3).

It is clear that $A_1(2) = \{\{n, 2\}, \{3, 1\}\}$, $A_2(2) = \{\{5, 1\}\}$, $A_3(2) = \{\{5, 3\}\}$ and $A_4(2) = \{\{7, 3\}\}$. The following theorem proves the result for $t = 2$.

Theorem 4.12 *Let $m, n \in \mathbb{N}$ be two integers such that $n > m$. There exists a tournament T of order n such that $\{k^+(v); v \in T\} = \{n, m\}$ if and only if $\{n, m\} \notin \bigcup_{i=1}^{4} A_i(2)$.*

Proof. By Theorems 4.10 and 4.11, if $(n, m) \notin \{(n, 2), (n, 4), (3, 1), (5, 1), (5, 3), (7, 3), (9, 5)\}$ then there exists an n-tournament such that $\{k^+(v); v \in T\} = \{n, m\}$. It is clear that there are no tournaments on 3 or 5 vertices such that $\{k^+(v); v \in T\} = \{|V(T)|, 1\}$. Suppose that there exists a tournament T such that $\{k^+(v); v \in T\} = \{n, 2\}$. Let $v \in T$ such that $k^+(v) = 2$; then $d^+(v) = 1$. Set $N^+(v) = \{u\}$. Then $d^+_{N^-(v)}(u) = 0$ and so $k^+(u) = 1$, a contradiction.

Suppose that there exists a tournament such that $\{k^+(v) : v \in T\} \in \{\{5, 3\}, \{7, 3\}\}$. By Theorem 4.10, T is strong. Let $x \in T$ such that $k^+(x) = 3$. Then $d^+(x) = 1$. Set $N^+(x) = \{y\}$ and $N^{++}(x) = \{z\}$. We have $N^-(y) = V(T) \setminus \{x, y, z\}$. Since T is strong, it follows that $d^+(z) \ge 2$, and so $k^+(y) \ge 4$. Then $k^+(y) = n$ and $N^+(z) = V(T) \setminus \{y, z\}$. But $k^+(v) \ge 4$ for all $v \in T \setminus \{x, y, z\}$, and so $k^+(v) = n$ for all $v \in T \setminus \{y, z\}$; thus $T \setminus \{x, y, z\}$ is an all-kings tournament, a contradiction. For $m = 4$, if $n \notin \{6, 8\}$, let X be an all-kings $(n-4)$-tournament and define an n-tournament T such that $V(T) = V(X) \cup \{x, y, z, t\}$ and $E(T) = E(X) \cup \{(x, y), (y, z), (y, t), (z, t), (z, x), (t, x)\} \cup \{(v, x), (v, y), (v, z), (t, v), v \in X\}$. We have $k^+(x) = 4$ and $k^+(u) = n$ for all $u \in T - x$. Otherwise, $n \in \{6, 8\}$, let X be a tournament on $n-4$ vertices such that $k(X) = n-5$. Let w be the non-king vertex of X and define the tournament T such that $V(T) = V(X) \cup \{x, y, z, t\}$ and $E(T) = E(X) \cup \{(x, y), (y, z), (y, t), (z, t), (z, x), (t, x), (w, z)\} \cup \{(v, x), (v, y), (t, v), v \in X\} \cup \{(z, v) : v \in K(X)\}$. We have $k^+(x) = 4$ and $k^+(u) = n$ for all $u \in T - x$, and so T is the desired tournament. The tournament T in Figure 5 verifies that $\{k^+(v) : v \in T\} = \{9, 5\}$. In fact, $k^+(y_1) = 5$ and $k^+(y_i) = 9$ for all $i \in \{2, \ldots, 9\}$. □

Finally, we settle completely the question asked by El Sahili in Problem 1:

Theorem 4.13 *Let $t \ge 2$ be an integer and let $m_1 > m_2 > \cdots > m_t$ be t positive integers. There exists a tournament T such that $\{k^+(v) : v \in T\} = \{m_1, m_2, \ldots, m_t\}$ if and only if $\{m_1, m_2, \ldots, m_t\} \notin \bigcup_{i=1}^{4} A_i(t)$.*

Here is an overview of the proof. We use induction on t. The necessary condition is done by contradiction. That is, supposing for a set of integers belonging to some

