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The degree-diameter problem for plane graphs with pentagonal faces

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Abstract

The degree-diameter problem consists of finding the maximum number of vertices n of a graph with diameter d and maximum degree Δ. This problem is well studied, and has been solved for plane graphs of low diameter in which every face is bounded by a 3-cycle (triangulations), and plane graphs in which every face is bounded by a 4-cycle (quadrangulations). In this paper, we solve the degree diameter problem for plane graphs of diameter 3 in which every face is bounded by a 5-cycle (pentagulations). We prove that if $\Delta \geq 8$, then $n \leq 3\Delta - 1$ for such graphs. This bound is sharp for Δ odd.

1 Introduction

The well-known **degree-diameter problem** asks for the maximum order $n(\Delta, d)$ of a graph with maximum degree Δ and diameter d. By considering a Δ-regular breadth-first tree, we easily obtain a trivial upper bound on $n(\Delta, d)$ known as the **Moore Bound**. The graphs attaining this bound for $\Delta > 2$ and $d > 1$ are called **Moore Graphs**, and there are only finitely many of them: the Petersen graph, the Hoffman-Singleton graph and, conjecturally, some 'missing' Moore graph(s) of diameter 2 and maximum degree 57 [1,4,11,15]. These Moore graphs are not planar, and the upper bounds attained on $n(\Delta, d)$ for planar graphs are substantially smaller than the Moore bound.

In [10], Hell and Seyffarth exactly solve the degree-diameter problem for planar graphs of diameter 2, showing that $n(\Delta, 2) = \frac{3}{2}\Delta + 1$ for such graphs. Further results for planar graphs are obtained in [9] by Fellows, Hell and Seyffarth. They give bounds on $n(\Delta, 3)$ and show that for each fixed diameter d, there exists some constant c such that $n(\Delta, d) \leq c\Delta^{\lfloor d/2 \rfloor}$. For planar graphs with even diameter and large maximum degree, the degree-diameter problem was solved exactly by Tishchenko in [19]. In [16],

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Nevo, Pineda-Villavicencio and Wood extend the result of [19] to all diameters. They also improve the state of the art in the degree-diameter for graphs embedded on surfaces by showing that for a graph with large Δ embedded in a surface of genus g, there is some constant c and a function $f(g)$ such that $n(\Delta, d) \leq cf(g)(\Delta - 1)^{\lfloor d/2 \rfloor}$.

Further refining the problem, we consider plane graphs in which every face is bounded by a circuit or cycle of the same length ρ. When $\rho = 3$, we obtain the well-studied maximal planar graphs / triangulations. Seyffarth proved in [17] that a triangulation of diameter 2 and $\Delta \geq 8$ has at most $\frac{3}{2}\Delta + 1$ vertices, and this bound is sharp. Interestingly, the bound is the same as the bound for general planar graphs obtained in [10], and this fact is critical to the proof in [10]. Plane graphs with $\rho = 4$ are maximal planar bipartite graphs, or quadrangulations. For quadrangulations, Dalfó, Huemer and Salas prove the sharp bounds $n(\Delta, 2) = \Delta + 2$, $n(\Delta, 3) = 3\Delta - 1$ when Δ is odd and $n(\Delta, 3) = 3\Delta - 2$ when Δ is even [3]. They also give approximate bounds on $n(\Delta, d)$ for quadrangulations with $d > 3$ and Δ large. In [8], the present author considered plane graphs in which ρ was (almost) as large as possible for fixed diameter d, obtaining the following sharp bounds: $n(\Delta, d) = 2d + 1$ when $\rho = 2d + 1$ and $n(\Delta, d) = \Delta(d - 1) + 2$ when $\rho = 2d$. The extremal graphs were also characterized.

The degree-diameter problem has been studied for graphs and triangulations on other surfaces, see [14, 18], as well as for highly structured graphs such as triangular and honeycomb networks [12, 13]. In recent work, the problem was tackled for outerplanar graphs [5], and a generalization of the degree-diameter problem is the subject of the 2022 paper [20]. For a comprehensive overview of the degree-diameter problem, see Miller and Širáň's survey [15]. For the early version of this work, and related research, see [7].

We call a plane graph in which every face is bounded by a cycle of length 5 a **pentagulation**. In this paper, we prove that $n(\Delta, 3) = 3\Delta - 1$ for pentagulations with $\Delta \geq 8$. The paper begins with definitions and basic lemmas in Section 2. In Section 3, we prove that a diameter 3 pentagulation is triangle-free. The structure of 4-cycles and separating 5-cycles is explored in Section 4. Section 5 introduces the notion of dislocated 4-cycles, a concept central to the proof of the main theorem. The proof that $n(\Delta, 3) \leq 3\Delta - 1$ for pentagulations is very involved, so we split it into three sections. Section 6 considers pentagulations with a pair of dislocated 4-cycles, Section 7 proves the bound for pentagulations with a 4-cycle, but no dislocated pair, and Section 8 proves that a diameter 3 pentagulation with $\Delta \geq 8$ contains at least one 4-cycle, and gives examples to show the bound is sharp for Δ odd. We conclude and give some further questions in Section 9.

2 Preliminaries

For most definitions used, see [6]. Let $G = (V, E)$ be a graph, and S, T two subsets of V. The distance between vertices u and v is denoted $\boldsymbol{d(u, v)}$, and we let $\boldsymbol{d(u, S)} = \min\{d(u, w) : w \in S\}$. For a subgraph H, we overload notation and denote $\boldsymbol{d(u, H)} = d(u, V(H))$. We say S **dominates** T if every vertex of T is adjacent to some vertex of S, and S dominates the whole graph G if S dominates V. Let $\boldsymbol{N_i(v)}$ be the set of vertices at distance i from v. A cycle C in a plane graph G partitions the plane into an **interior** bounded region denoted **Int** $\boldsymbol{(C)}$, an **exterior** unbounded region **Ext** $\boldsymbol{(C)}$, and the cycle C itself. Denote **Int** $[\boldsymbol{C}] = \mathrm{Int}(C) \cup C$, and **Ext** $[\boldsymbol{C}] = \mathrm{Ext}(C) \cup C$. If both $\mathrm{Int}(C)$ and $\mathrm{Ext}(C)$ contain vertices, then C is a **Jordan separating cycle**. Consider a subgraph H of a graph G. A **chord** of H in G is an edge uv such that $u, v \in V(G)$ and $uv \in E(G) - E(H)$. The **girth** of a graph is the length of its shortest cycle.

It is well known that a plane graph is 2-connected if and only if each face is bounded by a cycle, so all pentagulations are 2-connected.

Lemma 2.1. *Let G be a pentagulation of diameter 3, and C a cycle of G. If C is a Jordan separating cycle, then C dominates its interior, or dominates its exterior. Further, if C has length 3 or 4, then it is a Jordan separating cycle.*

Proof. Suppose that C is a Jordan separating cycle, and that $u \in \mathrm{Int}(C)$, $v \in \mathrm{Ext}(C)$ are two vertices not dominated by C. Any $u-v$ geodesic contains at least one vertex of C, so $d(u, v) \geq 4$, contradicting the diameter of G.

Suppose C has length 3 or 4. Its interior and exterior both contain at least one face. Since a facial cycle has five vertices, we have $|V(\mathrm{Int}[C])| \geq 5$ and $|V(\mathrm{Ext}[C])| \geq 5$. Thus $\mathrm{Int}(C)$ and $\mathrm{Ext}(C)$ both contain at least one vertex, so C is a Jordan separating cycle. □

Lemma 2.2. *Every cycle of length 6 or 7 in a pentagulation is a Jordan separating cycle.*

Proof. Let C be a cycle of length 6 or 7 in a pentagulation G. The cycle C does not bound any face of G, so its interior either contains a vertex, or has some chord e. Since the length of C is at most 7, $C \cup \{e\}$ induces a cycle of length 3 or 4. Applying Lemma 2.1, we see that $\mathrm{Int}(C)$ contains some vertex. Similarly, $\mathrm{Ext}(C)$ contains a vertex. □

For a cycle C of length 5 in a pentagulation G, there are three distinct possibilities:

1. The cycle C Jordan separates G,
2. C is a facial cycle that separates G, but necessarily does not Jordan separate G,
3. C is a facial cycle that does not separate G.

3 There are no 3-cycles

The following lemmas show that no 3-cycle in a pentagulation dominates its interior (or exterior). We phrase our lemmas in terms of cycle interiors, but the same results are easily seen to hold for exteriors.

Lemma 3.1. *Let G be a pentagulation. If C is a 3-cycle in G, then no single vertex of C dominates the interior of C.*

Proof. For the sake of contradiction, let $C : v_1, v_2, v_3$ be a 3-cycle, the interior of which is dominated by the single vertex v_1. Choose C to be minimal, so there is no 3-cycle C' such that v_1 dominates the interior of C', and for which $\text{Int}(C') \subset \text{Int}(C)$. By Lemma 2.1, the cycle C Jordan separates G, so there is some vertex $u \in \text{Int}(C)$. By assumption, u and v_1 are adjacent. As G is a pentagulation, and thus 2-connected, the vertex u has some neighbor other than v_1 in $\text{Int}[C]$. This neighbor is not v_2, as then v_1, v_2, u is a 3-cycle, contradicting the minimality of C. Similarly, u and v_3 are not adjacent. (see Figure 1).

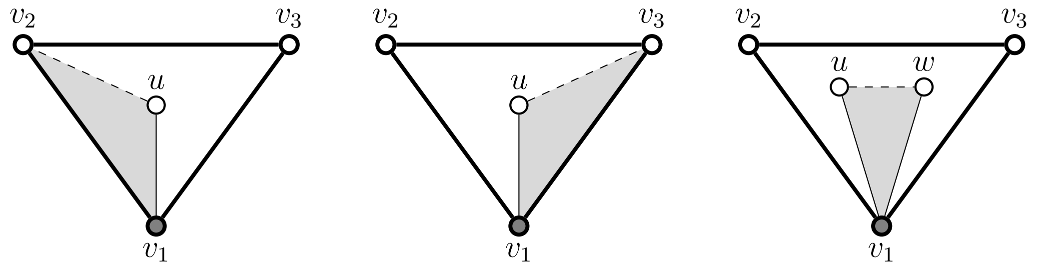


Figure 1: Some steps in the proof of Lemma 3.1.

Thus there is some other vertex w in $\text{Int}(C)$ that is adjacent to u. Since v_1 dominates $\text{Int}(C)$, the vertices v_1, u and w form a 3-cycle, contradicting the minimality of C. □

Lemma 3.2. *Let G be a pentagulation, and let C be a 3-cycle in G. The interior of C is not dominated by any two vertices of C.*

Proof. Let $C = v_1, v_2, v_3$ be a 3-cycle. Assume to the contrary and without loss of generality that every vertex in $\text{Int}(C)$ is dominated by $\{v_1, v_2\}$. We claim that no vertex in $\text{Int}(C)$ is adjacent to v_3. Assume to the contrary there is a vertex v adjacent to v_3. Without loss of generality, v is adjacent to v_1 as well, since $\{v_1, v_2\}$ dominates $\text{Int}(C)$. Thus the triangle v_1, v, v_3 is dominated by v_1, contradicting Lemma 3.1 and proving the claim.

The edge v_1v_2 lies on the boundary of two faces, one of which is in the interior of C. Call this interior face f, and note that the boundary of f is a 5-cycle. By Lemma 3.1, the interior of C is not dominated by a single vertex, so both v_1 and v_2 have some neighbor in $\text{Int}(C)$. Thus the cycle bounding f is of the form u, v_1, v_2, w, x, where u, w and x are vertices in the interior of C. As $\{v_1, v_2\}$ dominates $\text{Int}(C)$, the vertex x is adjacent to either v_1 or v_2. If x is adjacent to v_1, then u, x, v_1 is a triangle whose interior is dominated by v_1, and similarly if x is adjacent to v_2 then w, x, v_2 is a triangle whose interior is dominated by v_2. Both possibilities contradict Lemma 3.1, completing the proof. □

Lemma 3.3. *Let G be a pentagulation and C be a 4-cycle in G. Then no vertex of C dominates* $\text{Int}(C)$.

Proof. Let $C = v_1, v_2, v_3, v_4$ be a 4-cycle. Assume for the sake of contradiction that v_1 dominates $\text{Int}(C)$, and choose C to be minimal, i.e., there is no 4-cycle C' dominated by v_1 such that $\text{Int}(C') \subset \text{Int}(C)$. By Lemma 2.1, $\text{Int}(C) \neq \emptyset$. Let u be the neighbor of v_1 in the interior of C such that uv_1 and v_1v_2 both lie on the boundary of some common face. Since G is 2-connected, u is adjacent to some vertex w in $\text{Int}[C] - \{v_1\}$. Up to symmetry, there are three possible choices for the vertex w.

Case 1: $w = v_2$ or $w = v_4$.

If $w = v_2$, we obtain a 3-cycle v_1, u, v_2, the interior of which is dominated by v_1, contradicting Lemma 3.1. The situation is similar if u is adjacent to v_4.

Case 2: $w = v_3$.

The interior of the 4-cycle v_1, u, v_3, v_2 is dominated by v_1, contradicting minimality of C.

Case 3: w is a vertex in $\text{Int}(C)$.

By assumption, the vertex v_1 dominates $\text{Int}(C)$, so v_1 and w are adjacent. Thus v_1, u, w is a 3-cycle whose interior is dominated by v_1, contradicting Lemma 3.1. □

Lemma 3.4. *Let C be a 4-cycle in a pentagulation. No pair of vertices of C, that are adjacent in C, dominate* $\text{Int}(C)$.

Proof. Assume for the sake of contradiction that $C = v_1, v_2, v_3, v_4$ is a 4-cycle in a pentagulation whose interior is dominated by $\{v_1, v_2\}$. By Lemma 3.3, both v_1 and v_2 have at least one neighbor in $\text{Int}(C)$ — for if one of these two vertices had no neighbor in $\text{Int}(C)$, the other would dominate $\text{Int}(C)$. Thus there is a face f in the interior of C, bounded by a 5-cycle of the form u, v_1, v_2, w, x, where u and w are vertices in $\text{Int}(C)$ and x is a vertex in $\text{Int}[C]$. If x is either v_3 or v_4, then $\text{Int}[C]$ contains a triangle whose interior is dominated by v_1 or v_2 respectively, contradicting Lemma 3.1. If x lies in $\text{Int}(C)$, then it is adjacent to either v_1 or v_2. If x is adjacent to v_1, then v_1, u, x is a triangle whose interior is dominated by v_1, and if x is adjacent to v_2, then the interior of the triangle v_2, w, x is dominated by v_2. In any case, we obtain a triangle whose interior is dominated by a single vertex, contradicting Lemma 3.1. □

Lemma 3.5. *A 3-cycle in a pentagulation does not dominate its interior (or exterior).*

Proof. Let $C : v_1, v_2, v_3$ be a 3-cycle in a pentagulation G. Assume for the sake of contradiction that C dominates its interior. By Lemmas 3.1 and 3.2, no proper subset of $V(C)$ dominates $\text{Int}(C)$, so every vertex of C has at least one neighbor in $\text{Int}(C)$. Thus there exists a neighbor u of v_1 in $\text{Int}(C)$. Since G is 2-connected, the vertex u has some neighbor w in $\text{Int}[C] - \{v_1\}$. By Lemma 3.2, the vertex w is neither v_2 nor v_3, as this induces a 3-cycle whose interior is dominated by two vertices. By Lemma 3.1, w is not adjacent to v_1, as this creates a 3-cycle whose interior is dominated by v_1. By Lemma 3.4, neither v_2 nor v_3 is adjacent to w, since this induces a 4-cycle, the interior of which is dominated by two adjacent vertices. Thus u does not have a neighbor in $\text{Int}[C] - \{v_1\}$, a contradiction. □

Lemma 3.5 and Lemma 2.1 easily yield the following corollary, which we make extensive use of.

Corollary 3.6. *Pentagulations of diameter* 3 *contain no* 3*-cycles.*

4 The structure of separating cycles

We have shown that diameter 3 pentagulations do not contain 3-cycles (and, hence, that any 4-cycle or 5-cycle in a such a pentagulation is chordless). In this section, we describe the structure of 4-cycles and separating 5-cycles in diameter 3 pentagulations.

Lemma 4.1. *If a pentagulation contains a Jordan separating* 5*-cycle C, then the interior of C is dominated by neither a single vertex of C, nor by an adjacent pair of vertices in C.*

Proof. Let $C = v_1, v_2, v_3, v_4, v_5$ be a Jordan separating cycle of a pentagulation G. We first prove that $\text{Int}(C)$ is not dominated by a single vertex of C. Assume to the contrary that v_1 dominates $\text{Int}(C)$, and let u be a neighbor of v_1 in $\text{Int}(C)$. Since G is 2-connected, u has some neighbor in $\text{Int}[C] - \{v_1\}$. If u is adjacent to any neighbor of v_1 (including v_2 and v_5), then G contains a triangle, contradicting Corollary 3.6. If u is adjacent to v_3 or v_4, we obtain a 4-cycle whose interior is dominated by the single vertex v_1, contradicting Lemma 3.3. Thus u has no neighbor in $\text{Int}[C] - \{v_1\}$, a contradiction.

Now assume to the contrary that $\{v_1, v_2\}$ dominates $\text{Int}(C)$. Let u be a neighbor of v_1 in the interior of C, and note that u has some neighbor in $\text{Int}[C] - \{v_1\}$. As in the previous argument, u is not adjacent to any neighbor of v_1. If u is adjacent to either v_3 or v_4, then G contains a 4-cycle whose interior is either dominated by the single vertex v_1, or by the adjacent pair $\{v_1, v_2\}$, contradicting Lemma 3.3 or Lemma 3.4, respectively. If u is adjacent to some neighbor of v_2, then G contains a 4-cycle whose interior is dominated by the adjacent pair $\{v_1, v_2\}$, yielding a contradiction. □

Lemma 4.2. *Let C be a* 4*-cycle of a pentagulation. If C dominates its interior, then no two vertices which are adjacent in C both have neighbors in* $\text{Int}(C)$.

Proof. Let $C = v_1, v_2, v_3, v_4$ be a 4-cycle in a pentagulation, and suppose that C dominates its interior. Assume to the contrary, and without loss of generality, that both v_1 and v_2 have some neighbor in $\text{Int}(C)$. The edge v_1v_2 lies on some face in the interior of C. This face is bounded by a 5-cycle of the form u, v_1, v_2, w, x, where u and w are neighbors of v_1 and v_2 respectively, and $x \in \text{Int}[C]$. Since C dominates its interior, the vertex x is either a vertex of C, or is adjacent to a vertex of C. If x is a vertex of C, or if x is adjacent to v_1 or v_2, then there is some 3-cycle in $\text{Int}[C]$ that dominates its interior, contradicting Lemma 3.5. If x is adjacent to v_3 or v_4, then there is some 4-cycle in $\text{Int}[C]$ whose interior is dominated by two adjacent vertices of the 4-cycle, contradicting Lemma 3.4. □

Lemma 4.3. *Let C be a 6-cycle in a pentagulation. If the interior of C is dominated by two vertices u and v of C such that $d_C(u, v) = 3$, then no chord of C lies in the interior of C.*

Proof. Let $C = v_1, v_2, v_3, v_4, v_5, v_6$ be a 6-cycle in a pentagulation, the interior of which is dominated by $\{v_1, v_4\}$. Assume to the contrary that $e = v_iv_j$, with $|j - i| > 1 \pmod 6$, is a chord of C contained in $\text{Int}[C]$. If $|j-i| = 2$, then the chord induces a 3-cycle in C that dominates its interior, contradicting Lemma 3.5. Thus $|j - i| = 3$. If $e = v_1v_4$ then the chord induces a 4-cycle whose interior is dominated by two adjacent vertices, contradicting Lemma 3.4. If $e \neq v_1v_4$, then $e = v_2v_5$ or $e = v_3v_6$, and $C \cup \{e\}$ either induces the 4-cycle v_2, v_3, v_4, v_5 or the 4-cycle v_3, v_4, v_5, v_6. In either case there is a 4-cycle dominated by just v_3, contradicting Lemma 3.3. □

Lemma 4.4. *Let C be a 6-cycle in a pentagulation. If $\text{Int}(C)$ is dominated by two vertices u and v with $d_C(u, v) = 3$, then there exists some vertex in $\text{Int}(C)$ that is adjacent to both u and v.*

Proof. Let G be a pentagulation. Assume to the contrary that $C = v_1, v_2, v_3, v_4, v_5, v_6$ is a 6-cycle in G whose interior is dominated by $\{v_1, v_4\}$, and that no vertex in $\text{Int}(C)$ is adjacent to both v_1 and v_4. Choose C to be a minimal counterexample. That is, there is no 6-cycle C' that has its interior dominated by $\{v_1, v_4\}$, and that does not contain any neighbor of both v_1 and v_4 in $\text{Int}(C')$, and that satisfies $\text{Int}(C') \subset \text{Int}(C)$. The cycle C is chordless by Lemma 4.3, and is a Jordan separating cycle by Lemma 2.2, so there exists some vertex w in $\text{Int}(C)$. Without loss of generality, the vertex w is adjacent to v_1. Since G is 2-connected, there is some neighbor x of w in $\text{Int}[C] - \{v_1, v_4\}$. The vertex x is neither v_2 nor v_6, as this would create a triangle v_1, w, x, v_1 that dominates its interior, contradicting Lemma 3.5. Further, x is neither v_3 nor v_5 as either case induces a 4-cycle whose interior is dominated by v_1, contradicting Lemma 3.3. So x lies in $\text{Int}(C)$, and is adjacent to either v_1 or v_4. If x is adjacent to v_1, then v_1, x, w is a triangle, the interior of which is dominated by v_1, contradicting Lemma 3.1. Thus x is adjacent to v_4, and the two internally disjoint paths v_1, v_2, v_3, v_4 and v_1, w, x, v_4, induce a 6-cycle in $\text{Int}[C]$. The interior of this 6-cycle is dominated by $\{v_1, v_4\}$, and by assumption there is not a common neighbor of both v_1 and v_4 in the interior of this cycle, contradicting the minimality of C. □

Corollary 4.5. *Let C be a Jordan separating 5-cycle in a pentagulation. If $\text{Int}(C)$ is dominated by two non-adjacent vertices u and v of C, then there is some vertex in $\text{Int}(C)$ that is adjacent to both v and u.*

Proof. Let G be a pentagulation, and let $C = v_1, v_2, v_3, v_4, v_5$ be a Jordan separating 5-cycle in G whose interior is dominated by $\{v_1, v_3\}$. Since C is Jordan separating, there exists a vertex w in $\mathrm{Int}(C)$ that is, without loss of generality, adjacent to v_1. If w is adjacent to v_3, we are done. Suppose w is not adjacent to v_3. Since G is 2-connected, w has some neighbor x in $\mathrm{Int}[C] - \{v_1\}$. The vertex x is not any neighbor of v_1, as then v_1, w, x is a triangle that dominates its interior, contradicting Lemma 3.5. Note that $x \neq v_4$, as this would induce a 4-cycle dominated by v_1, contradicting Lemma 3.3. Thus x is a vertex in $\mathrm{Int}(C)$ that is adjacent to v_3. The internally disjoint paths v_1, v_5, v_4, v_3 and v_1, w, x, v_3 induce a 6-cycle whose interior is dominated by $\{v_1, v_3\}$. By Lemma 4.4, the interior of this 6-cycle contains some vertex adjacent to both v_1 and v_3, completing the proof. □

Lemma 4.6. *Let G be a pentagulation. If C is a 4-cycle that dominates its interior, then every vertex u in $\mathrm{Int}(C)$ has degree 2.*

Proof. Let G be a pentagulation, let $C = v_1, v_2, v_3, v_4$ be a 4-cycle in G that dominates its interior, and let w be a vertex in $\mathrm{Int}(C)$. Since C dominates its interior, we assume without loss of generality that w is adjacent to v_1. Because G is 2-connected, w has at least one neighbor in $\mathrm{Int}[C] - \{v_1\}$. Assume contrary to the lemma that $d(w) > 2$. Thus w has at least two distinct neighbors x_1 and x_2 in $\mathrm{Int}[C] - \{v_1\}$. Neither x_1 nor x_2 is adjacent to v_1, as this would induce a triangle in $\mathrm{Int}[C]$ that dominates its interior, contrary to Lemma 3.5. Therefore, each vertex x_i is either a vertex in $\mathrm{Int}(C)$, or the vertex v_3.

Suppose $x_1 = v_3$, then $x_2 \neq v_3$. Up to swapping the labels on v_2 and v_4, the vertex x_2 lies inside the cycle v_1, w, v_3, v_2. Since C dominates its interior, x_2 is adjacent to v_1, v_2 or v_3. If x_2 is adjacent to v_1 or v_3, this induces a triangle. If x_2 is adjacent to v_2, the interior of the 4-cycle v_1, w, x_2, v_2 is dominated by $\{v_1, v_2\}$, contradicting Lemma 3.4. Thus $x_1 \neq v_3$, and similarly $x_2 \neq v_3$.

Since C dominates its interior, each vertex x_i is adjacent to some vertex in $\{v_2, v_3, v_4\}$. The vertex x_1 is not adjacent to v_2, as this induces a 4-cycle x_1, v_2, v_1, w whose interior is dominated by $\{v_1, v_2\}$, contradicting Lemma 3.4. Similarly, x_1 is not adjacent to v_4, and x_2 is not adjacent to either v_2 or v_4. We conclude that both x_1 and x_2 are neighbors of v_3 in $\mathrm{Int}(C)$. But this induces a 4-cycle x_1, w, x_2, v_3 that is dominated by v_3, contradicting Lemma 3.3. □

By Lemma 2.1, any 4-cycle in a pentagulation of diameter 3 dominates either its interior or exterior. The next theorem gives a complete description of the structure of this dominated region. An example of such a region is given by Figure 2.

Theorem 4.7. *Let G be a pentagulation, and C a 4-cycle in G. If C dominates its interior, then there exist two non-adjacent vertices u and v of C, and a positive integer k such that the induced subgraph $G[Int[C]]$ consists of exactly:*

(1) *the cycle C,*

(2) *k $u - v$ paths of length 3, and*

(3) *$k - 1$ $u - v$ paths of length 2.*

All the paths in (2) and (3) are internally disjoint, do not contain any vertices of $C - \{u, v\}$, and the paths of length 2 and 3 alternate.

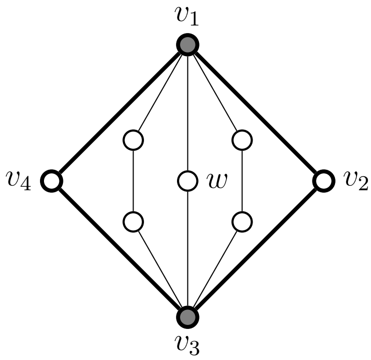


Figure 2: A 4-cycle dominating its interior which has $k = 2$ paths of length 3 and $k - 1 = 1$ paths of length 2 between two non-adjacent vertices v_1 and v_3, illustrating Theorem 4.7.

Proof. Let G be a pentagulation, and $C : v_1, v_2, v_3, v_4$ a 4-cycle in G that dominates its interior. By Lemmas 3.3 and 3.4, exactly two non-adjacent vertices of C have neighbors in $\text{Int}(C)$. Suppose without loss of generality that these two vertices are v_1 and v_3. The interior of C is chordless, as a chord would induce a 3-cycle that dominates its interior, contradicting Lemma 3.5. By Lemma 4.6, any vertex in $\text{Int}(C)$ has degree 2. Further, any vertex in $\text{Int}(C)$ is adjacent to either v_1 or v_3, and there is no 3-cycle in the interior of C by Lemma 3.5. Thus every vertex in $\text{Int}(C)$ lies on a $v_1 - v_3$ path of length 2 or 3, and these paths are internally disjoint. Since G is a pentagulation and every face is bounded by a 5-cycle, the paths of length 2 and 3 must alternate. □

By Corollary 3.6, no diameter 3 pentagulation contains a triangle. Figure 3 shows two diameter 3 pentagulations containing 4-cycles.

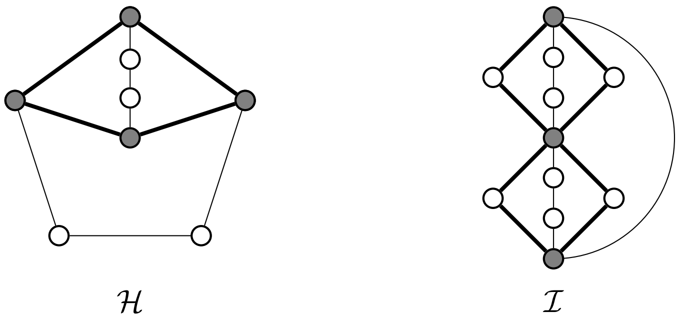


Figure 3: Two diameter 3 pentagulations that contain 4-cycles, $\mathcal{H}$ and $\mathcal{I}$. Pairs of non-adjacent grey vertices dominate regions bounded by bold 4-cycles.

5 Singling out a square with dislocated 4-cycles

In order to describe the structure of diameter 3 pentagulations, we need a new concept: *dislocated* 4-cycles. In Figure 2, consider the three 4-cycles $C_1 : v_1, v_2, v_3, v_4$;

$C_2 : v_1, w, v_3, v_4$ and $C_3 : v_1, w, v_3, v_2$. Although these three cycles are distinct, both C_2 and C_3 are just 'substructures' of C_1, formed by C_1 and the alternating paths in its interior (Theorem 4.7). Heuristically, two 4-cycles in a pentagulation are dislocated when—unlike the cycles in Figure 2—they cannot be considered part of the same collection of alternating paths. For example, the two bold 4-cycles in Figure 3, graph $\mathcal{I}$ are dislocated.

Consider two distinct 4-cycles, C_1 and C_2, in a pentagulation G. We say that C_1 and C_2 are **dislocated** 4-cycles if there exist two regions $R_1 \in \{\mathrm{Int}(C_1), \mathrm{Ext}(C_1)\}$ and $R_2 \in \{\mathrm{Int}(C_2), \mathrm{Ext}(C_2)\}$, as well as two pairs of vertices $\{u_1, v_1\} \subset V(C_1)$ and $\{u_2, v_2\} \subset V(C_2)$, such that all three of the following conditions hold:

1. The regions R_1 and R_2 are dominated by $\{u_1, v_1\}$ and $\{u_2, v_2\}$, respectively,
2. The sets $\{u_1, v_1\}$ and $\{u_2, v_2\}$ are not equal,
3. The intersection $R_1 \cap R_2$ is empty.

Note that by Lemma 3.4, the edge u_1v_1 is not in $E(C_1)$, and u_2v_2 is not in $E(C_2)$.

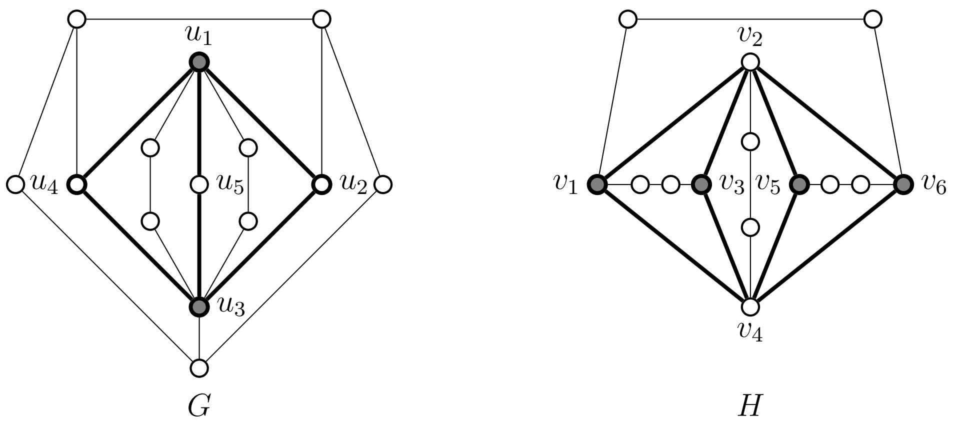


Figure 4: In G, there is no pair of dislocated 4-cycles. In H, any pair of 4-cycles in which both cycles dominate their interior or exterior is dislocated.

For an example, consider Figure 4. No two of these cycles in G are dislocated, as they fail either condition (2) or (3) of the definition. In H, any pair C and D of 4-cycles such that C and D both dominate one of their two regions is a dislocated pair.

6 Bounding the order, part I: An abundance of 4-cycles

In this section, we consider pentagulations containing two or more dislocated 4-cycles. But first, we handle a simple case, for which we recall the well-known theorem stating that if a graph of order n and maximum degree Δ is dominated by γ vertices, then $n \leq \gamma(\Delta + 1)$ (see, for example, Theorem 10.6 of [2]).

Lemma 6.1. *Let G be a pentagulation of order n and maximum degree $\Delta \geq 3$. If any 4-cycle of G dominates G, then $n \leq 3\Delta - 1$.*

Proof. Let G be a pentagulation of order n and maximum degree Δ that is dominated by the 4-cycle $C : v_1, v_2, v_3, v_4$. Since $\mathrm{Int}(C)$ is dominated by C, we have without loss of generality, by Theorem 4.7, that every vertex of $\mathrm{Int}(C)$ lies on a $v_1 - v_3$ path of length 2 or 3. There are at most $\frac{\Delta-1}{2}$ paths of length 3 in $\mathrm{Int}(C)$, and at most $\frac{\Delta-3}{2}$ paths of length 2 in $\mathrm{Int}(C)$. Because $\mathrm{Ext}(C)$ is dominated by C, we have by Theorem 4.7 that every vertex of $\mathrm{Ext}(C)$ lies on either a $v_1 - v_3$ path, or a $v_2 - v_4$ path, and any such path has length 2 or 3. If the vertices of $\mathrm{Ext}(C)$ lies on $v_1 - v_3$ paths, then $\{v_1, v_3\}$ dominates $\mathrm{Ext}(C)$, so G is dominated by two vertices. Thus $n \leq 2\Delta + 2 \leq 3\Delta - 1$.

Therefore the vertices of $\mathrm{Ext}(C)$ lie on $v_2 - v_4$ paths. As before, the number of paths of length 3 is bounded above by $\frac{\Delta-1}{2}$, and the number of paths of length 2 is at most $\frac{\Delta-3}{2}$. Each path of length 3 in $\mathrm{Int}(C)$ ($\mathrm{Ext}(C)$) contributes 2 to the number $|V(\mathrm{Int}(C))|$ ($|V(\mathrm{Ext}(C))|$), and each path of length 2 contributes 1 to $|V(\mathrm{Int}(C))|$ ($|V(\mathrm{Int}(C))|$). Thus:

$$\begin{aligned} n &= |V(C)| + |V(\mathrm{Int}(C))| + |V(\mathrm{Ext}(C))| \\ &\leq 4 + 2\left[2\left(\frac{\Delta-1}{2}\right) + 1\left(\frac{\Delta-3}{2}\right)\right] \\ &= 3\Delta - 1. \end{aligned}$$

□

In the proofs of Lemmas 6.2 and 6.4 to follow, we refer to specific vertices and faces of the graphs $\mathcal{H}$ and $\mathcal{I}$ by the labels given in Figure 5.

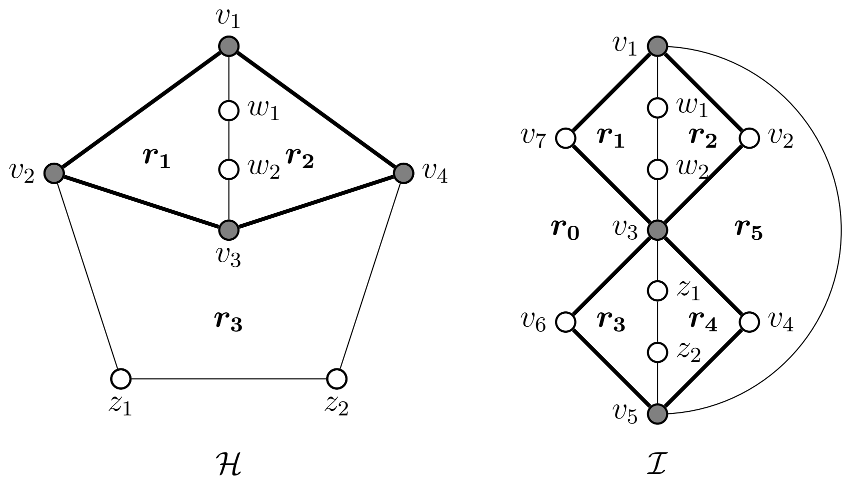


Figure 5: The graphs $\mathcal{H}$ and $\mathcal{I}$, with the labels used in the proofs of Lemmas 6.2 and 6.4.

Lemma 6.2. *Let G be a pentagulation of diameter 3, order n and maximum degree Δ. If G contains $\mathcal{H}$ as a subgraph, then $n \leq 3\Delta - 1$.*

Proof. Assume G contains $\mathcal{H}$ (Figure 3) as a subgraph, and let $C : v_1, v_2, v_3, v_4$ be the 4-cycle of H. Label the remaining vertices of $\mathcal{H}$ so that v_1, w_1, w_2, v_3 and v_2, z_1, z_2, v_4

are paths of length 3 (see Figure 5), with w_1 and w_2 lying in $\mathrm{Int}(C)$ and z_1 and z_2 lying in $\mathrm{Ext}(C)$. Since G has diameter 3, we know that, without loss of generality, the cycle C dominates its interior by Lemma 2.1. Assume to the contrary that C does not dominate its exterior. Then there is a vertex $u \in \mathrm{Ext}(C)$ such that $d(u, C) \geq 2$. If u lies in the outer face of $\mathcal{H}$, then $d(u, w_2) \geq 4$. If u lies in r_3, then $d(u, w_1) \geq 4$. In either case, we obtain a contradiction, so C dominates its exterior and is thus a dominating 4-cycle. That $n \leq 3\Delta - 1$ follows immediately from Lemma 6.1. □

Theorem 6.3. *Let G be a pentagulation of diameter 3, order n, and maximum degree $\Delta \geq 3$. If G contains two dislocated 4-cycles, C_1 and C_2, then G contains $\mathcal{I}$ as a subgraph (see Figure 3), or $n \leq 3\Delta - 1$.*

Proof. Let G be a pentagulation of diameter 3, order n and maximum degree $\Delta \geq 3$. Suppose that G contains two dislocated 4-cycles $C_1 : v_1, v_2, v_3, v_4$ and $C_2 : u_1, u_2, u_3, u_4$. We consider all the possible configurations for the two dislocated 4-cycles. Note that if any 4-cycle dominates G, or if G contains an $\mathcal{H}$ subgraph, then $n \leq 3\Delta - 1$ by Lemmas 6.1 and 6.2. Assume without loss of generality that C_1 dominates its interior. By Theorem 4.7, and without loss of generality, the region $\mathrm{Int}(C_1)$ is dominated by $\{v_1, v_3\}$, and there exist vertices w_1 and w_2 in $\mathrm{Int}(C_1)$ such that $P_1 : v_1, w_1, w_2, v_3$ is a path in G.