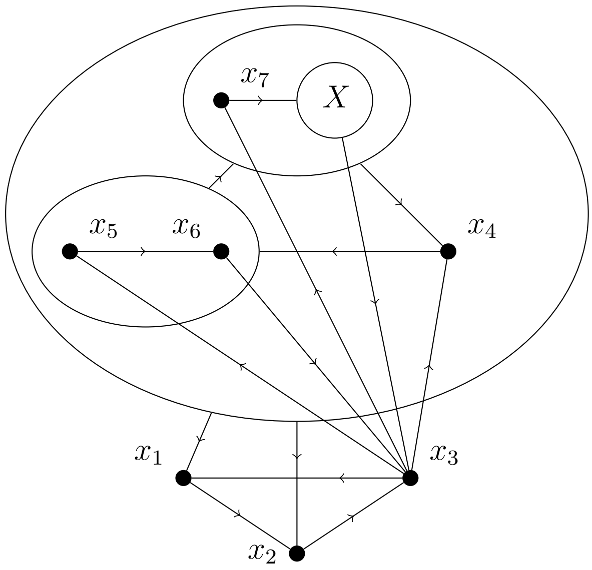


Figure 6: Tournament W with X an all-kings $(n-7)$-tournament

$A_i(t)$, there is a representative tournament; we remove some vertices from the tournament to obtain a new tournament having a set of king degrees belonging to some $A_i(t-1)$, which is a contradiction.

For the sufficient condition, we assume that the set $\{m_1, \ldots, m_t\}$ is not an exception. If $m_t = 1$, the existence of a tournament with the set of king degrees $\{m_1 - 1, m_2 - 1, \ldots, m_{t-1} - 1\}$ is equivalent to the existence of a tournament with the set of king degrees $\{m_1, \ldots, m_t\}$. Thus, we restrict our attention to the case where $m_t \geq 3$.

The proof is divided into two main cases based on whether there exists some i such that a tournament T_1 with king degrees $\{m_1 - m_{i+1}, \ldots, m_i - m_{i+1}\}$ exists.

- **Case 1:** If such an i exists, we analyze the existence of another tournament T_2 with king degrees $\{m_{i+1}, \ldots, m_t\}$. If T_2 exists, we combine T_1 and T_2 using $T_1 \to T_2$ to construct the desired tournament. Otherwise, we use induction to construct a tournament having a set of $t-1$ king degrees, most of which are $m_i - k$, for some constant k. Then by adding a "small" tournament to the first one, we increase the $m_i - k$ to m_i and we obtain some vertices of king degrees, the missing m_i's.

- **Case 2:** If no such i exists, the structure of the integers provides valuable information: a large subset of these integers forms a consecutive decreasing sequence, which can be represented by a transitive tournament. That is, if the integers from m_i to m_{i+k} are consecutive, then the set $\{m_i - m_{i+k} + 1, \ldots, m_{i+k} - m_{i+k} + 1\}$ is nothing but $\{k+1, k, \ldots, 1\}$, and then a transitive tournament of order $k+1$ whose vertices can reach other $m_{i+k} - 1$ vertices by a directed path of length no more than 2 guarantees the existence of the integers from m_i to m_{i+k} in the set of king degrees of the whole tournament. The remaining integers correspond to tournaments with specific vertex properties. Using this structure, we construct the entire tournament.

Below is the proof of Theorem 4.13.

Proof. We demonstrate the result by induction on t. By Theorem 4.12, the result holds for $t = 2$. Indeed, $A_1(2) = \{\{m_1, 2\}, \{3, 1\}\}$, $A_2(2) = \{\{5, 1\}\}$, $A_3(2) = \{\{5, 3\}\}$ and $A_4(2) = \{\{7, 3\}\}$. Suppose that it is true until $t - 1$. For the necessary condition, to show a contradiction, suppose that there exists a tournament T such that $\{k^+(v); v \in T\} = \{m_1, m_2, \dots, m_t\}$ and $\{m_1, m_2, \cdots_t\} \in \bigcup_{i=1}^4 A_i(t)$. If $m_t = 2$, let x be a vertex of king degree 2 and y be its out-neighbor; then $k^+(y) = 1$, a contradiction. Otherwise, $m_t \in \{1, 3\}$. If $|\{v \in T : k^+(v) = m_t\}| = m_t$, by Lemma 3.1, $x \to y$ for all $x, y \in T$ such that $k^+(x) > m_t$ and $k^+(y) = m_t$. Set $T' = T \setminus \{v \in T : k^+(v) = m_t\}$; then we have $\{k^+_{T'}(v) : v \in T'\} = \{m_1 - m_t, \dots, m_{t-1} - m_t\} \in \bigcup_{i=1}^4 A_i(t-1)$, a contradiction. Otherwise $|\{v \in T; k^+(v) = m_t\}| < m_t$, and then $m_t = 3$. By Remark 4.3, $|\{v \in T : k^+(v) = 3\}| = 1$. Let x be the vertex of king degree 3, let y be its unique out-neighbor and z its second out-neighbor. Note that z is the unique out-neighbor of y and z has an out-neighbor distinct from x. Thus $k^+_{T-x}(v) = k^+_T(v) - 1$ for all $v \in T - x$, so $\{k^+_{T-x}(v) : v \in T - x\} \in A_3(t-1) \cup A_4(t-1)$, a contradiction.