Case 1: The cycles C_1 and C_2 have exactly two adjacent vertices in common.
By symmetry, we may assume without loss of generality that $v_2 = u_1$ and $v_3 = u_4$ (see Figure 6, (1)).

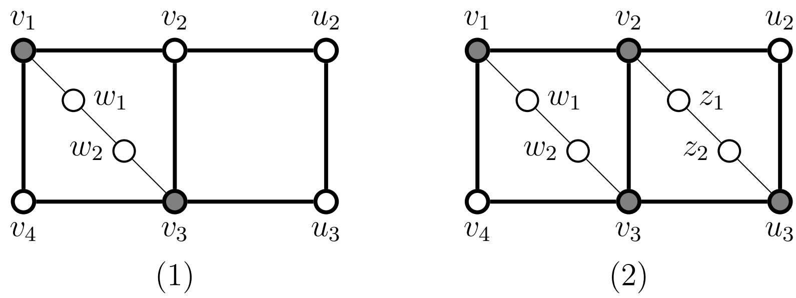


Figure 6: Two dislocated 4-cycles, C_1 and C_2, that share an edge, as in Case 1 of the proof of Theorem 6.3.

Since C_1 and C_2 are dislocated, both u_2 and u_3 lie in $\mathrm{Ext}(C_1)$. By Corollary 3.6, the pentagulation G is triangle-free, so $d_G(w_1, C_2) = 2$. Since C_2 dominates either its interior or exterior, we have that C_2 dominates its interior. By Theorem 4.7, there exist vertices z_1 and z_2 in $\mathrm{Int}(C_2)$ such that either $P_2 : v_2, z_1, z_2, u_3$ is a path in G, or $P_2' : u_2, z_1, z_2, v_3$ is a path in G. If G contains the path P_2', then there is a $z_1 - w_1$ path R of length at most 3 in G. Since G is triangle-free, the vertex w_1 is only adjacent to v_1 and w_2, and z_1 is only adjacent to u_2 and z_2. Thus, since G is a plane graph and $d_G(w_1, z_1) \leq 3$, v_1 and u_2 are adjacent. This induces a triangle, which is impossible. Therefore G contains the path P_2, not the path P_2' (see Figure 6, (2)). Since G has diameter 3, there exists some $w_1 - z_2$ path of length at most 3. By the same argument as in the prior paragraph, we deduce that v_1 and u_3 are adjacent. But now we have induced $\mathcal{H}$ as a subgraph of G, with the 4-cycle of $\mathcal{H}$ corresponding to the 4-cycle of G on $v_1, v_2 = u_1, v_3 = u_4, u_3$. By Lemma 6.2, we have $n \leq 3\Delta - 1$.

Case 2: The dislocated cycles C_1 and C_2 have exactly three vertices in common.
Up to symmetry, there are two different ways that C_1 could share three vertices with C_2: the cycles may share both the dominating vertices v_1 and v_3, or only one of them.

Case 2.1: The vertices v_1 and v_3 are in both C_1 and C_2.
Assume without loss of generality that $v_1 = u_1$, $v_2 = u_4$ and $v_3 = u_3$ (see Figure 7 (1)).

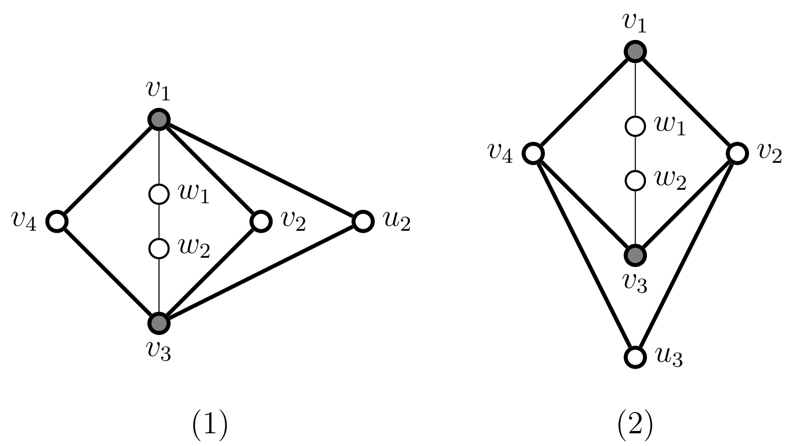


Figure 7: Case 2.1 in the proof of Theorem 6.3 has the two dislocated 4-cycles C_1 and C_2 sharing v_1, v_2 and v_3. Case 2.2 has the cycles sharing v_2, v_3 and v_4.

Since C_1 and C_2 are dislocated, the set $\{u_2, v_2\}$ dominates either the interior or exterior of C_2. We claim the set dominates the interior of C_2. By Lemma 4.2, the vertex v_2 does not have any neighbor in $\text{Int}(C_1)$, and thus has no neighbors in $\text{Ext}(C_2)$. By Lemma 3.3, no single vertex of C_2 dominates the exterior of C_2, so the set $\{v_2, u_2\}$ does not dominate $\text{Ext}(C_2)$, proving the claim.

Since $\{u_2, v_2\}$ dominates $\text{Int}(C_2)$, there are two vertices z_1 and z_2 in $\text{Int}(C_2)$ such that $P_2 : v_2, z_1, z_2, u_2$ is a path in G. The vertices of $C_1 \cup C_2 \cup P_1 \cup P_2$ induce an $\mathcal{H}$ subgraph in G. Thus $n \leq 3\Delta - 1$ by Lemma 6.2.

Case 2.2: Only one of v_1 and v_3 is common to both C_1 and C_2.
Assume without loss of generality that $v_2 = u_2$, $v_3 = u_1$ and $v_4 = u_4$ (see Figure 7 (2)). Since G is triangle-free, the distance $d_G(w_1, C_2) = 2$, so C_2 does not dominate its exterior and thus dominates its interior. By Theorem 4.7, there are vertices z_1 and z_2 in $\text{Int}(C_2)$ such that either $P_2 : v_3, z_1, z_2, u_3$ is a path of G, or $P_2' : v_2, z_1, z_2, v_4$ is a path of G. In the latter case, we obtain an $\mathcal{H}$ subgraph on $C_1 \cup C_2 \cup P_1 \cup P_2'$. In the former case, we have $d(w_1, z_2) > 3$.

Case 3: The cycles C_1 and C_2 have exactly one vertex in common.
Since C_1 and C_2 only share one vertex, and G is triangle-free, either $d(w_1, V(C_2)) \geq 2$ or $d(w_2, V(C_2)) \geq 2$. As such, C_2 does not dominate its exterior, and thus dominates its interior. Up to symmetry, there are four possible cases.

Case 3.1: The dislocated cycles C_1 and C_2 share the vertex $v_2 = u_4$ and $\mathrm{Int}(C_2)$ is dominated by $\{u_1, u_3\}$ (see Figure 8 (1)).

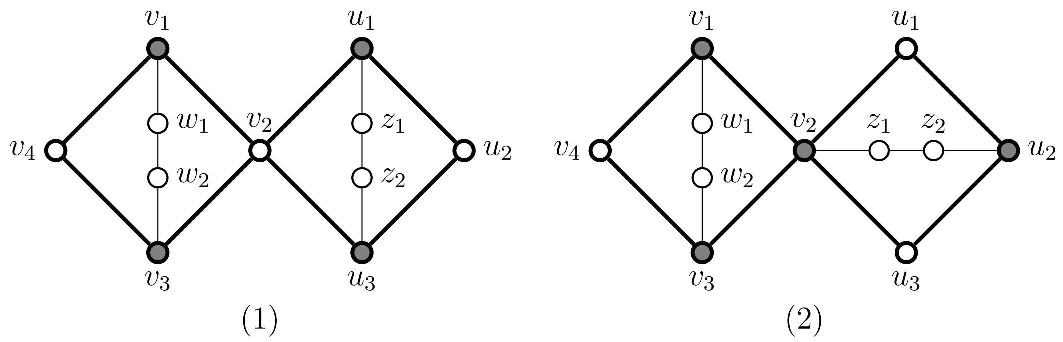


Figure 8: In both figures, the dislocated 4-cycles C_1 and C_2 share the vertex $v_2 = u_4$. We have (1) when the interior of C_2 is dominated by $\{u_1, u_3\}$, as in Case 3.1, and we have (2) when the interior of C_2 is dominated by $\{u_2, u_4\}$, as in Case 3.2 of the proof of Theorem 6.3.

By Theorem 4.7, there is a vertex z_1 in $\mathrm{Int}(C_2)$ that is adjacent to u_1, but not to any other vertex of C_2. But then $d_G(w_1, z_1) > 3$, a contradiction.

Case 3.2: The dislocated cycles C_1 and C_2 share the vertex $v_2 = u_4$ and $\mathrm{Int}(C_2)$ is dominated by $\{u_2, u_4\}$ (see Figure 8 (2)).
By Theorem 4.7, there are two vertices z_1 and z_2 in the interior of C_2 such that $P_2 : v_2, z_1, z_2, u_2$ is a path in G. Since G is a triangle-free plane graph, and both $d_G(z_2, w_1) \leq 3$ and $d_G(z_2, w_w) \leq 3$, we have that u_2 is adjacent to both v_1 and v_3. Thus G contains $\mathcal{H}$ as a subgraph.

Case 3.3: The dislocated cycles C_1 and C_2 share the vertex $v_3 = u_1$ and $\mathrm{Int}(C_2)$ is dominated by $\{u_2, u_4\}$ (see Figure 9 (1)).

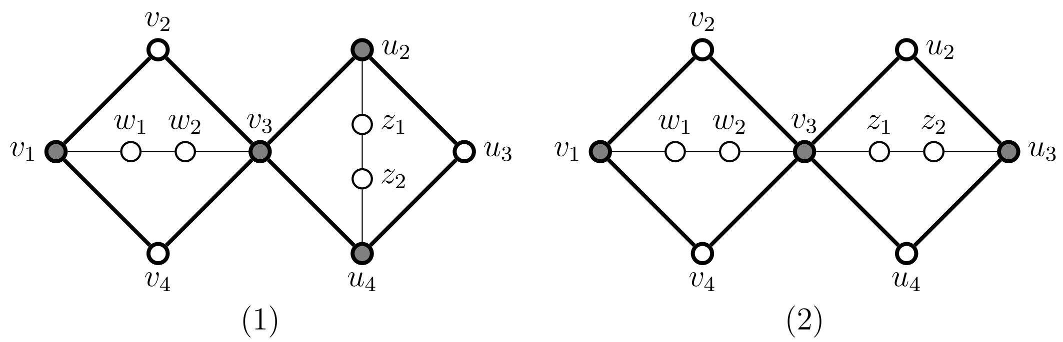


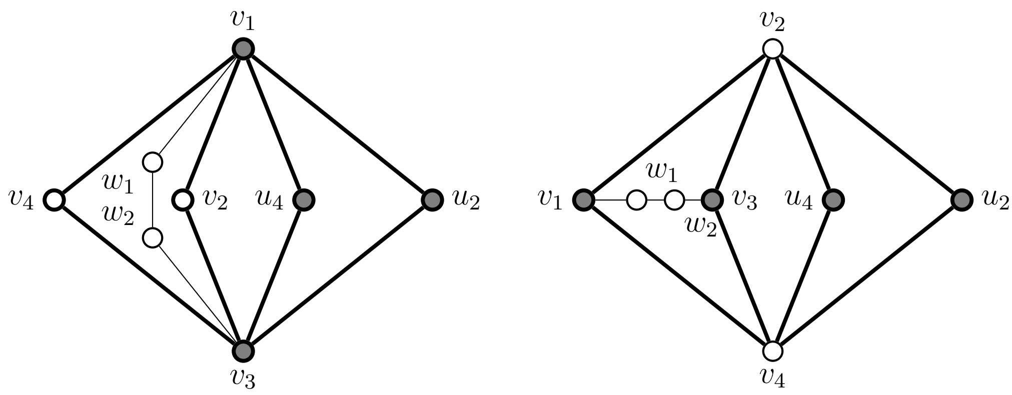
Figure 9: In both figures, the dislocated 4-cycles C_1 and C_2 share the vertex $v_3 = u_1$. When the interior of C_2 is dominated by u_2 and u_4, as in Case 3.3 of the proof of Theorem 6.3, (1) occurs. When the interior of C_2 is dominated by u_1 and u_3, as in Case 3.4, (2) occurs.

Reversing the roles of the cycles C_1 and C_2, we observe that this case is identical to Case 3.2, hence G contains $\mathcal{H}$ as a subgraph, so $n \leq 3\Delta - 1$.

Case 3.4: The dislocated cycles C_1 and C_2 share the vertex $v_3 = u_1$ and $\text{Int}(C_2)$ is dominated by $\{u_1, u_3\}$ (see Figure 9 (2)).
By Theorem 4.7, there are vertices z_1 and z_2 in $\text{Int}(C_2)$ such that $P_2 : v_3, z_1, z_2, u_3$ is a path in G. Since $d(w_1, z_2) \leq 3$, we have that v_1 and u_3 are adjacent. Thus $\mathcal{I}$ is a subgraph of G.

Case 4: The dislocated cycles C_1 and C_2 are disjoint.
In this case, no vertex of C_2 is adjacent to w_1, so C_2 dominates its interior. By Theorem 4.7, and without loss of generality, there are vertices z_1 and z_2 in the interior of C_2 and edges u_1z_1, z_1z_2 and z_2u_3. Since G has diameter 3, we have that $d_G(w_i, z_j) \leq 3$ for any indices i and j in $\{1, 2\}$. Since G is triangle-free, it contains all four edges of the form u_iw_k, where i and k are in $\{1, 3\}$. However, noting the 4-cycle on v_1, u_1, v_3, u_3, we see that G contains $\mathcal{H}$ as a subgraph.

Case 5: The dislocated cycles C_1 and C_2 share exactly two non-adjacent vertices.



(1) (2)

Figure 10: In (1), the dislocated 4-cycles C_1 and C_2 share vertices $v_1 = u_1$ and $v_3 = u_3$, as in Case 5.1 of Theorem 6.3. In Figure (2), the cycles share vertices $v_2 = u_1$ and $v_4 = u_3$, as in Case 5.2.

Up to symmetry, there are two subcases to consider. Either $v_1 = u_1$ and $v_3 = u_3$ are common to both C_1 and C_2, or the vertices $v_2 = u_2$ and $v_4 = u_4$ are. In both cases, since C_1 and C_2 are dislocated, the set $\{u_2, u_4\}$ of vertices dominates the interior of C_2 (it does not dominate the exterior, as neither is adjacent to w_1). Thus, in both cases, by Theorem 4.7, there are vertices z_1 and z_2 in $\text{Int}(C_2)$ such that $P_2 : u_2, z_1, z_2, u_4$ is a path in G.

Case 5.1: The vertices v_1 and v_3 are common to C_1 and C_2 (see Figure 10 (1)).
Consider the cycle $C : v_1, v_2, v_3, u_4$. Since z_1 is not adjacent to a vertex of C, the cycle C dominates its interior. If $\{v_1, v_3\}$ dominates $\text{Int}(C)$, then C and C_2 are dislocated 4-cycles sharing three vertices, and by Case 2 we have that $n \leq 3\Delta - 1$. Similarly, if $\{v_2, u_4\}$ dominates $\text{Int}(C)$, then C and C_1 are dislocated.

Case 5.2: The vertices v_2 and v_4 are common to C_1 and C_2 (see Figure 10 (2)).
Denote by C' the cycle on v_2, v_3, v_4, u_4. By the argument of the preceding paragraph, C' and C_1 are dislocated 4-cycles. Thus, by Case 2, $n \leq 3\Delta - 1$. □

Lemma 6.4. *Let G be a pentagulation of diameter 3, order n and maximum degree Δ. If G contains $\mathcal{I}$ as a subgraph, then $n \leq 3\Delta - 1$.*

Proof. Let G be a pentagulation of diameter 3, order n and maximum degree Δ that contains $\mathcal{I}$ as a subgraph. Let the vertices of $\mathcal{I}$ be labeled as they are in Figure 5, such that the vertices w_1 and z_1 lie in the interiors of the 4-cycles $C_1 : v_1, v_2, v_3, v_7$ and $C_2 : v_3, v_4, v_5, v_6$, respectively. Since G is triangle-free (Corollary 3.6), the subgraph $\mathcal{I}$ is an induced subgraph of G. Therefore, $d_G(z_2, C_1) = 2$, and by a similar argument, $d_G(w_1, C_2) = 2$. Hence the cycles C_1 and C_2 dominate their interiors by Lemma 2.1. In particular, the set $\{v_1, v_3\}$ dominates $\mathrm{Int}(C_1)$, and $\{v_3, v_5\}$ dominates $\mathrm{Int}(C_2)$. We refine our choice of embedding of G (or equivalently, our choice of subgraph isomorphic to $\mathcal{I}$), so that the interiors of the cycles C_1 and C_2 are maximal. In other words, there does not exist a 4-cycle C_1' such that $\mathrm{Int}(C_1) \subset \mathrm{Int}(C_1')$ and $\mathrm{Int}(C_1')$ is dominated by $\{v_1, v_3\}$, and likewise for C_2. Assume for the sake of contradiction that $n > 3\Delta - 1$. Suppose that every vertex of $V(G) - V(\mathcal{I})$ is adjacent to at least one of v_1, v_3 or v_5. Then:

$$\begin{aligned} n &= |V(\mathcal{I})| + |V(G) - V(\mathcal{I})| \\ &\leq 11 + (d(v_1) - 4) + (d(v_3) - 6) + (d(v_5) - 4) \\ &\leq 11 + 3\Delta - 14 \ = \ 3\Delta - 3 \ < \ 3\Delta - 1. \end{aligned}$$

Thus assume that G contains vertices in $V(G) - V(\mathcal{I})$ that are not adjacent to any of v_1, v_3 or v_5. Let x be such a vertex, and label the faces $r_0, r_1, \ldots, r_5$ of $\mathcal{I}$ as they are labeled in Figure 5. The regions $r_1 \cup r_2$, and $r_3 \cup r_4$ are dominated by the 4-cycles C_1 and C_2, respectively, and as such any vertex added to these regions is adjacent to a vertex in the set $\{v_1, v_3, v_5\}$. Thus we assume that x is not in any of the regions r_1, r_2, r_3 or r_4. By the symmetry of r_0 and r_5, we assume without loss of generality that x is in r_5. If x is adjacent to v_2 and v_4, then we induce a 4-cycle $C : v_2, x, v_4, v_3$ which shares an edge with the cycle C_1. Since $d(w_1, C) = 2$, C dominates its interior. Thus C and C_1 are dislocated 4-cycles that share an edge, so $n \leq 3\Delta - 1$ by Theorem 6.3, a contradiction. Hence we assume that x is not adjacent to both v_2 and v_4. There are two cases to consider.

Case 1: The vertex x is not adjacent to either v_2 or v_4.
Since the diameter of G is 3, x is within distance 3 of each of w_1, w_2, z_1, z_2. Thus x has neighbors y_1, y_2 and y_3 in r_5 such that y_1v_1, y_2v_3 and y_3v_5 are all edges in G. Note that $y_1 \neq y_3$ as this induces a triangle with vertex set $\{v_1, y_1, v_5\}$. We claim that $y_1 \neq y_2$. Assume to the contrary that $y_1 = y_2$, and let C be the 4-cycle on v_1, v_2, v_3, y_1, v_1. Note that $d_G(z_2, C) = 2$, so C dominates its interior. By the maximality of C_1, we deduce that C and C_1 are dislocated 4-cycles that share more than one vertex. Thus $n \leq 3\Delta - 1$ by Theorem 6.3, proving the claim. Similarly $y_2 \neq y_3$, so the three vertices y_1, y_2 and y_3 are distinct. The paths $Q_1 : v_1, y_1, x$, $Q_2 : v_3, y_2, x$ and $Q_3 : v_5, y_3, x$ divide r_5 up into three sub-regions. Let r_6 denote the region with vertices $v_1, v_2, v_3, y_2, x, y_1$ on its boundary, let r_7 be bounded by $v_3, y_2, x, y_3, v_5, v_4$, and let r_8 be bounded by v_1, y_1, x, y_3, v_5.

We claim that the subgraph $\mathcal{I}' = \mathcal{I} \cup Q_1 \cup Q_2 \cup Q_3$ of G is an induced subgraph. Any edge between two vertices on the boundary of any region $r_0, \ldots, r_4$ induces a triangle, which is not possible since G is triangle-free. Similarly, no edge crosses r_8. Any edge crossing r_6 either creates a triangle, which is not possible, or a 4-cycle C such that C_1 and C are two dislocated 4-cycles which share at least two vertices. By Theorem 6.3, we have $n \leq 3\Delta - 1$, contrary to assumption. The argument that no edges cross the region r_7 is similar to the argument for r_6, just replace the role of C_1 with C_2. This proves the claim.

If there exists a vertex in r_6, it is adjacent to v_1 or v_3 since it is within distance 3 of z_2. Similarly, any vertex in r_7 is adjacent to v_3 or v_5 as it is within distance 3 of w_1. No vertex lies in r_8, as it would be adjacent to both v_1 and v_5 to be within distance 3 of w_2 and z_1 respectively, inducing a triangle on y_4, v_1, v_5. Any vertex of r_0 is at distance 3 or less from x, and thus is adjacent to one of v_1, v_3 or v_5. The subgraph $\mathcal{I}'$ has 15 vertices, and every vertex of $G - \mathcal{I}'$ is adjacent to one of v_1, v_3 or v_5. Noting that $d_{\mathcal{I}'}(v_1) = 5$, $d_{\mathcal{I}'}(v_3) = 7$ and $d_{\mathcal{I}'}(v_5) = 5$, we can bound the order of G:

$$\begin{aligned} n &\leq 15 + (d(v_1) - 5) + (d(v_3) - 7) + (d(v_5) - 5) \\ &\leq 3\Delta - 2 \ < \ 3\Delta - 1. \end{aligned}$$

Case 2: The vertex x is adjacent to v_2.
By assumption, x is not adjacent to any of v_1, v_3, v_4 or v_5, and $d(x, z_2) \leq 3$. As no two vertices on the boundary of r_5 are adjacent, there exists some vertex y_1 in r_5 such that there is a path $S_1 : v_2, x, y_1, v_5$ in G. We claim that $\mathcal{I} \cup S_1$ is an induced subgraph of G. Since G is triangle-free, no edges crosses a region bounded by a 5-cycle. Thus the only possible region of $\mathcal{I} \cup S_1$ with a chord is the region bounded by the two paths S_1 and v_2, v_3, v_4, v_5. However, any edge between the vertices bounding this region creates either a triangle, which is impossible, or two 4-cycles A_1 and A_2. In all cases, every vertex of A_1 and A_2 is distance at least 2 from w_1, so A_1 and A_2 dominate their interiors. Thus, for some i and j in $\{1, 2\}$, the cycles C_i and A_j are a pair of dislocated 4-cycles that share at least two vertices. By Theorem 6.3, we have $n \leq 3\Delta - 1$, proving the claim.

Because $d_G(y_1, w_2) \leq 3$, and since $\mathcal{I} \cup S_1$ is an induced subgraph of G, there exists some vertex y_2 in $r_5 - \{x, y_1\}$ such that G contains the path $S_2 : y_1, y_2, v_3$. Let $\mathcal{I}'' = \mathcal{I} \cup S_1 \cup S_2$, and note that the paths S_1 and S_2 divide r_5 into three sub-regions: $r_6 = \mathrm{Int}(v_1, v_2, x, y_1, v_5)$, $r_7 = \mathrm{Int}(v_2, v_3, y_2, y_1, x)$ and $r_8 = \mathrm{Int}(v_3, y_2, y_1, v_5, v_4)$. We show that any vertex in $G - \mathcal{I}''$ is adjacent to one of v_1, v_3 or v_5. Since G is triangle-free, and every face of $\mathcal{I}''$ is bounded by a 5-cycle, $\mathcal{I}''$ is an induced subgraph of G. As such, the only vertices on the boundary of r_6 within distance 2 of w_2 are v_1 and v_2. The region r_6 is empty by Lemma 4.1, as it is dominated by two adjacent vertices. Similarly r_7 is empty, as the only vertices on the boundary of r_7 within distance 2 of w_1 are the adjacent pair v_2 and v_3. Any vertex in r_8 is adjacent to either v_3 or v_5, as it is distance at most 3 from w_1. Any vertex in r_0 is adjacent to one of v_1, v_3 or v_5 as it is distance at most 3 from x. Note that $\mathcal{I}''$ has 14 vertices, and that $d_{\mathcal{I}''}(v_1) = 4$, $d_{\mathcal{I}''}(v_3) = 7$ and $d_{\mathcal{I}''}(v_5) = 5$. Any vertex of $G - \mathcal{I}''$ is adjacent to one of v_1, v_2 or v_3, so we can bound the order of G:

$$n \leq 14 + (d(v_1) - 4) + (d(v_3) - 7) + (d(v_5) - 5) \leq 3\Delta - 2.$$

In every case, we have derived a contradiction, completing the proof. □

Theorem 6.5 follows immediately from Lemma 6.1, Theorem 6.3 and Lemma 6.4.

Theorem 6.5. *Let G be a pentagulation of diameter 3, order n and maximum degree $\Delta \geq 8$. If G contains either a dominating 4-cycle, or two dislocated 4-cycles, then $n \leq 3\Delta - 1$.*

7 Bounding the order, part II: The lonely 4-cycle

We show that if a pentagulation contains some 4-cycle, but no dislocated pair of them, then it satisfies $n \leq 3\Delta - 1$. Throughout this section, we work with pentagulations of diameter 3 that contain some 4-cycle C. Assume without loss of generality that C dominates its interior. This motivates the following terminology. The 4-cycle C of a plane graph is **interior maximal** if it dominates its interior, and there does not exist any other 4-cycle C' such that C' dominates its interior, and $\text{Int}(C) \subset \text{Int}(C')$.

Lemma 7.1. *Let G be a pentagulation of diameter 3 that does not contain two dislocated 4-cycles, and let C be an interior maximal 4-cycle of G. If D is any cycle in $\text{Ext}[C]$ of length at most 7, then D is chordless.*

Proof. Assume to the contrary D has some chord e. By Corollary 3.6, $D \cup \{e\}$ has no 3-cycle, so $D \cup \{e\}$ induces a 4-cycle. Either this 4-cycle contradicts the maximality of C, or is dislocated from C, and both cases yield a contradiction. □

Lemma 7.2. *Let G be a pentagulation of diameter 3 that does not contain two dislocated 4-cycles, and let C be an interior maximal 4-cycle of G. If D is any 5-cycle in G such that both $\text{Int}(D) \subset \text{Ext}(C)$ and $\text{Int}(D)$ is dominated by two or fewer vertices of D, then $\text{Int}(D)$ does not contain any vertex of G.*

Proof. By Lemma 4.1, the interior of D is not dominated by either a single vertex of D, or an adjacent pair of vertices in D. Assume to the contrary that there is a vertex w in $\text{Int}(D)$, and let u and v be two non-adjacent vertices of D that dominate $\text{Int}(D)$. By Corollary 4.5, the vertex w is adjacent to both u and v. Thus, there exists some 4-cycle A in $\text{Int}[D]$ that dominates its interior. The cycle A either contradicts the maximality of C, or A and C are dislocated. □

Theorem 7.3. *Let G be a pentagulation of diameter 3, order n and maximum degree $\Delta \geq 8$. If G contains a 4-cycle, then $n \leq 3\Delta - 1$.*

Proof. Assume to the contrary that G contains a 4-cycle $C_1 = v_1, v_2, v_3, v_4$, and has order $n > 3\Delta - 1$. By Theorem 6.5, there are no two dislocated 4-cycles in G. Assume without loss of generality that C_1 is interior maximal, and that $\text{Int}(C_1)$ is dominated by $\{v_1, v_3\}$. By Theorem 4.7, there exist vertices w_1 and w_2 in $\mathrm{Int}(C_1)$ such that $P_1 : v_1, w_1, w_2, v_3$ is a path in G. If every vertex of G is adjacent to either v_1 or v_3, then $n \leq 2\Delta < 3\Delta - 1$, so there exists some vertex of G in $\mathrm{Ext}(C_1)$ which is not adjacent to v_1 or to v_3. We consider two cases, according to whether or not the vertices v_2 and v_4 have neighbors in $\mathrm{Ext}(C_1)$.

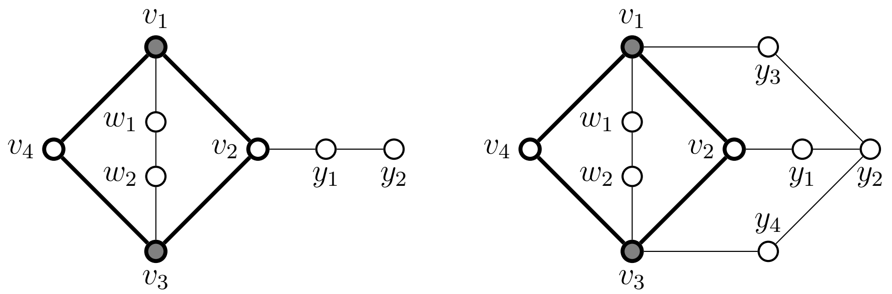


Figure 11: In Case 1, since the vertex y_1 is not an end-vertex, there exists some neighbor y_2 of y_1 (1). Since the diameter of G is 3, it contains $y_2 - w_1$ and $y_2 - w_2$ paths, forcing the subgraph $\mathcal{G}$ (2).

Case 1: The vertex v_2 has at least one neighbor in $\mathrm{Ext}(C_1)$.
Let y_1 be a vertex in the exterior of C_1 that is adjacent to v_2. The vertex y_1 is not adjacent to either v_1 or v_3 as this induces a triangle, contradicting Corollary 3.6. Further, y_1 is not adjacent to v_4 as this induces a 4-cycle on the vertices v_2, y_1, v_3, v_4, contradicting the fact that G does not contain two dislocated 4-cycles. Since G is 2-connected, there is some vertex y_2 in $\mathrm{Ext}(C_1)$ to which y_1 is adjacent (see Figure 11 (1)).

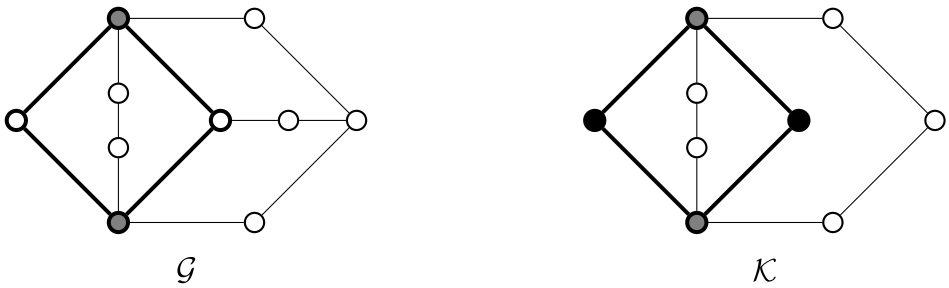


Figure 12: If G is a diameter 3 pentagulation that contains some 4-cycle, but no two dislocated 4-cycles, it must contain one of $\mathcal{G}$ or $\mathcal{K}$ as a subgraph, by Cases 1 and 2 respectively in the proof of Theorem 7.3. The black vertices of $\mathcal{K}$ are not adjacent to any vertices of $G - \mathcal{K}$.

Note that $d(y_2) \geq 2$, and there exist $y_2 - w_1$ and $y_2 - w_2$ paths of length at most 3. Since G is triangle-free, the vertices y_2 and v_2 are not adjacent. Further, y_2 is not adjacent to either v_1 or v_3, as this induces a 4-cycle dislocated from C_1 on the vertices v_1, v_2, y_1, y_2 or v_3, v_2, y_1, y_2 respectively. Finally, y_1 is not adjacent to v_4, as this induces $\mathcal{H}$ as a subgraph of G, which yields a contradiction by Lemma 6.2.

Since no $y_2 - w_1$ or $y_2 - w_2$ geodesic can be formed with the vertices mentioned thus far, there exist vertices y_3 and y_4 in $\mathrm{Ext}(C_1)$ such that y_2y_3, y_3v_1, y_2y_4 and y_4v_3 are edges in G (see Figure 11 (2)). Note that $y_3 \neq y_4$, as this would again induce $\mathcal{H}$ as a subgraph of G. Let $\mathcal{G}$ denote the subgraph of G constructed thus far (see Figure 12). Applying Lemma 7.1, we deduce that $\mathcal{G}$ is an induced subgraph of G. Thus, the only two vertices of the 5-cycle $C_3 : v_1, v_2, y_1, y_2, y_3$ within distance 2 of w_2 are v_1 and v_2, so $\{v_1, v_2\}$ dominates $\mathrm{Int}(C_2)$. Hence, by Lemma 4.1, there is no vertex in $\mathrm{Int}(C)$. Similarly, there is no vertex in the region bounded by the cycle $C_4 : v_2, y_1, y_2, y_4, v_3$. Any vertex of G not adjacent to v_1 or v_3 for which we have not yet accounted lies in the external region of the cycle $C_2 : v_1, y_3, y_2, y_4, v_3, v_4$. There are four subcases to consider.

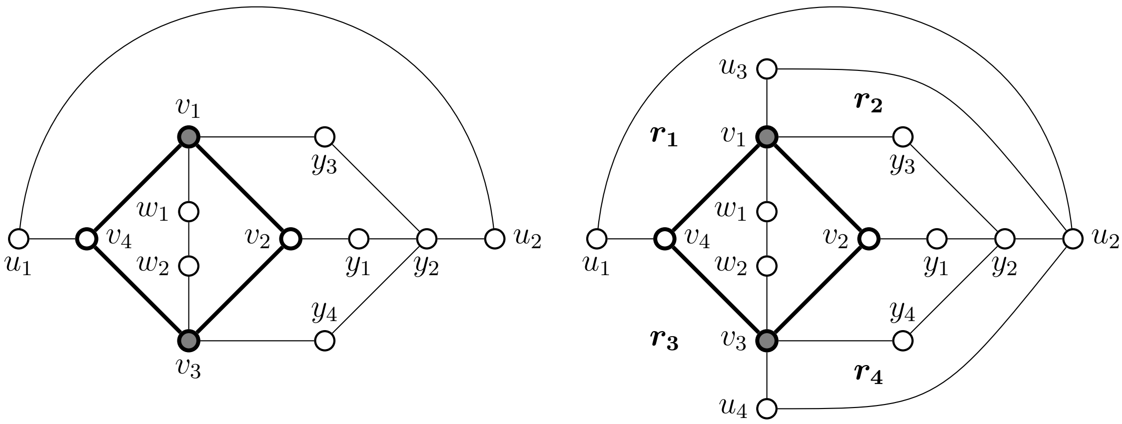
Case 1.1: There exists some vertex u_1 in $\mathrm{Ext}(C_2)$ adjacent to v_4.
Since G is triangle-free, u_1 is not adjacent to either v_1 or v_3. Because G does not contain two dislocated 4-cycles, u_1 is adjacent to neither y_3 nor y_4. Thus, any $u_1 - y_1$ geodesic contains the vertex y_2. Either u_1 is adjacent to y_2, or there exists a vertex u_2 in the exterior of C_2 such that $P_2 : u_1, u_2, y_2, y_1$ is a geodesic in G. If u_1 and y_2 are adjacent, then $\mathrm{Ext}(C_2)$ is subdivided into 2 regions: the region r_1 with vertices u_1, v_4, v_1, y_3 and y_2 on its boundary, and the region r_2 with u_1, v_4, v_3, y_4 and y_2 on its boundary. The subgraph $\mathcal{G} \cup \{u_1, u_1v_4, u_1y_2\}$ is an induced subgraph of G, so the only vertices on the boundary of r_1 within distance 2 of w_2 are the adjacent pair v_1 and v_4. The region r_1 is dominated by two adjacent vertices of the 5-cycle bounding it, so by Lemma 4.1, r_1 is empty. Similarly, the region r_2 is empty, so every vertex of G not yet mentioned is adjacent to either v_1 or v_3, and we can bound the order of G:

$$\begin{aligned} n &= |V(\mathcal{G}) \cup \{u_1\}| + |V(G) - V(\mathcal{G}) - \{u_1\}| \\ &\leq 11 + (d(v_1) - 4) + (d(v_3) - 4) \ \leq \ 2\Delta + 3 \ \leq \ 3\Delta - 1. \end{aligned}$$

This contradicts our assumption, and so the geodesic contains u_2 (see Figure 13). Let $\mathcal{G}' = \mathcal{G} \cup P_2 \cup \{u_1v_4\}$. By Lemma 7.1, $\mathcal{G}'$ is an induced subgraph of G. Since $d(u_2, w_1) \leq 3$, there is some vertex u_3 that is adjacent to both u_2 and v_1. Similarly, because $d(u_2, w_2) \leq 3$, there exists a vertex $u_4 \neq u_3$ that is adjacent to u_2 and v_4 (see Figure 13).

The region $\mathrm{Ext}(C_2)$ is divided into four subregions, all of which are bounded by 5-cycles. Label these regions: $r_1 = \mathrm{Int}(u_1, v_4, v_1, u_3, u_2)$, $r_2 = \mathrm{Int}(v_1, u_3, u_2, y_2, y_3)$, $r_3 = \mathrm{Ext}(u_1, v_4, v_3, u_4, u_2)$, $r_4 = \mathrm{Int}(v_3, y_4, y_2, u_2, u_4)$ (see Figure 13). The only two vertices on the boundary of r_1 within distance 2 of w_2 are v_1 and v_4. Thus the adjacent pair $\{v_1, v_4\}$ dominates r_1, and by Lemma 4.1, r_1 is empty. Similarly, r_3 is empty. The only vertex on the boundary of r_2 within distance 2 of w_2 is v_1, and so r_2 is dominated by v_1. By Lemma 4.1, the regions r_2 and r_4 are empty. We deduce that all vertices of G not yet mentioned lie in the interior of C_1, and hence are adjacent to either v_1 or v_3. This allows us to bound the order of G:

$$\begin{aligned} n &= |V(\mathcal{G}') \cup \{u_3, u_4\}| + |V(G) - V(\mathcal{G}') - \{u_3, u_4\}| \\ &\leq 14 + (d(v_1) - 5) + (d(v_3) - 5) \ \leq \ 2\Delta + 4 \ \leq \ 3\Delta - 1. \end{aligned}$$



$\mathcal{G}'$ $G[V(\mathcal{G}') \cup \{u_3, u_4\}]$

Figure 13: In Case 1.1 of the proof of Theorem 7.3, we assume that there is a vertex u_1 adjacent to v_4. As a result, we obtain first that $\mathcal{G}'$ is a subgraph of G (left), and then that G also contains the vertices u_3 and u_4 (right).