Suppose now that $\{m_1, m_2, \dots, m_t\} \notin \bigcup_{i=1}^4 A_i(t)$. We are going to prove that there exists a tournament T such that $\{k^+(v) : v \in T\} = \{m_1, m_2, \dots, m_t\}$. If $m_t = 1$ then $\{m_1 - 1, m_2 - 1, \dots, m_{t-1} - 1\} \notin \bigcup_{i=1}^4 A_i(t-1)$ since otherwise $\{m_1, m_2, \dots, m_t\} \in \bigcup_{i=1}^4 A_i(t)$, a contradiction. Hence, by induction, there exists a tournament T' such that $\{k^+(v) : v \in T'\} = \{m_1 - 1, m_2 - 1, \dots, m_{t-1} - 1\}$. The tournament $T = T' \to x$ gives the result, so we may assume that $m_t \geq 3$.

Case 1: If there exists $i_0 \in \{1, \dots, t-1\}$ such that $\{m_1 - m_{i_0+1}, \dots, m_{i_0} - m_{i_0+1}\} \notin \bigcup_{i=1}^4 A_i(i_0)$, suppose that i_0 is chosen minimum. If there exists a tournament T_2 such that $\{k^+(v) : v \in T_2\} = \{m_{i_0+1}, \dots, m_t\}$, let T_1 be a tournament such that $\{k^+(v) : v \in T_1\} = \{m_1 - m_{i_0+1}, \dots, m_{i_0} - m_{i_0+1}\}$; the tournament $T = T_1 \to T_2$ gives the result. Otherwise, such a tournament T_2 does not exist, by induction, $\{m_{i_0+1}, \dots, m_t\} \in \bigcup_{i=1}^4 A_i(t - i_0)$, i.e. $\{m_{i_0+1}, \dots, m_t\} \in \{\{t - i_0 + 3, t - i_0 + 1, t - i_0, \dots, 3\}, \{t - i_0 + 5, t - i_0 + 1, t - i_0, \dots, 3\}, \{4\}\}$.

Subcase 1.1: If $m_{t-1} - m_t = 2$ i.e. $\{m_{i_0+1}, \dots, m_t\} = \{5, 3\}$, then $\{m_1 - m_t + 2, \dots, m_{t-1} - m_t + 2\} = \{m_1 - 1, m_2 - 1, \dots, m_{t-2} - 1, 4\} \notin \bigcup_{i=1}^4 A_i(t-1)$ and thus Lemma 4.8 gives the result.

Subcase 1.2: Say $m_{t-1} - m_t = 4$. If $m_t = 3$, let T_1 be a tournament such that $\{k^+(v) : v \in T_1\} = \{m_1 - 3, m_2 - 3, \dots, m_{t-1} - 3\}$ and let T_2 be a circuit triangle. The tournament $T = T_1 \to T_2$ gives the result. Otherwise, $m_t = 4$ and Lemma 4.9 gives the result.

Subcase 1.3: If $m_{t-1} - m_t \in \mathbb{N} \setminus \{2, 4\}$ then there exists a tournament T_1 such that $\{k^+(v) : v \in T_1\} = \{m_1 - m_{t-1} + m_t, m_2 - m_{t-1} + m_t, \dots, m_{t-2} - m_{t-1} + m_t, m_t\}$. Indeed, if $m_t = 4$ then $\{m_1 - m_{t-1} + m_t, m_2 - m_{t-1} + m_t, \dots, m_{t-2} - m_{t-1} + m_t, m_t\} \notin \bigcup_{i=1}^4 A_i(t-1)$ and if $m_t = 3$ then $m_{t-1} - m_t = 1$ and thus:

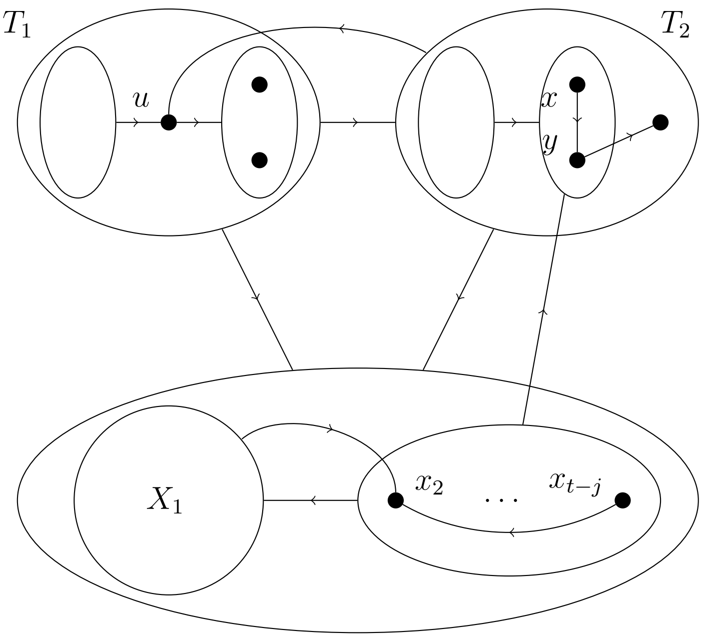
$\{m_1 - m_{t-1} + m_t, m_2 - m_{t-1} + m_t, \ldots, m_{t-2} - m_{t-1} + m_t, m_t\} \in \bigcup_{i=1}^4 A_i(t-1)$ implies that $\{m_1, \ldots, m_t\} \in \bigcup_{i=1}^4 A_i(t)$. Let T_2 be an all-kings tournament of order $m_{t-1} - m_t$, let x be a vertex of T_1 such that $k^+_{T_1}(x) = m_t$, and define a tournament T such that $V(T) = V(T_1) \cup V(T_2)$ and $E(T) = E(T_1) \cup E(T_2) \cup \{(v, y) : v \in T_2, y \in N^+[x]\} \cup \{(y, v) : y \in N^-(x), v \in T_2\}$; by Lemma 4.5, the result follows.

Case 2: Assume that $\{m_1 - m_{i_0+1}, \ldots, m_{i_0} - m_{i_0+1}\} \in \bigcup_{i=1}^4 A_i(i_0)$ for all $i_0 \in \{1, \ldots, t-1\}$.

Subcase 2.1: Let $m_{t-1} - m_t \in \{1, 3\}$. If there exists a tournament T_1 such that $\{k^+(v) : v \in T_1\} = \{m_1 - m_{t-1} + m_t, \ldots, m_{t-2} - m_{t-1} + m_t, m_t\}$, let y be a vertex in T_1 such that $k^+(y) = m_t$. Let T_2 be an all-kings $(m_{t-1} - m_t)$-tournament and define the tournament T such that $V(T) = V(T_1) \cup V(T_2)$ and $E(T) = E(T_1) \cup E(T_2) \cup \{(x, v) : x \in T_2, v \in N^+[y]\} \cup \{(v, x) : v \in N^-(y), x \in T_2\}$. By Lemma 4.5, T gives the result. Otherwise, such a tournament T_1 does not exist, by induction, $m_t = m_{t-1} - m_t = 3$, i.e. $\{m_1, m_2, \ldots, m_t\} = \{m_1, t+3, t+2, \ldots, 7, 6, 3\}$ where $m_1 \in \{t+5, t+7\}$. Indeed, there exists i such that $\{m_1 - m_{t-1} + m_t, \ldots, m_{t-2} - m_{t-1} + m_t, m_t\} \in A_i(t-1)$, and then $m_t = 3$ and $i \in \{3, 4\}$, so if $m_{t-1} - m_t = 1$ then $\{m_1, m_2, \ldots, m_t\} \in A_i(t)$, a contradiction. If $t = 3$, the tournament W in Figure 6 gives the result. Otherwise, let T_1 be a tournament such that $\{k^+(v) : v \in T_1\} = \{m_1 - 1, t+2, t+1, \ldots, 7, 6, 3\}$. Since $\{m_1 - 4, t-1, t-2, \ldots, 3\} \in A_3(t-2) \cup A_4(t-2)$, by Lemma 4.1 and Remark 4.3, $|\{v \in T_1 : k^+(v) = 3\}| = 1$. Set $\{v \in T_1 : k^+(v) = 3\} = \{x\}$, $N^+(x) = \{y\}$ and $N^+(y) = \{z\}$. Let $w \in T_1$ such that $k^+(w) = 6$ and suppose that it is chosen such that $w = y$ if $k^+(y) = 6$. Set $X = \{x, y, w\} \cup N^+(w)$; we have $X = N^+[x] \cup N^+[w]$. Let v be a vertex and define the tournament T such that $V(T) = V(T_1) \cup \{v\}$ and $E(T) = E(T_1) \cup \{(v, u) : u \in X\} \cup \{(u, v) : u \in T_1 - X\}$. By definition of T, we have $k^+_T(u) = k^+_{T_1}(u) + 1$ for all $u \in T_1 - X$, $k^+_T(v) = 7$, $k^+_T(w) = 6$ and $k^+_T(x) = 3$. Moreover, by Lemma 3.2, each vertex $u \in X - \{x, w\}$ reaches w by a directed path of length 2; then u has at least one out-neighbor in $T_1 - X$ and so $k^+_T(u) = k^+_{T_1}(u) + 1$. Thus T gives the result.