Case 1.2: There is some vertex u_1 in $\text{Ext}(C_2)$ that is adjacent to y_2, but no vertex in $\text{Ext}(C_2)$ adjacent to v_4.
Since G is triangle-free, u_1 is adjacent to neither y_3 nor y_4. Because G does not contain two dislocated 4-cycles, u_1 is adjacent to neither v_1 nor v_3. Because $d(u_1, w_1) \leq 3$ and $d(u_1, w_2) \leq 3$, there are vertices u_2 and u_3 in $\text{Ext}(C_2)$ such that $Q_1 : u_1, u_2, v_1$ and $Q_2 : u_1, u_3, v_3$ are paths in G. Note that $u_2 \neq u_3$, as this would induce a 4-cycle on the vertex set $\{u_2, v_1, v_4, v_3\}$. This 4-cycle is either dislocated from C_1, contradicting our assumption, or it is not dislocated from C_1, contradicting the maximality of C_1. Denote by $\mathcal{G}^*$ the graph $\mathcal{G} \cup Q_1 \cup Q_2 \cup \{y_2u_1\}$, and observe that $\mathcal{G}^*$ is chordless by Lemma 7.1 (see Figure 14).

Consider the cycle $C_5 : v_1, u_2, u_1, y_2, y_3$. The only vertex on the boundary of $\text{Int}(C_5)$ that is within distance 2 of w_2 is v_1, so v_1 dominates $\text{Int}(C_5)$. By Lemma 4.1, $\text{Int}(C_5)$ is empty. Similarly, the interior of the cycle $C_6 : v_3, u_3, u_1, y_2, y_4$ is empty. Observe that if every vertex of $G - \mathcal{G}^*$ were adjacent to v_1 or v_3, then the order of G would be bounded as follows:

$$n = |V(\mathcal{G}^*)| + |V(G) - V(\mathcal{G}^*)|$$
$$n \leq 13 + (d(v_1) - 5) + (d(v_3) - 5) \;\leq\; 2\Delta + 3 \;\leq\; 3\Delta - 1.$$

This contradicts our assumption, and thus there is a vertex x_1 of $G - \mathcal{G}^*$ not adjacent to v_1 or v_3. This vertex lies in the face of $\mathcal{G}^*$ bounded by $C_7 = u_2, u_1, u_3, v_3, v_4, v_1$, which we will refer to, without loss of generality, as the exterior of C_7. Since $\mathcal{G}^*$ is an induced subgraph of G, the distance $d_G(y_1, C_7) = 2$, and $\{v_1, v_3, u_1\}$ is the set of vertices of C_7 that are at distance exactly 2 from y_1. Because G has diameter 3, we conclude that x_1 is adjacent to u_1. Since G is both triangle-free and does not contain a pair of dislocated 4-cycles, the vertex x_1 is not adjacent to any of the vertices of

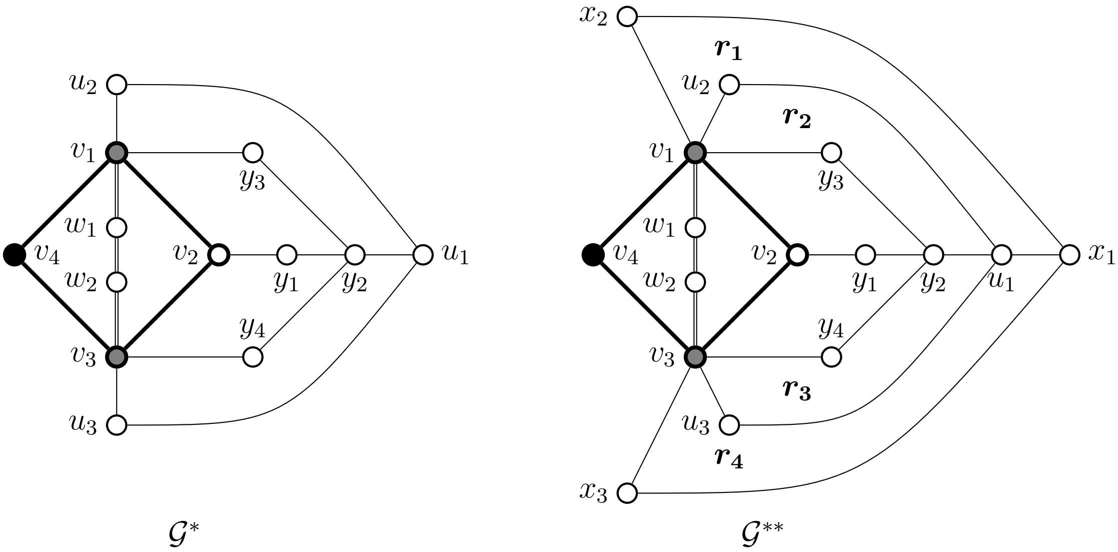


Figure 14: In Case 1.2, we obtain first that $\mathcal{G}^*$, and then $\mathcal{G}^{**}$, are subgraphs of G. The black vertex v_4 does not have any neighbors in G besides v_1 and v_3.

$V(C_7) - \{u_1\}$. As $d_G(x_1, w_1) \leq 3$ and $d_G(x_1, w_2) \leq 3$, there exist vertices x_2 and x_3 in $\mathrm{Ext}(C_7)$ such that $Q_3 : x_1, x_2, v_1$ and $Q_4 : x_1, x_3, v_3$ are paths in G. These two vertices are distinct, for if they were not, the 4-cycle on x_2, v_1, v_4, v_3 would be dislocated from C_1, a contradiction. Let $\mathcal{G}^{**} = \mathcal{G}^* \cup Q_3 \cup Q_4$ (see Figure 14 ($\mathcal{G}^{**}$)). We now label the regions of $\mathcal{G}^{**}$ as follows. Let $r_1 = \mathrm{Int}(v_1, x_2, x_1, u_1, u_2)$, $r_2 = \mathrm{Int}(v_1, u_2, u_1, y_2, y_3)$, $r_3 = \mathrm{Int}(v_3, u_3, u_1, y_2, y_4)$, $r_4 = \mathrm{Int}(v_3, x_3, x_1, u_1, u_3)$ and $r_0 = \mathrm{Ext}(v_1, x_2, x_1, x_3, v_3, v_4)$. Other than r_0, all of these regions are bounded by 5-cycles. The regions r_1 and r_2 are both empty, as the only vertex on either of their boundaries within distance 2 of w_2 is v_1, and by Lemma 4.1, no single vertex of a Jordan separating 5-cycle dominates the interior of that cycle. Similarly, the regions r_3 and r_4 are empty as the only vertex on their boundaries within distance 2 of w_1 is v_3. Any vertex of r_0 is adjacent to one of v_1 or v_3, as these are the only two vertices on the boundary of r_0 within distance 2 of y_1. Thus all vertices of $G - \mathcal{G}^{**}$ are adjacent to either v_1 or v_3. This yields the following contradiction, and shows that no vertex of $\mathrm{Ext}(C_2)$ is adjacent to y_2:

$$\begin{aligned} n &= |V(\mathcal{G}^{**})| + |V(G) - V(\mathcal{G}^{**})| \\ &\leq 16 + (d(v_1) - 6) + (d(v_3) - 6)) \ \leq\ 2\Delta - 4 \ \leq\ 3\Delta - 1. \end{aligned}$$

Case 1.3: There exists some vertex u_1 in $\mathrm{Ext}(C_2)$ that is adjacent to y_3, and no vertex of $\mathrm{Ext}(C_2)$ is adjacent to either y_2 or v_4.

Since G contains neither any 3-cycles, nor any pair of dislocated 4-cycles, the vertex u_1 is not adjacent to any vertex of $C_2 - \{v_3\}$. Thus there are only two ways we can have $d(u_1, w_2) \leq 3$: either G contains the edge u_1v_3, or there is some vertex u_2 in $\mathrm{Ext}(C_2)$ such that $S_1 : y_3, u_1, u_2, v_3$ is a path in G (see Figure 15).

Suppose that u_1 and v_3 are adjacent. Denote by S_2 the path y_3, u_1, v_3, and let $\mathcal{G}^\flat = \mathcal{G} \cup S_2$. By Lemma 7.1, $\mathcal{G}^\flat$ is an induced subgraph of G. The path S_2

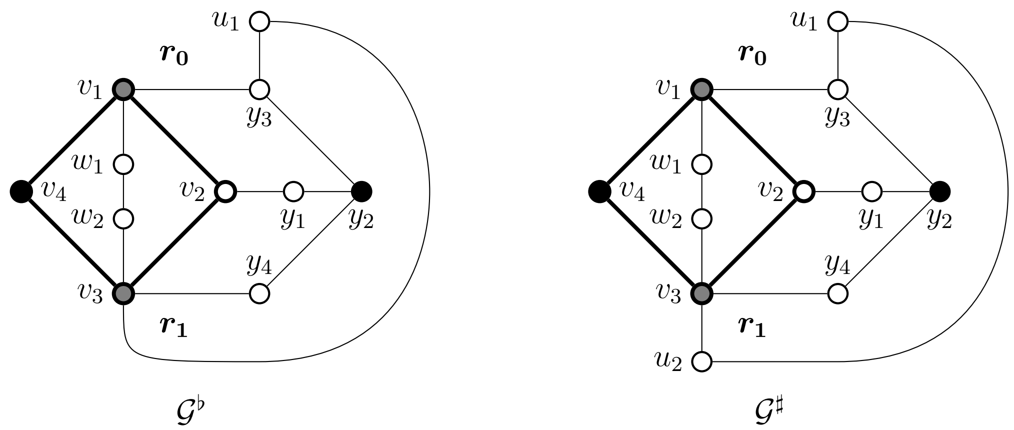


Figure 15: Case 1.3 assumes that there is a vertex u_1 adjacent to y_1. In this case, either $\mathcal{G}^\flat$ or $\mathcal{G}^\sharp$ is a subgraph of G. The black vertices may not have neighbors in G not shown in the diagrams.

divides $\text{Ext}(C_2)$ into two regions bounded by 5-cycles, $r_0 = \text{Ext}(v_1, y_3, u_1, v_3, v_4)$ and $r_1 = \text{Int}(y_3, u_1, v_3, y_4, y_2)$. The only vertices on the boundary of r_0 within distance 2 of y_1 are v_1, v_3 and y_3, so any vertex in r_0 is adjacent to one of these three. The only vertices on the boundary of r_1 within distance 2 of w_1 are v_3 and y_3, so the set $\{v_3, y_3\}$ dominates r_1, and we can bound the order of G.

$$
\begin{aligned}
n &= |V(\mathcal{G}^\flat)| + |V(G) - V(\mathcal{G}^\flat)| \\
&\leq 11 + (d(v_1) - 4) + (d(v_3) - 5) + (d(y_3) - 3) \ \leq \ 3\Delta - 1.
\end{aligned}
$$

Since this contradicts our assumption, the graph G contains the path S_1. Let $\mathcal{G}^\sharp = \mathcal{G} \cup S_1$, and observe by Lemma 7.1 that $\mathcal{G}^\sharp$ is an induced subgraph of G. The region $\text{Ext}(C_2)$ is divided into two sub-regions bounded by 6-cycles, $r_0 = \text{Ext}(v_1, y_3, u_1, u_2, v_3, v_4)$ and $r_1 = \text{Int}(y_3, u_1, u_2, v_3, y_4, y_2)$. The are only two vertices, y_3 and v_3, on the 6-cycle bounding r_1 within distance 2 of w_1. Thus $\{y_3, v_3\}$ dominates r_1, and so by Lemma 4.4, there is some vertex u_3 in r_1 that is adjacent to both y_3 and v_3. Let $\mathcal{G}^{\sharp\sharp} = \mathcal{G}^\sharp \cup \{u_3, u_3y_3, u_3v_3\}$. The only vertices on the boundary of r_0 within distance 2 of y_1 are v_1, v_3 and y_3, so every vertex of r_0 is adjacent to one of these three vertices. Thus:

$$
\begin{aligned}
n &= |V(\mathcal{G}^{\sharp\sharp})| + |V(G) - V(\mathcal{G}^{\sharp\sharp})| \\
&\leq 13 + (d(v_1) - 4) + (d(v_3) - 6) + (d(y_3) - 4) \ \leq \ 3\Delta - 1.
\end{aligned}
$$

This contradicts our assumption, and hence y_3 does not have a neighbor in $\text{Ext}(C_2)$. By the same argument, the vertex y_4 also does not have a neighbor in $\text{Ext}(C_2)$.

Case 1.4: The vertices v_4, y_2, y_3 and y_4 do not have any neighbors in $\text{Ext}(C_2)$. By cases 1.1 to 1.3, the only vertices of C_2 that can have neighbors in $\text{Ext}(C_2)$ are v_1 and v_3. Further, both v_1 and v_3 are at distance 2 from y_1, so any vertex in $\text{Ext}(C_2)$ is adjacent to either v_1 or v_3 in order to be within distance 3 of y_1. Hence we get the following bound on n:

$$n = |V(\mathcal{G})| + |V(G) - V(\mathcal{G})|$$
$$\leq 10 + (d(v_1) - 4) + (d(v_3) - 4) \ \leq \ 2\Delta + 2 \ \leq \ 3\Delta - 1.$$

In all subcases, $n \leq 3\Delta - 1$, and so the vertex v_2 does not have a neighbor in $\mathrm{Ext}(C_1)$. By symmetry, we further conclude that v_4 does not have any neighbors in $\mathrm{Ext}(C_1)$.

Case 2: Neither v_2 nor v_4 have any neighbors in G besides v_1 and v_3.
As $n > 3\Delta - 1$, there is some vertex y_1 in G that is not adjacent to either v_1 or v_3. Note that $d(y_1, C_1) > 1$, but $d(y_1, w_1) \leq 3$ and $d(y_1, w_2) \leq 3$. Therefore, there exist vertices y_2 and y_3 in the exterior of C_1 such that $P_2 : y_1, y_2, v_1$ and $P_3 : y_1, y_3, v_3$ are paths in G (see Figure 16 ($\mathcal{K}$)). Note that $y_2 \neq y_3$. If $y_2 = y_3$, then there is a 4-cycle on y_2, v_1, v_2, v_3, contradicting either the maximality of C_1, or the assumption that G does not contain two dislocated 4-cycles. Let $\mathcal{K} = C_1 \cup P_1 \cup P_2 \cup P_3$, and name the cycle $C_2 : v_1, y_2, y_1, y_3, v_3, v_4$ (see Figure 16). Observe that, by Lemma 7.1, the subgraph $\mathcal{K}$ is an induced subgraph of G. Since $n > 3\Delta - 1$ by assumption, there exists some vertex u_1 in $G - \mathcal{K}$ that is not adjacent to either v_1 or v_3. We may assume without loss of generality that u_1 is in $\mathrm{Ext}(C_2)$. The vertex u_1 is not adjacent to both of y_2 and y_3 as this creates a 4-cycle dislocated from C_1, contradicting our assumption. There are two cases to consider.

Case 2.1: The vertex u_1 is adjacent to y_2.
Since G contains neither triangles nor dislocated 4-cycles, u_1 is not adjacent to any vertex of $C_2 - \{y\}$. Since $d_G(u_1, w_2) \leq 3$, there is some vertex u_2 in $\mathrm{Ext}(C_2)$ such that $Q_1 : y_2, u_1, u_2, v_3$ is a path in G. By Lemma 7.1, the graph $\mathcal{K} \cup Q_1$ is an induced subgraph of G. Thus the interior of the 6-cycle $C_3 : y_2, u_1, u_2, v_3, y_3, y_1$ is dominated by y_2 and v_3, as these are the only vertices of the cycle within distance 2 of w_1. By Lemma 4.4, there exists a vertex u_3 in $\mathrm{Int}(C_3)$ such that $Q_2 : y_2, u_3, v_3$ is a path in G. The path Q_2 divides the region bounded by C_3 into two regions, each bounded by a 5-cycle. By Corollary 4.5, neither region contains any vertex of G. Let $\mathcal{K}'$ denote the graph $\mathcal{K} \cup Q_1 \cup Q_2$ (see Figure 16), and observe by Lemma 7.1 that it is an induced subgraph of G.

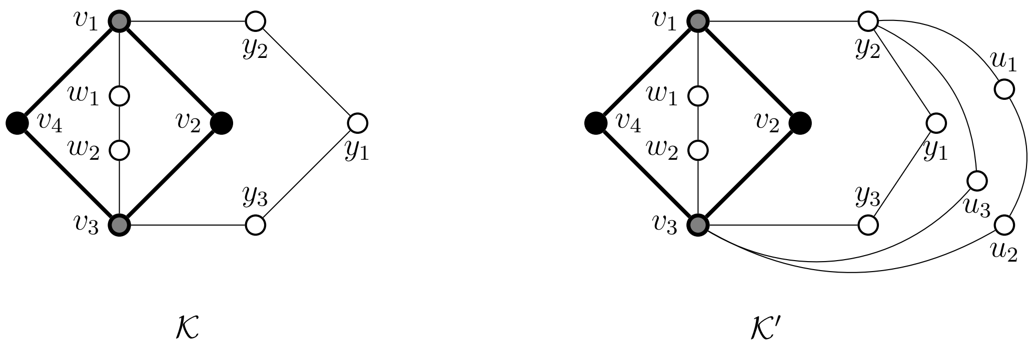


Figure 16: In Case 2, neither v_4 nor v_2 have neighbors other than v_1 and v_3. In this Case, G contains $\mathcal{K}$ as a subgraph. In Case 2.1, G contains $\mathcal{K}'$ as a subgraph.

If every vertex of $G - \mathcal{K}'$ is adjacent to one of v_1, v_3 or y_2, then we obtain the following contradiction:

$$n \leq 12 + (d(v_1) - 4) + (d(v_3) - 6) + (d(y_2) - 4) \ \leq \ 3\Delta - 2.$$

So there exists some vertex x_1 not adjacent to any of v_1, v_3 or y_2. Noting the symmetry between the interior of the cycle $C_4 : v_1, y_2, y_1, y_3, v_3, v_2$ and the exterior of the cycle $C_5 : v_1, y_2, u_1, u_2, v_3, v_4$, we may assume without loss of generality that x_1 is in the interior of C_4.

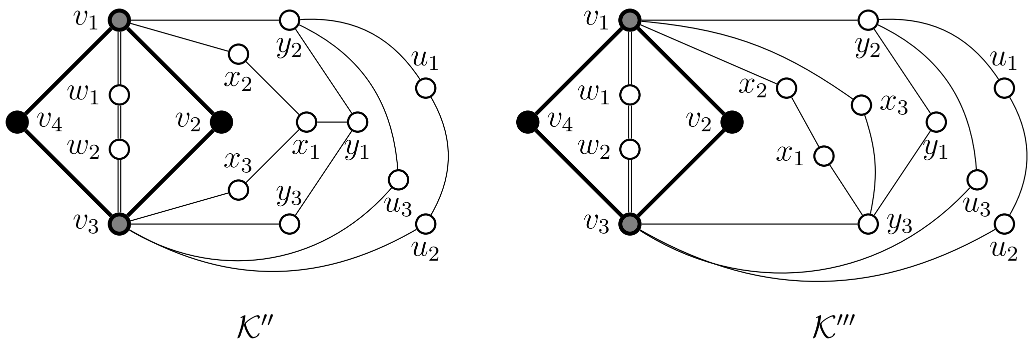


Figure 17: In Case 2.1.1, G has the graph $\mathcal{K}''$ as a subgraph. In Case 2.1.2, the graph $\mathcal{K}'''$ is a subgraph of G.