Subcase 2.2: If $m_{t-1} - m_t = 2$. If $m_t \notin \{4, 6\}$, by induction, there exists a tournament T_1 such that $\{k^+(v) : v \in T_1\} = \{m_1 - m_t + 2, \ldots, m_{t-1} - m_t + 2\}$. Thus Lemma 4.8 gives the result. Otherwise, $m_t \in \{4, 6\}$, so let T_1 be an all-kings tournament of order $m_1 - m_2 + 3$ such that there exists $u \in T_1$ with $d^+(u) = 2$ (this tournament exists by Remark 4.6, in fact, as we are in case 2, $m_1 - m_2 \in \{2, 4\}$). Let $j = \max\{i : m_i - m_{i+1} \in \{2, 3\}, 1 \leq i \leq t-2\}$. We have $m_j - m_{j+1} + 1 \in \{3, 4\}$. We distinguish two cases:

Subcase 2.2.1: Suppose there exists a tournament T_2 such that $\{k^+(v); v \in T_2\} = \{m_2 - m_{j+1} + 1, m_3 - m_{j+1} + 1, \ldots, m_j - m_{j+1} + 1\}$. There exist two vertices x and y in T_2 such that $|N^+[x] \cup N^+[y]| = 3$. Indeed, if $m_j - m_{j+1} + 1 = 3$, let $x \in T_2$ such that $k^+(x) = 3$; then x has a unique out-neighbor y, and we have $N^+[x] \cup N^+[y] = K^+(x)$. And if $m_j - m_{j+1} + 1 = 4$, since $\{m_1 - m_{j+1}, \ldots, m_j - m_{j+1}\} \in \bigcup_{i=1}^4 A_i(j)$, $\{m_2 - m_{j+1}, \ldots, m_j - m_{j+1}\} = \{j+1, \ldots, 4, 3\}$ and so $\{m_2 - m_{j+1} + 1, \ldots, m_j - m_{j+1} + 1\} = \{j+2, \ldots, 5, 4\}$; by Lemma 4.7, T_2 can be taken such that there exist two vertices x and y in T_2 with $|N^+[x] \cup N^+[y]| = 3$. Consider the tournament T



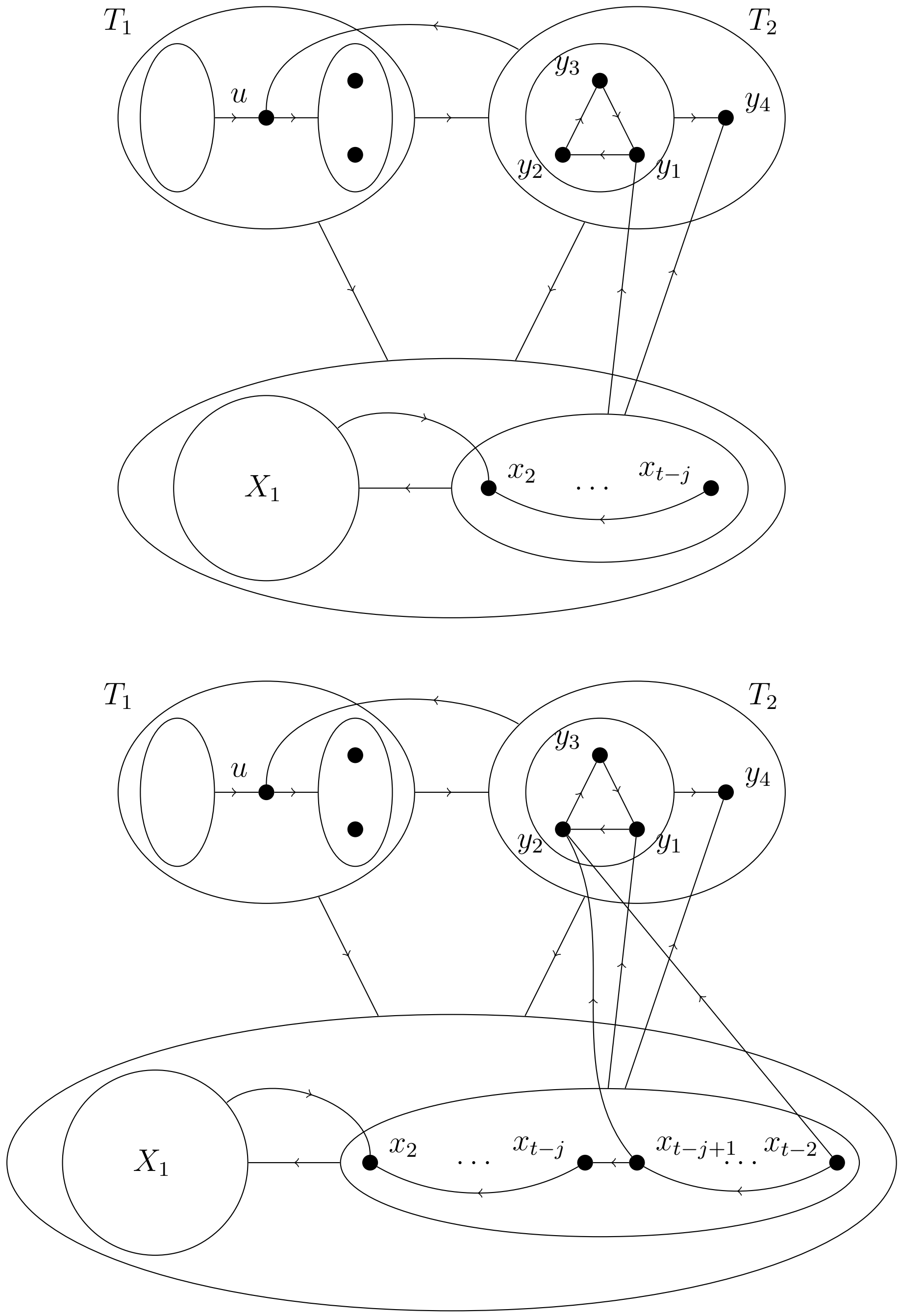
T_1: all-kings $(m_1 - m_2 + 3)$-tournament T_2: $(m_2 - m_{j+1} + 1)$-tournament
X_1: all-kings $(m_t - 3)$-tournament $T[x_2, \dots, x_{t-j}]$: transitive tournament

Figure 7: Tournament T defined in the Subcase 2.2.1

represented in Figure 7. We have $k_T^+(x_1) = m_t$ for all $x_1 \in X_1$, $k_T^+(x_i) = m_{t-i+1}$ for all $i \in \{2, \dots, t-j\}$, $k_T^+(v) = k_{T_2}^+(v) + m_{j+1} - 1$ for all $v \in T_2$ and $k_T^+(v) = m_1$ for all $v \in T_1$. The result follows.

Subcase 2.2.2: If there exists $l \in \{1, 2, 3, 4\}$ such that $\{m_2 - m_{j+1} + 1, m_3 - m_{j+1} + 1, \dots, m_j - m_{j+1} + 1\} \in A_l(j-1)$ then $A_l(j-1) = A_2(1) = \{\{4\}\}$ or $\{m_2 - m_{j+1} + 1, m_3 - m_{j+1} + 1, \dots, m_j - m_{j+1} + 1\} = \{j+2, j, j-1, \dots, 4, 3\}$. If $A_l(j-1) = \{\{4\}\}$, let T_2 be a tournament such that $V(T_2) = \{y_1, y_2, y_3, y_4\}$ and $E(T_2) = \{(y_1, y_2), (y_2, y_3), (y_3, y_1), (y_i, y_4) : i \in \{1, 2, 3\}\}$. Consider the first tournament T represented in Figure 8. We have $k_T^+(x_i) = m_{t-i+1}$ for all $i \in \{1, 2, \dots, t-j\}$, $k_T^+(v) = k_{T_2}^+(v) + m_{j+1} - 1 = m_2$ for all $v \in T_2 - y_4$, $k_T^+(y_4) = m_3$ and $k_T^+(v) = m_1$ for all $v \in T_1$. The result follows.