Case 2.1.1: The vertex x_1 is adjacent to y_1.
Since G contains neither triangles nor dislocated 4-cycles, x_1 has no neighbors in $C_4 - \{y_1\}$. Since there exist $x_1 - w_1$ and $x_1 - w_2$ geodesics, there are vertices x_2 and x_3 in $\text{Int}(C_4)$ such that $Q_3 : y_1, x_1, x_2, v_1$ and $Q_4 : y_1, x_1, x_3, v_3$ are paths in G. Since C_1 is maximal and G does not contain dislocated 4-cycles, the vertices x_2 and x_3 are distinct. Denote $\mathcal{K}'' = \mathcal{K}' \cup Q_3 \cup Q_4$ (see Figure 17).

The exterior of the cycle on $v_1, y_2, u_1, u_2, v_3, v_4$ is dominated by $\{v_1, v_3, y_2\}$, as these are the only vertices of the cycle within distance 2 of x_1. The two regions bounded by the 5-cycles on v_1, y_2, y_1, x_1, x_2 and v_3, y_3, y_1, x_1, x_3 do not contain any vertices by Lemma 4.1, as only v_1 of the former cycle is within distance 2 of w_2, and only v_3 of the latter is within distance 2 of w_1. Finally, the 6-cycle on the vertices $v_1, x_2, x_1, x_3, v_3, v_2$ is dominated by v_1 and v_3, as these are the only two vertices of the cycle within distance 2 of u_1. Thus every vertex of $G - \mathcal{K}''$ is adjacent to v_1, v_3 or y_2, and we obtain a contradiction:

$$\begin{aligned} n &= |V(\mathcal{K}'')| + |V(G) - V(\mathcal{K}'')| \\ &\leq 15 + (d(v_1) - 5) + (d(v_3) - 7) + (d(y_2) - 4) \ \leq \ 3\Delta - 1. \end{aligned}$$

Case 2.1.2: The vertex x_1 is adjacent to y_3.
The vertex x_1 is not adjacent to any vertex of $\mathcal{K} - \{y_3\}$. Since $d_G(x_1, w_1) \leq 3$, there exists a vertex x_2 such that $Q_5 : y_3, x_1, x_2, v_1$ is a path in G. Consider the 6-cycle $C_6 : v_1, y_2, y_1, y_3, x_1, x_2$. The only vertices of C_6 within distance 2 of w_2 are v_1 and y_3. So by Lemma 4.4, there is a vertex x_3 in $\text{Int}(C_6)$ such that $Q_6 : v_1, x_3, y_3$ is a path in G. The path Q_6 divides $\text{Int}(C_6)$ into two regions bounded by 5-cycles, both

dominated by $\{v_1, y_3\}$. Denote $C_7 : v_1, y_2, u_1, u_2, v_3, v_4$. The only vertices of C_7 within distance 2 of x_1 are v_1 and v_3, so $\text{Ext}(C_7)$ is dominated by $\{v_1, v_3\}$. The interior of the 6-cycle on $v_1, x_2, x_1, y_3, v_3, v_2$ is dominated by v_1 and v_3, as these are the only two vertices of the cycle within distance 2 of u_1. Thus, letting $\mathcal{K}''' = \mathcal{K}' \cup Q_5 \cup Q_6$ (see Figure 17), we derive a contradiction:

$$\begin{aligned} n &= |V(\mathcal{K}''')| + |V(G) - V(\mathcal{K}''')| \\ &\leq 15 + (d(v_1) + 6) + (d(v_3) - 6) + (d(y_3) - 4) \ \leq \ 3\Delta - 1. \end{aligned}$$

Case 2.1.3: The vertex x_1 is not adjacent to any vertex of $\mathcal{K}'$.
By the same argument as in Case 2.1.1, there are distinct vertices x_1 and x_2 in $\text{Int}(C_2)$ such that $Q_7 : x_1, x_2, v_1$ and $Q_8 : x_1, x_3, v_3$ are paths in G. Denote $\mathcal{K}^* = \mathcal{K}' \cup Q_7 \cup Q_8$ and consider the cycle $C_8 : x_1, x_2, v_1, y_2, y_1, y_3, v_3, x_3$. By Lemma 7.1, the interior of C_8 is the only region of $\mathcal{K}^*$ that may contain a chord of $\mathcal{K}^*$. Because G contains neither triangles nor dislocated 4-cycles, and x_1 is not adjacent to y_1, the only possible chords of $\mathcal{K}^*$ are x_2y_3 and x_3y_2. Since $d(u_1, x_1) \leq 3$, either x_3 is adjacent to y_2, or there is some vertex y_4 adjacent to both x_1 and y_2.

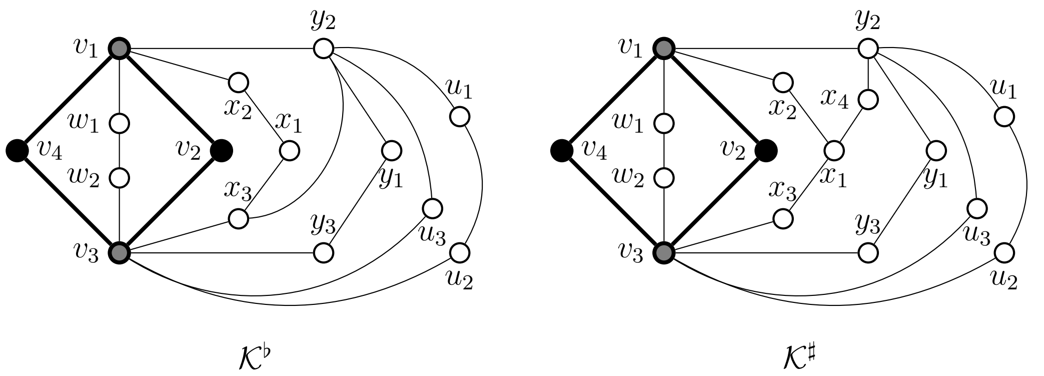


Figure 18: In the first sub-case of 2.1.3, the vertices y_2 and x_3 are adjacent, and G contains the subgraph $\mathcal{K}^\flat$. In the second sub-case, there is a vertex x_4 adjacent to both x_1 and y_2, and G contains the subgraph $\mathcal{K}^\sharp$.

Subcase 2.1.3 - 1: The vertices x_3 and y_2 are adjacent.
Observe by Lemma 7.1 that $\mathcal{K}^* \cup \{x_3y_2\}$ is an induced subgraph of G. Since $d(x_2, y_3) \leq 3$, there is a vertex x_4 adjacent to both v_3 and x_2. Denote $\mathcal{K}^\flat = \mathcal{K}^* \cup \{x_4, x_3y_2, v_3x_4, x_2x_4\}$. The exterior of the cycle on $v_1, y_2, u_1, u_2, v_3, v_4$ is dominated by v_1, v_3 and y_2, as these are the only vertices of the cycle within distance 2 of x_1. The interior of the 5-cycle on v_1, x_2, x_4, v_3, v_2 is dominated by v_1 and v_3, as only these vertices of the cycle are within distance 2 of u_1. The cycle on x_2, x_1, x_3, v_3, x_4 is dominated by x_3 and v_3 as these are the only two vertices within distance 2 of u_1, and so by Lemma 4.1 the interior of this cycle contains no vertices. The interior of the 5-cycle on y_2, y_1, y_3, v_3, x_3 is dominated by v_3 and y_2, as only these vertices of the cycle are at distance 2 from w_1. The interior of the 5-cycle on v_1, y_2, x_3, x_1, x_2 is also empty by Lemma 4.1, as only y_2 and x_3 are within distance 2 of y_3. Since the vertices of G not in $\mathcal{K}^\flat$ are all adjacent to one of v_1, v_3 or y_2, we can bound the order

of G.

$$n = |V(\mathcal{K}^\flat)| + |V(G) - V(\mathcal{K}^\flat)|$$
$$\leq 16 + (d(v_1) - 5) + (d(v_3) - 8) + (d(y_2) - 5) \ \leq \ 3\Delta - 2.$$

Subcase 2.1.3 - 2: The graph G contains a vertex x_4 that is adjacent to x_1 and y_2. Let $\mathcal{K}^\sharp$ be the subgraph $\mathcal{K}^* \cup \{x_4, x_1x_4, y_2x_4\}$ of G, and observe by Lemma 7.1 that $\mathcal{K}^\sharp$ is an induced subgraph of G. The exterior of the cycle on $v_1, y_2, u_1, u_2, v_3, v_4$ is dominated by v_1, v_3 and y_2, as these are the only vertices of the cycle within distance 2 of x_1. The 7-cycle on $y_2, y_1, y_3, v_3, x_3, x_1, x_4$ is dominated by y_2 and v_3 as these are the only vertices within distance 2 of w_1. The interior of the 5-cycle on v_1, y_2, x_4, x_1, x_2 is empty by Lemma 4.1, as it is dominated by v_1, the only vertex of the cycle within distance 2 of w_2. The interior of the 6-cycle on $v_1, x_2, x_1, x_3, v_3, v_2$ is dominated by v_1 and v_3, the only vertices of the cycle within distance 2 of u_1. Every vertex of G that is not in $\mathcal{K}^\sharp$ is adjacent to one of v_1, v_3 or y_2, so the order of G is bounded above:

$$n = |V(\mathcal{K}^\sharp)| + |V(G) - V(\mathcal{K}^\sharp)|$$
$$\leq 16 + (d(v_1) - 5) + (d(v_3) - 7) + (d(y_2) - 5) \leq 3\Delta - 1.$$

Case 2.2: The vertex u_1 is not adjacent to y_2 or y_3.
Since $d_G(u_1, w_1) \leq 3$ and $d_G(u_1, w_2) \leq 3$, there exist vertices u_2 and u_3 in G such that $S_1 : u_1, u_2, v_1$ and $S_2 : u_1, u_3, v_3$ are paths in G. The vertices u_2 and u_3 are distinct, by the maximality of C_1 and the fact that G contains no dislocated 4-cycles. By Case 2.1, neither y_2 nor y_3 can have a neighbor in $G - \mathcal{K}$ which is not adjacent to v_1 or to v_3. By symmetry, neither u_2 nor u_3 can have any neighbor in $G - \{u_1\}$ that is not adjacent to v_1 or to v_3. Since G contains neither triangles nor dislocated 4-cycles, the only possible chords of the cycle on $v_1, u_2, u_1, u_3, v_3, y_3, y_1, y_2$ are y_1u_1, y_2u_3 and y_3u_2. Up to symmetry, this leaves three possible ways to construct a $u_1 - y_1$ geodesic in G: with the edge y_2u_3, with the edge u_1y_1, or by (possibly repeated) subdivision of the edge u_1y_1. We let $\mathcal{L} = \mathcal{K} \cup S_1 \cup S_2$ (see Figure 19).

Case 2.2.1: The vertices y_2 and u_3 are adjacent.
By Lemma 7.1, the subgraph $\mathcal{L} \cup \{y_2u_3\}$ is an induced subgraph of G. Since $d_G(y_3, u_2) \leq 3$, there exists some vertex x_1 in G such that either $S_3 : y_3, x_1, v_1$ or $S_4 : y_3, v_3, x_1, u_2$ is a path in G. Up to relabeling of the vertices and choosing the region bounded by $v_1, y_2, y_1, y_3, v_3, v_2$ to be the exterior region of our subgraph, these possibilities are the same. Hence we assume without loss of generality that S_3 is a $y_3 - u_2$ geodesic, and we denote by $\mathcal{L}'$ the graph $\mathcal{L} \cup \{y_2u_3\} \cup S_3$ (see Figure 19). The interior of the 5-cycle on v_1, v_2, v_3, y_3, x_1 is dominated by v_1 and v_3 as these are the only vertices of the cycle within distance 2 of u_1. The interiors of the two 5-cycles on v_1, y_2, y_1, y_3, x_1 and v_1, u_2, u_1, u_3, y_2 are dominated by the pairs v_1, y_3 and v_1, u_3 respectively, as these are the only vertices on the cycles within distance 2 of w_2. The interior of the 5-cycle on y_2, u_3, v_3, y_3, y_1 is dominated by y_2 and v_3, these being the only vertices of the cycle within distance 2 of w_1. By Lemma 7.2, all four of the regions mentioned are empty. All vertices of G not in $\mathcal{L}'$ lie in the exterior of

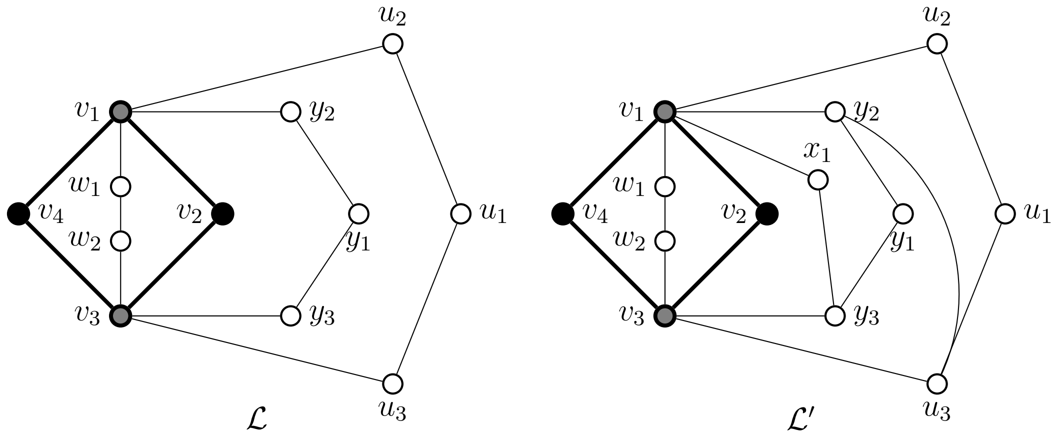


Figure 19: The graph G contains the subgraph $\mathcal{L}$ in Case 2.2. It contains the subgraph $\mathcal{L}'$ in Case 2.2.1.

the cycle on $v_1, u_2, u_1, u_3, v_3, v_4$. The vertices of this cycle within distance 2 of y_1 are v_1, v_3 and u_3. Hence:

$$\begin{aligned} n &= |V(\mathcal{L}')| + |V(G) - (V(\mathcal{L}'))| \\ &\le 13 + (d(v_1) - 6) + (d(v_3) - 5) + (d(u_3) - 3) \le 3\Delta - 1. \end{aligned}$$

This contradicts our assumption, so y_2 and u_3 are not adjacent. By symmetry, y_3 and u_2 are not adjacent.

Case 2.2.2: The vertices u_1 and y_1 are adjacent.
Note the interiors of the two 5-cycles on v_1, u_2, u_1, y_1, y_2 and v_3, u_3, u_1, y_1, y_3 are dominated by only the vertices v_1 and v_3 respectively, these being the only vertices of the cycles within distance 2 of w_2 and w_1 respectively. Thus by Lemma 4.1, both interiors are empty. Since $n > 3\Delta - 1$, there exists some vertex x_1 in $G - \mathcal{L}$ that is not adjacent to v_1 or v_3. By symmetry between the exterior of the cycle on $v_1, u_2, u_1, u_3, v_3, v_4$ and the interior of the cycle on $v_1, y_2, y_1, y_3, v_3, v_2$, we assume without loss of generality that x_1 is in the interior of the latter cycle. By Case 2.1, the vertex x_1 is not adjacent to y_2 or y_3. By the same argument as the one at the start of Case 2.2, there exist distinct vertices x_2 and x_3 in G such that $S_5 : x_1, x_2, v_1$ and $S_6 : x_1, x_3, v_3$ are paths in G. Let $\mathcal{L}''$ denote the graph $\mathcal{L} \cup \{y_1 u_1\} \cup S_5 \cup S_6$. Using both Lemma 7.1, and the fact that G contains neither triangles nor dislocated 4-cycles, we see that the only possible chords of $\mathcal{L}''$ are $x_1 y_1$, $x_2 y_3$ and $x_3 y_2$. The only possibilities for an $x_1 - u_1$ geodesic of length at most 3 require that G contains the edge $x_1 y_1$, or path x_1, z_1, y_1, containing some new vertex z_1. Let $\mathcal{L}^\flat = \mathcal{L}'' \cup \{x_1 y_1\}$ and $\mathcal{L}^\sharp = \mathcal{L}'' \cup \{z_1, x_1 z_1, z_1 y_1\}$ (see Figure 20).

Suppose that G contains the path x_1, z_1, y_1. By Lemma 7.1, the subgraph $\mathcal{L}^\sharp$ is an induced subgraph of G. Since $d_G(z_1, w_1) \le 3$ and $d_G(z_1, w_2) \le 3$, there exist vertices z_2 and z_3 such that $S_7 : z_1, z_2, v_1$ and $S_8 : z_1, z_3, v_3$ are paths in G. By swapping the labels $z_1 \leftrightarrow x_1$, $z_2 \leftrightarrow x_2$ and $z_3 \leftrightarrow x_3$, we obtain $\mathcal{L}^\flat$ as a subgraph of G. Thus to complete the proof of Case 2.2.2, it suffices to prove the following claim.

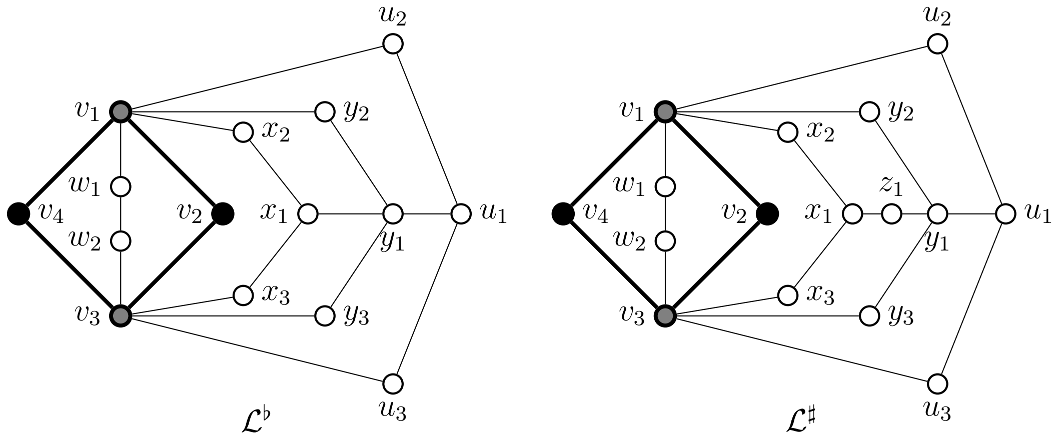


Figure 20: In Cases 2.2.2 and 2.2.3, the graph G always contains $\mathcal{L}^\flat$ as a subgraph. If, in Case 2.2.2, G contains $\mathcal{L}^\sharp$ as a subgraph, it will inevitably also have a $\mathcal{L}^\flat$ subgraph.

Claim: If G contains $\mathcal{L}^\flat$ as a subgraph, then $n \leq 3\Delta - 1$.