Otherwise, $\{m_1, \dots, m_t\} = \{m_1, m_t + t + 1, m_t + t - 1, m_t + t - 2, \dots, m_t + t - j + 2, m_t + t - j, m_t + t - j - 1, \dots, m_t + 2, m_t\}$ where $m_1 \in \{m_t + t + 3, m_t + t + 5\}$. Let T_2 be a tournament such that $V(T_2) = \{y_1, y_2, y_3, y_4\}$ and $E(T_2) = \{(y_1, y_2), (y_2, y_3), (y_3, y_1), (y_i, y_4) : i \in \{1, 2, 3\}\}$. Consider the second tournament T represented in Figure 8. We have $k_T^+(x_1) = m_t$ for all $x_1 \in X_1$, $k_T^+(x_i) = m_t + i = m_{t-i+1}$ for all $i \in \{2, \dots, t-j\}$, $k_T^+(x_i) = m_t + i + 1 = m_{t-i+1}$ for all $i \in \{t-j+1, \dots, t-2\}$, $k_T^+(v) = m_2$ for all $v \in T_2 - y_4$, $k_T^+(y_4) = m_4$ and $k_T^+(v) = m_1$ for all $v \in T_1$. The result follows. □



T_1: all-kings $(m_1 - m_2 + 3)$-tournament X_1: all-kings $(m_t - 3)$-tournament
$T[x_i\text{'s}]$: transitive tournament

Figure 8: The tournaments defined in Subcase 2.2.2: at the top is the first one defined, and below is the second one.

Acknowledgments

I would like to thank Professor Amine El Sahili and Dr. Maydoun Mortada for their valuable observations. I also wish to express my appreciation to the referees for their thoughtful comments.

References

[1] J. R. Duncan and M. S. Jacobson, On tournaments with Domination Number exactly k, *J. Comb. Math. Comb. Comput.* 31 (1999), 129–136.

[2] A. El Sahili, Seminars on Graph Theory, 2021.

[3] P. Erdős, On Schütte's problem, *Math. Gaz.* 47 (1963), 220–222.

[4] R. L. Graham and J. H. Spencer, A constructive solution to a tournament problem, *Canad. Math. Bull.* 14 (1) (1971), 45–48.

[5] F. Harary, R. Z. Norman and D. Cartwright, "Structural Models: An Introduction to the Theory of Directed graphs", John Wiley and Sons, Inc. New York, 1965, 294pp.

[6] H. G. Landau, On dominance relations and the structure of animal societies: III The condition for a score structure, *Bull. Math. Biophys.* 15 (1953), 143–148.

[7] X. Lu, D. Wang and C. K. Wong, Note On the bounded domination number of tournaments, *Discrete Math.* 220 (2000), 257–261.

[8] S. B. Maurer, The King Chicken Theorems, *Math. Mag.* 53 (1980), 67–80.

[9] K. B. Reid, Every vertex a king, *Discrete Math.* 38 (1982), 93–98.

[10] K. B. Reid, Tournament with prescribed numbers of kings and serfs, *Congr. Numer.* 29 (1980), 809–826.

[11] K. B. Reid, A. A. McRae, S. M. Hedetniemi and S. T. Hedetniemi, Domination and irredundance in tournaments, *Australas. J. Combin.* 29 (2004), 157–172.

[12] E. Szekeres and G. Szekeres, On a problem of Schütte and Erdős, *Math. Gaz.* 49 (1965), 290–293.

[13] Y. Yu, J. Di and M. Lin, Kings in Tournaments, *Math. Medley* 33 (1) (2006), 28–32.

[14] Y. Yu, Kings in Tournaments (2), *Math. Medley* 33 (2) (2006), 15–18.

(Received 21 Apr 2023; revised 18 May 2024, 21 Dec 2024)