Consider the subgraph $\mathcal{L}^\flat$, and note that it is an induced subgraph of G by Lemma 7.1. There exist $x_2 - u_3$ and $x_3 - u_2$ geodesics of length at most 3 in G. Since $\mathcal{L}^\flat$ is an induced subgraph of G, there are only two possible $x_2 - u_3$ geodesics, both of which use some vertex t_1 in $G - \mathcal{L}^\flat$. These possible geodesics are $X_1 : x_2, v_1, t_1, u_3$ and $X_2 : x_2, t_1, v_3, u_3$. Up to relabeling of the vertices, and making the face of $\mathcal{L}^\flat$ bounded by $v_1, x_2, x_1, x_3, v_3, v_2$ the outer face of the graph, the two plane graphs $\mathcal{L}^\flat \cup X_1$ and $\mathcal{L}^\flat \cup X_2$ are the same. Thus we assume without loss of generality that X_1 is a geodesic in G. By Lemma 7.1, the subgraph $\mathcal{L}^\flat \cup X_1$ is an induced subgraph of G. The only possible $x_3 - u_2$ geodesic is $X_3 : x_3, t_2, v_1, u_2$, where t_2 is not among the vertices mentioned thus far. Let $\mathcal{L}^* = \mathcal{L}^\flat \cup X_1 \cup X_2$, and observe that it is an induced subgraph of G by Lemma 7.1. The interior of the 5-cycle on v_1, t_2, x_3, v_3, v_2 is dominated by v_1 and v_3, these being the only vertices of the cycle within distance 2 of u_1. The interior of the 5-cycle on v_1, x_2, x_1, x_3, t_2 is dominated by v_1 and x_3, as these are the only vertices of the cycle within distance 2 of w_2. Similarly, the two regions bounded by 5-cycles that contain the vertex t_1 are also dominated by just two vertices. The interiors of the two 5-cycles on v_1, y_2, y_1, x_1, x_2 and v_3, y_3, y_1, x_1, x_3 are dominated by only v_1 and v_3 respectively, these being the only vertices of each cycle within distance 2 of w_1 and w_1, respectively. Thus, all the regions mentioned above are empty by Lemma 7.2. As such, every vertex of $G - \mathcal{L}^*$ is in the interior of C_1, and hence adjacent to v_1 or to v_3. Hence we prove the claim with the following contradiction:

$$\begin{aligned} n &= |V(\mathcal{L}^*)| + |V(G) - V(\mathcal{L}^*)| \\ &\leq 17 + (d(v_1) - 8) + (d(v_3) - 6) \\ &\leq 2\Delta + 3 \ \leq \ 3\Delta - 1. \end{aligned}$$

Case 2.2.3: The $y_1 - u_1$ geodesic is the single edge y_1u_1, subdivided either once or twice into a path of length 2 or 3 respectively. Assume there exists some vertex x_1 in $G - \mathcal{L}$ on the path $Y_1 : y_1, x_1, u_1$ in G, and note that $\mathcal{L} \cup Y_1$ is an induced subgraph of G by Lemma 7.1. Since the distance between x_1 and the vertices w_1 and w_2 is at most 3, there are paths x_1, x_2, v_1 and x_1, x_3, v_3 in G. But now we see that $\mathcal{L}^\flat$ is a subgraph of G, and $n \leq 3\Delta - 1$ by the claim in Case 2.2.2. If we instead assume that there are vertices x_1 and z_1 on the path $Y_2 : y_1, x_1, z_1, u_1$, we again see that $\mathcal{L} \cup Y_2$ is an induced subgraph of G, and that G contains paths x_1, x_2, v_1 and x_1, x_3, v_3. Similarly, the graph G will also have paths z_1, z_2, v_1 and z_1, z_3, v_3, and we see that G contains $\mathcal{L}^\flat$ as a subgraph. Again invoke the claim in Case 2.2.2 to complete the proof. □

8 Bounding the order, part III: Not a 4-cycle in sight

In this section, we show that a pentagulation G of diameter 3, order n and maximum degree $\Delta \geq 8$ contains at least one 4-cycle. The restriction $\Delta \geq 8$ is used heavily. As demonstrated by the rightmost graph in Figure 33, pentagulations of diameter 3 and $\Delta \leq 6$ need not have 4-cycles.

Lemma 8.1. *Let G be a pentagulation with girth 5, and let v be a vertex of G. Then $N(v)$ is an independent set, every vertex of $N_2(v)$ has a unique neighbor in $N(v)$, and every vertex of $N(v)$ has at least one neighbor in $N_2(v)$.*

Proof. Since G contains no triangles, $N(v)$ is an independent set. Because G contains no 4-cycles, any vertex of $N_2(v)$ has exactly one neighbor in $N(v)$. As G is 2-connected and triangle-free, every vertex of $N(u)$ has a neighbor in $N_2(v)$. □

Lemma 8.2. *If G is a pentagulation of girth 5, then G is either the cycle C_5, or G does not contain two adjacent vertices of degree 2.*

Proof. Assume to the contrary that G is a pentagulation of girth 5 other than C_5 that contains two adjacent vertices x and y of degree 2. Let w be the single vertex of $N_1(x) - \{y\}$ and z the vertex of $N_1(y) - \{x\}$. The path $P : w, x, y, z$ lies on the boundary of two distinct faces f_1 and f_2 of G, each bounded by 5-cycles. Thus there exist two distinct vertices u and v that are both adjacent to w and z. Hence there is a 4-cycle u, w, v, x, contradicting the girth of G. □

Consider a vertex v in a pentagulation G. Let $\boldsymbol{\mathcal{F}}$ be the subgraph of G consisting of the edges and vertices that lie on the boundary of any face incident with v. Given two vertices x and y of $N_2(v)$, call an $x - y$ path Q of length k a ***k*-chord** (with respect to v) if both $(Q - \{x, y\}) \cap N_2(v) = \emptyset$ and $E(Q) \cap E(\mathcal{F}) = \emptyset$.

For example, consider the subgraph of a girth 5 pentagulation shown in Figure 21. The path $P : w_1, w_5$ is a 1-chord with respect to v, while $Q : w_5, z, w_8$ is a 2-chord. The edge w_1w_2 is not a 1-chord, since it belongs to $\mathcal{F}$. Notice that $\mathcal{F} \cup P$ contains a cycle $C_P : w_1, w_5, u_3, v, u_1$ formed by taking the union of the $w_1 - w_5$ 1-chord P

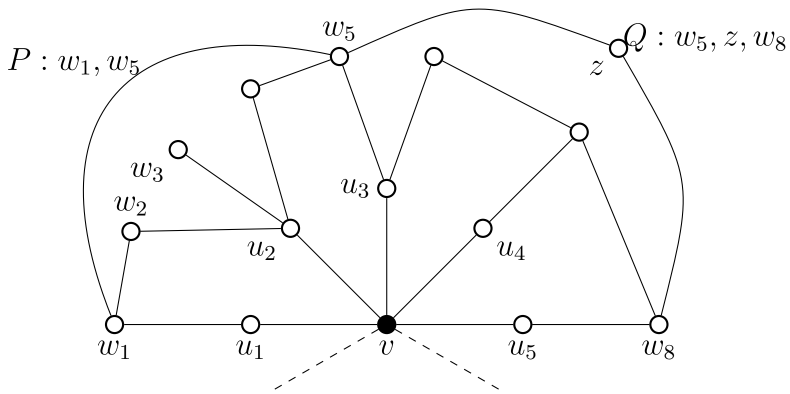


Figure 21: A vertex v in a pentagulation of girth five, and some of the edges and vertices near it. The dashed lines indicate some edges to parts of the graph not shown.

and the two unique $v - w_1$ and $v - w_5$ geodesics. One can construct another cycle $C_Q : w_5, z, w_8, u_5, v, u_3$ in the same fashion.

As the next lemma demonstrates, 1-chords and 2-chords with respect to some vertex will always induce cycles in the same manner that P and Q induce C_P and C_Q.

Lemma 8.3. *Let G be a pentagulation with girth 5, and let v be a vertex of G such that $d(v) \geq 8$. Given distinct vertices x and y of $N_2(v)$, let $P : x, \ldots, y$ be a k-chord of v, and let u_x and u_y denote the unique vertices in $N(v) \cap N(x)$ and $N(v) \cap N(y)$ respectively. If $k \leq 2$, then u_x and u_y are distinct, and P, u_y, v, u_x is a Jordan separating cycle.*

Proof. There are unique vertices u_x and u_y as described, by Lemma 8.1. Assume to the contrary that $k \leq 2$, but that $u_x = u_y$. The cycle P, u_y has length $k + 2 < 5$, which contradicts the fact that $g(G) = 5$. Thus $u_x \neq u_y$, and so $C_P : P, u_y, v, u_x$ is a cycle. It remains to show that C_P is Jordan separating. Since C_P is a cycle of length 5 or 6, and $E(P) \cap E(\mathcal{F}) = \emptyset$, the cycle C_P is neither a facial cycle (P does not share an edge with a face incident to v), nor does it have any chords (as the girth of G is 5). Thus C_P is a Jordan separating cycle. □

Let v be a vertex of a girth 5 pentagulation, and let the path $Q : x, \ldots, y$ be a k-chord, for $k \in \{1, 2\}$, with respect to v. If u_x and u_y are the unique vertices of $N(v)$ adjacent to x and y respectively, then the cycle $C_Q : Q, u_y, v, u_x$ is the **cycle under *Q***. The chord Q is said to be **minimal** if C_Q dominates its interior, and there does not exist any k-chord (of the same length) Q' such that $\text{Int}(C_{Q'}) \subset \text{Int}(C_Q)$.

Theorem 8.4. *Let G be a diameter 3, girth 5 pentagulation of maximum degree Δ, and let v be a vertex of G with maximum degree. If $\Delta \geq 8$, then there do not exist any 1-chords with respect to v.*

Proof. We assume to the contrary that there exist vertices w'_0 and w'_j in $N_2(v)$, and some 1-chord $Q' : w'_1, w'_j$ with respect to v. Label the vertices of $N(v) =$

$\{u'_0, u'_1, \ldots, u'_{\Delta-1}\}$ in clockwise order, so that u'_i and u'_{i+1} always lie on the boundary of the same face (subscripts taken modulo Δ). Let u'_0 and u'_j be the unique, distinct neighbors of w'_0 and w'_j respectively (these exist by Lemmas 8.1 and 8.3). Let $C_{Q'}$ denote the cycle under Q' with respect to v. By Lemma 8.3, $C_{Q'}$ is a Jordan separating cycle. Since the diameter of G is 3, the cycle $C_{Q'}$ dominates either its interior or its exterior. Embed G such that $C_{Q'}$ dominates its interior, and let Q be a minimal 1-chord in $\mathrm{Int}[C_{Q'}]$ (it is possible that $Q = Q'$). Relabel the vertices of $N(v)$ and $N_2(v)$ so that the start and end vertices of Q are labeled w_0 and w_j respectively, the neighbors u_i of $N(v)$ are still in clockwise order, and w_0u_0, w_ju_j are edges of $E(G)$. Let f_i be the face incident with v that has vertices u_i and u_{i+1} on its boundary.

Claim 1: The inequality $j < 3$ holds (i.e., the interior of C_Q contains at most two faces incident with v).

We first assume to the contrary that $j \geq 4$ (see Figure 22). Let w_2 be a vertex of $N_2(v) \cap N(u_2)$ (which exists by Lemma 8.1). Since C_Q dominates its interior, w_2 is adjacent to some vertex of C_Q. Because G has girth 5, w_2 is not adjacent to any of u_0, v or u_j. By the minimality of Q, w_2 is not adjacent to either w_0 or w_j, a contradiction.

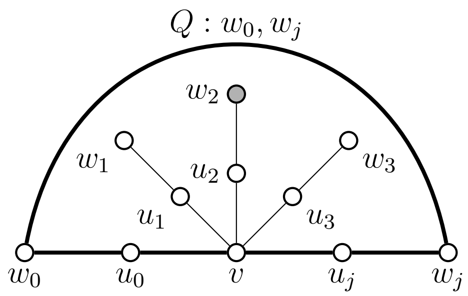


Figure 22: This figure shows Claim 1 of Theorem 8.4. The cycle C_Q under the 1-chord Q is bold, and the unique $N_2(v)$ neighbor w_2 of u_2 is grey.

Now suppose for the sake of contradiction that $j = 3$. Let w_1 be a vertex of $N(u_1) \cap N_2(v)$, and w_2 a vertex of $N(u_2) \cap N_2(v)$. By minimality of Q, w_1 is not adjacent to w_j. Since G has girth 5, w_1 is not adjacent to u_0, v or u_j. Because C_Q dominates its interior, w_1 is adjacent to w_0. Similarly, w_2 is adjacent to w_j, but not to w_0. This leaves two cases to consider.

Claim 1, Case 1: The degrees of u_1 and u_2 satisfy $d(u_1) = d(u_2) = 2$.
The path w_1, u_1, v, u_2, w_2 lies along the boundary of a face of G, so w_1 and w_2 are adjacent (see Figure 23 (1)). Thus the vertices w_0, w_1, w_2, w_j lie on a 4-cycle, contradicting the girth of G.

Claim 1, Case 2: either u_1 or u_2 has degree at least three.
Assume without loss of generality that u_1 has a vertex w'_1 of $N(u_1) \cap N_2(v)$ other than w_1 (see Figure 23 (2)). Since C_Q dominates its interior and G has no cycles of length 3 or 4, w'_1 is adjacent to either w_0 or w_j. The cycle under either the chord $w_0w'_1$ or the chord $w_jw'_1$ is contained strictly in $\mathrm{Int}[C_Q]$, contradicting the minimality

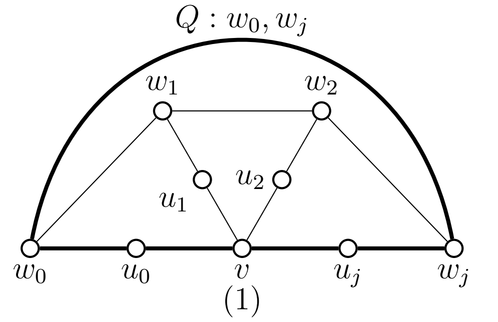


Figure 23: If $j = 3$ in the proof of Claim 1, there are two possibilities. Either both u_1 and u_2 have degree two (1), as in Claim 1 Case 1, or one of them has degree at least three (2), as in Claim 1 Case 2.

of Q and proving Claim 1.

Since $j < 3$, there are at least five neighbors u_3, u_4, …, $u_{\Delta-1}$ of v in $\mathrm{Ext}(C_Q)$. We consider cases, according to whether or not w_0 and w_j have neighbors in $\mathrm{Int}(C_Q)$.

Case 1: Neither w_0 nor w_j have any neighbors in $\mathrm{Int}(C_Q)$.
In $\mathrm{Int}[C_Q]$, the only neighbors of w_0 are u_0 and w_j, and the only neighbors of w_j are u_j and w_0. Thus the path $P : u_0, w_0, w_j, u_j$ lies on the boundary of a face contained in $\mathrm{Int}(C_Q)$, so there is a vertex x such that the cycle P, x bounds a face. By the assumption that w_0w_j is a 1-chord with respect to v, we have $x \neq v$. Thus there is a 4-cycle on v, u_0, x, u_j, a contradiction (see Figure 24).

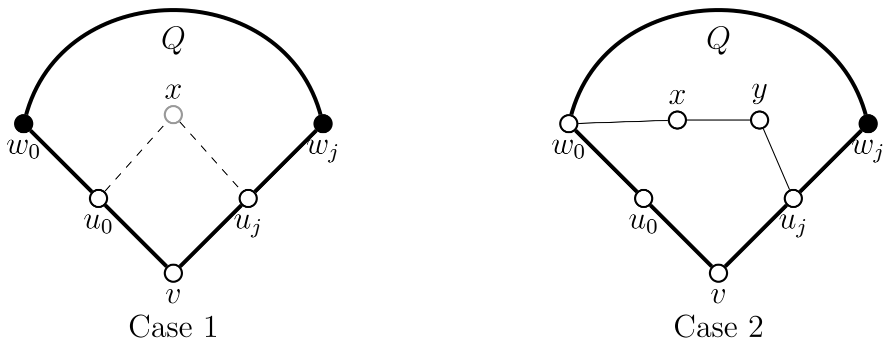


Figure 24: In Case 1, we assume that neither w_0 nor w_j has neighbors in $\mathrm{Int}(C_Q)$ (and colour these vertices black to indicate this). In Case 2, we assume that w_0 has a neighbor in $\mathrm{Int}(C_Q)$, but w_j does not.

Case 2: Either w_0 or w_j has a neighbor in $\mathrm{Int}(C_Q)$, but not both.
Assume without loss of generality that there is a vertex x in $\mathrm{Int}(C_Q)$ that is adjacent to w_0. If there are multiple vertices in $N_1(w_0) \cap \mathrm{Int}(C_Q)$, choose x such that the edges w_0w_j and w_0x lie on the boundary of a common face. Because w_j has no neighbor in $\mathrm{Int}(C_Q)$, the path $P : u_j, w_j, w_0, x$ lies on the boundary of some face f in the interior of C_Q. Thus there is some vertex y in $\mathrm{Int}[C_Q]$ such that the cycle P, y bounds f. As

G has girth 5, the vertex y is in $N_2(v)$ (see Figure 24). There are a number of cases to consider, based on the structure of the faces f_j and f_{j+1}.

Case 2.1: There is some vertex s in $N_1(w_j) \cap N_1(u_{j+1})$, and $d(u_{j+1}) = 2$.
Let t be the neighbor of s on the boundary of the face f_{j+1}, and observe that t and u_{j+2} are adjacent (see Figure 25). Since the girth of G is 5, we observe the following:

(1) the vertex w_j has no neighbors in the cycle v, u_j, w_j, s, u_{j+1} besides v and w_j;

(2) the vertex t is not adjacent to either w_0 or w_j;

(3) the vertex y is not adjacent to u_0, w_0 or w_j.

Thus there is no possible $y - t$ path of length 3 or less, a contradiction.

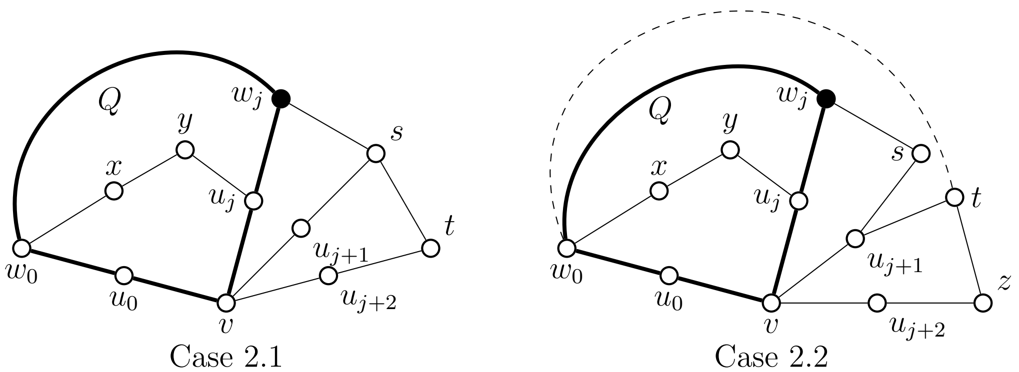


Figure 25: The diagram on the left illustrates Case 2.1, in which $d(u_{j+1}) = 2$ and the vertex of $N_2(v) \cap N(u_{j+1})$ is adjacent to w_j. On the right is Case 2.2, in which $d(u_{j+1}) > 2$, and some vertex of $N_2(v) \cap N(u_{j+1})$ is adjacent to w_j.

Case 2.2: There is a vertex s in $N_1(w_j) \cap N_1(u_{j+1})$, and $d(u_{j+1}) \geq 3$.
Since u_{j+1} has at least two neighbors in $N_2(v)$, the neighbor t of u_{j+1} on the boundary of f_{j+1} that is at distance 2 from v is distinct from s. Let z be the vertex of $N_2(v) - \{t\}$ incident with f_{j+1} (see Figure 25). Since G has girth 5, t is not adjacent to w_j. Since $d(t, y) \leq 3$, the vertices t and w_0 are adjacent.

Because the diameter of G is 3, the vertices t and w_0 are adjacent to ensure that $d(t, y) \leq 3$. The vertex z is not adjacent to any vertex within distance 2 of y by planarity, and the fact that G has girth 5. Thus $d(z, y) > 3$, contradicting the diameter of G.

Case 2.3: There is no vertex in $N_1(w_j) \cap N_1(u_{j+1})$.
Let s and t be the vertices of $N_2(v)$, incident with f_j, and adjacent to u_j and u_{j+1} respectively. Note that s and t are adjacent. If t is incident with the face f_{j+1}, then t has a neighbor z in $N(u_{j+2})$ that is also incident with f_{j+1} (see Figure 26 (1)). If t is not incident with f_{j+1}, then there is a vertex z' in $N(u_{j+1}) - \{t\}$ that is incident with f_{j+1} (see Figure 26 (2)). There are three ways to construct a $t - x$ geodesic of length at most 3.

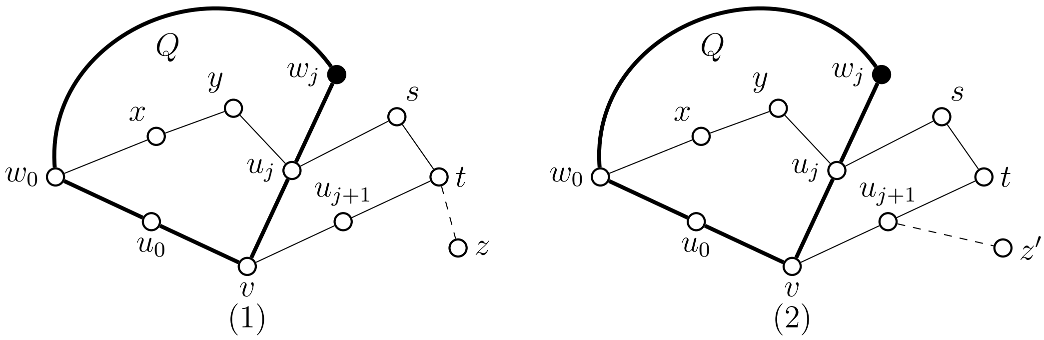


Figure 26: In Case 2.3, either $d(u_{j+1}) = 2$, and t has some neighbor z incident with f_{j+1} (1), or $d(u_{j+1}) > 2$, and u_{j+1} has some neighbor z' other than t that is incident with f_{j+1}.

Case 2.3.1: The vertices t and w_0 are adjacent.
In this case, t, w_0, x is a geodesic. The graph G contains one of the vertices z or z' described above, and has girth 5, and so either $d(z, y) > 3$ or $d(z', y) > 3$, respectively.

Case 2.3.2: There is a vertex $w_{\Delta-1}$ that is adjacent to t, w_0 and $u_{\Delta-1}$.
The path $t, w_{\Delta-1}, w_0, x$ is a $t-x$ geodesic (see Figure 27). One of z or z' is present in G, so by the planarity and girth constraints of G, either $d(z, y) > 3$ or $d(z', y) > 3$.

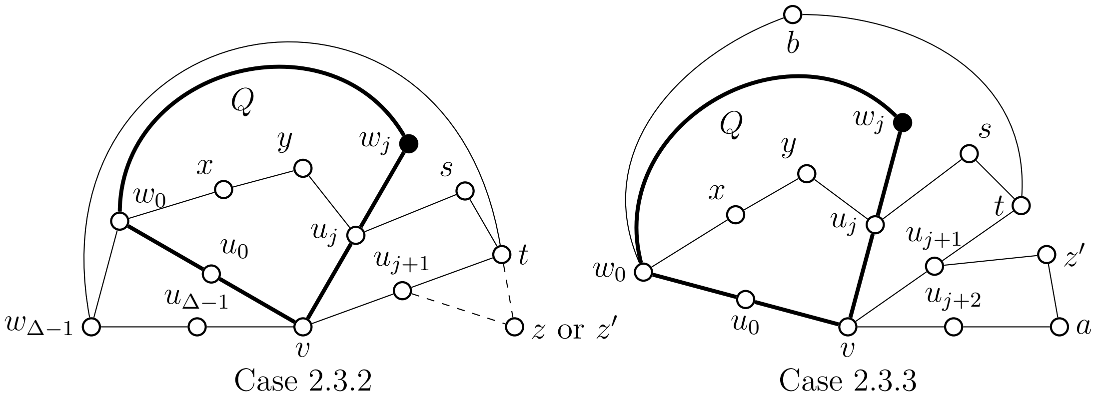


Figure 27: The left figure illustrates Case 2.3.2 in which t and $w_{\Delta-1}$ are adjacent. The right figure shows Case 2.3.3, under the assumption that G contains the vertex z' that is not adjacent to t.

Case 2.3.3: There is some vertex b, that is not adjacent to $u_{\Delta-1}$, but that is adjacent to both t and w_0. Thus the $t - x$ geodesic is t, b, w_0, x. If G contains z, which is adjacent to t, then z is not adjacent to w_0 as this induces a 4-cycle on z, w_0, b and t. Thus, if G contains z, we have the contradiction $d(z, y) > 3$. Therefore z' is a vertex of G. Let a be the vertex of $N_2(v) \cap N(z')$ that is incident with f_{j+1} (see Figure 27, Case 2.3.3). The only possible $y-z'$ geodesic is z', w_0, x, y, so z' and w_0 are adjacent.

As G is triangle-free, a and w_0 are not adjacent. Therefore $d(a, y) > 3$, concluding Case 2.

Case 3: The vertices w_0 and w_j each have a neighbor in $\mathrm{Int}(C_Q)$.
Let x and y be vertices in $\mathrm{Int}(C_Q)$ that are adjacent to w_0 and w_j, respectively. Since G has girth 5, x is not adjacent to any vertex of C_Q apart from w_0, and y is not adjacent to any vertex of C_Q besides w_j. There are two subcases to consider.

Case 3.1: At least one of the vertices u_0 and u_j has a neighbor in $\mathrm{Ext}(C_Q)$.
Assume without loss of generality that u_0 is adjacent to some vertex in $\mathrm{Ext}(C_Q)$. Let s be the neighbor of u_0 in $\mathrm{Ext}(C_Q)$ that is incident with the face $f_{\Delta-1}$, and let t be the other neighbor of s that is also incident with $f_{\Delta-1}$. Note that s is not adjacent to w_j, as this induces a 4-cycle on the vertices s, w_j, w_0 and u_0. There are two ways that G may contain an $s - y$ path of length at most 3, and we consider both as subcases.

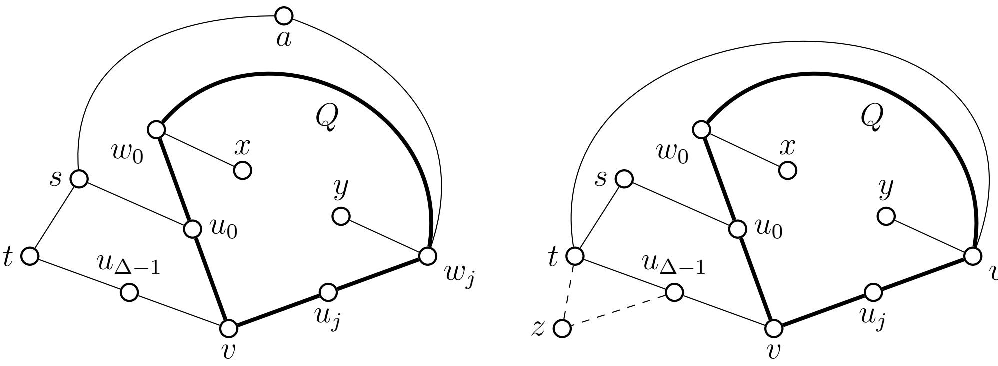


Figure 28: In Case 3.1.1, there is an $s-y$ path s, a, w_j, y containing some vertex a in $N_2(v) \cup N_3(v) - \{t\}$. In Case 3.1.2, the vertex t is adjacent to w_j, and s, t, w_j, y is an $s - y$ path of length 3.

Case 3.1.1: There is some vertex $a \neq t$ that is adjacent to both s and w_j.
The path s, a, w_j, y is the $s - y$ geodesic (see Figure 28). Since G has girth 5, $d(t, x) > 3$, a contradiction.

Case 3.1.2: The vertices t and w_j are adjacent.
The $s - y$ geodesic is s, t, w_j, y (see Figure 28). We consider the face $f_{\Delta-2}$. Either the vertex t is incident with this face, and there is a vertex z in $N_1(t) \cap N_1(u_{\Delta-2})$, or t is not incident with this face, and there is a vertex z' in $N_1(u_{\Delta-1}) \cap N_2(v)$. In both cases we derive a contradiction, as either $d(z, x) > 3$ or $d(z', x) > 3$.

Since Case 3.1 yields a contradiction, we may assume that neither u_0 nor u_j has a neighbor in $\mathrm{Ext}(C_Q)$. Since u_0 has no neighbor in $\mathrm{Ext}(C_Q)$, w_0 is incident with the face $f_{\Delta-1}$. Similarly, w_j is incident with f_j. Let s be the vertex of $N_1(w_0) - \{u_0\}$ that is incident with $f_{\Delta-1}$, and let w_{j+1} be the vertex of $N_1(w_j) - \{u_j\}$ that is incident with f_j.

Case 3.2: The vertex $u_{\Delta-1}$ has degree at least 3.
In this case, s is only incident with the face $f_{\Delta-1}$, and not the face $f_{\Delta-2}$. Let t denote the neighbor of $u_{\Delta-1}$ that is incident with $f_{\Delta-2}$, and we let z be the vertex of $N(t) - \{u_{\Delta-1}\}$ that is incident with $f_{\Delta-2}$ (see Figure 29).

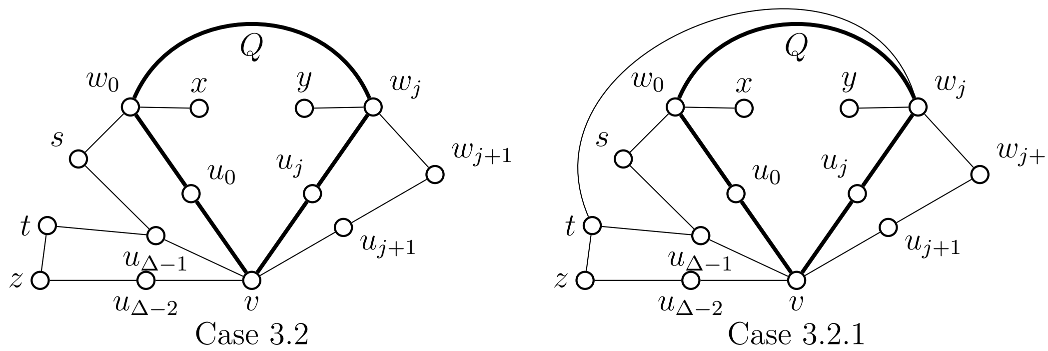


Figure 29: If $d(u_{\Delta-1}) > 2$, then distinct neighbors s and t of $u_{\Delta-1}$ are incident with the faces $f_{\Delta-1}$ and $f_{\Delta-2}$, respectively (Case 3.2). In Case 3.2.1, we consider the possibility that there is a $t - y$ path of the form t, w_j, y.

Considering the girth and planarity of G, there are only three possibilities for a $y - t$ geodesic.

Case 3.2.1: The vertices t and w_j are adjacent.
The $t - y$ geodesic is t, w_j, y (see Figure 29). Since G has girth 5, there is no $z - x$ path of length 3 or less, a contradiction.

Case 3.2.2: There is some vertex $a \neq z$ that is adjacent to both t and w_j.
It is possible that $a = w_{j+1}$, but this does not affect the argument. The $t-y$ geodesic is t, a, w_j, y. Similar to Case 3.2.1, $d(z, x) > 3$.

Case 3.2.3: The vertices z and w_j are adjacent.
The $t - y$ geodesic is t, z, w_j, y. Consider the vertex u_{j+1}. If it has degree 2, then there is a vertex $b \neq u_{j+1}$ that adjacent to w_{j+1} and incident with the face f_{j+1}. If $d(u_{j+1}) \geq 3$, then there exists a vertex $b' \neq w_{j+1}$ that is adjacent to u_{j+1} and incident with f_{j+1}. In either case, the vertex b or b' is not adjacent to w_j since G has girth 5. Whether G contains b or b', we obtain a contradiction, since either $d(b, x) > 3$ or $d(b', x) > 3$.

Case 3.3: The vertex $u_{\Delta-1}$ has degree 2.
The vertex s is the only neighbor of $u_{\Delta-1}$ besides v. Denote by t the vertex of $N_1(s) - \{u_{\Delta-1}\}$ that is incident with the face $f_{\Delta-2}$. Since G is a plane graph of girth 5, t is not adjacent to either w_0 or w_j. There are two subcases to consider: one for each way that G can exhibit a $t - y$ geodesic.

Case 3.3.1: The vertices t and w_{j+1} are adjacent.
The $t - y$ path is t, w_{j+1}, w_j, y. Either t or $u_{\Delta-2}$ has some neighbor z in $N_2(v)$ that has not yet been mentioned. We obtain a contradiction as $d(z, x) > 3$ (see Figure 30).

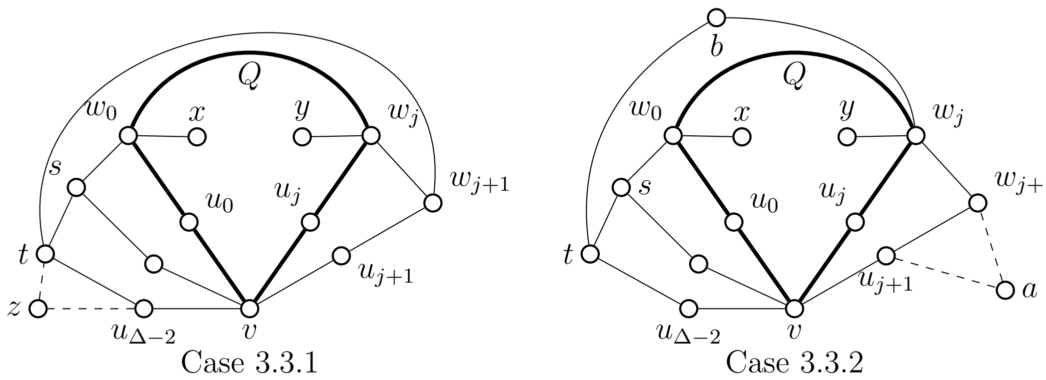


Figure 30: In Case 3.3, we assume that $u_{\Delta-1}$ has only two neighbors. In subcase 3.3.1, we consider what happens when the vertices t and w_{j+1} are adjacent.

Case 3.3.2: There is some vertex $b \neq w_{j+1}$ that is adjacent to both t and w_j. We have the $y-t$ geodesic t, b, w_j, y. Either $d(u_{j+1}) = 2$, and so w_{j+1} has a neighbor in $N_2(v)$ incident with f_{j+1}, or $d(u_{j+1}) \geq 3$ and u_{j+1} has a neighbor in $N_2(v) - \{w_{j+1}\}$ incident with f_{j+1}. In either case, call this neighbor a, and note that $d(a, x) > 3$.

In all cases, we derive a contradiction, completing the proof. □

Theorem 8.5. *Let G be a pentagulation of diameter 3, girth 5 and maximum degree Δ, and let v be a vertex of G with maximum degree. If $\Delta \geq 8$, then G does not have any 2-chords with respect to v.*

Proof. Assume for the sake of contradiction that there does exist some 2-chord with respect to v. Repeat the argument used at the start of the proof of Theorem 8.4, and adopt the same labeling convention for the vertices of $N(v)$ and $N_2(v)$, and for the faces incident with the vertex v. There is a minimal 2-chord $Q : w_0, a, w_j$, where w_0 and w_j are vertices of $N_2(v)$, the vertex a lies in $N_3(v)$, and the cycle C_Q under Q dominates its interior. The vertices u_0 and u_j are the unique vertices of $N(v) \cap N(w_0)$ and $N(v) \cap N(w_j)$, respectively.

Claim 1: The index j satisfies $j < 4$.
Assume to the contrary that $j \geq 4$, and observe by Lemma 8.1 that u_2 has some neighbor w_2 in $N_2(v)$. By Theorem 8.4, the vertex w_2 is adjacent to neither w_0 nor w_j. Since G has girth 5, w_2 is not adjacent to either u_0 or u_j. Since C_Q dominates its interior, w_2 is adjacent to a. Thus w_2, a, w_0 is a 2-chord, which contradicts the minimality of Q and proves Claim 1.

Claim 2: It is not possible that both w_0 and w_j have neighbors in $\text{Int}(C_Q)$.
Assume to the contrary that w_0 has some neighbor x in $\text{Int}(C_Q)$ and w_j has a neighbor y in $\text{Int}(C_Q)$. We have $x \neq y$: were $x = y$, there would be a 4-cycle on the vertices w_0, x, w_j, a. Since G has girth 5, x is not adjacent to a or w_j, and y is not adjacent to a or w_0. The face f_{j+2} is bounded by the 5-cycle $v, u_{j+2}, s, t, u_{j+3}$, where s and t are vertices of $N_2(v)$. Note that $d(t, y) \leq 3$. By Theorem 8.4, the vertex t is not

adjacent to any vertices of $N_2(v)$ apart from s, and possibly one other vertex that is incident with the face f_{j+3}. Hence G can only exhibit a $t-y$ path in one of two ways (see Figure 31):

(1) the vertices a and t are adjacent, and the geodesic is t, a, w_j, y, or

(2) there is some vertex b in $N_3(v)$ that is adjacent to both t and w_j, yielding a geodesic t, b, w_j, y.

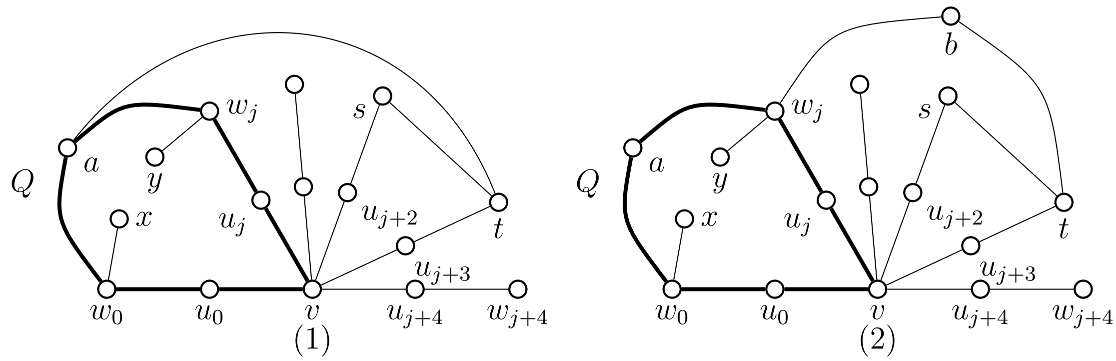


Figure 31: In Claim 2, since $N_2(v)$ has no 1-chords but G has diameter 3, either t, a, w_j, y is a $t-y$ path (shown on the left), or t, b, w_j, y is a $t-y$ path (shown right).

Since G has girth 5, and by Theorem 8.4, there are no 1-chords with respect to v. Thus in both case (1) and (2), $d(s, x) > 3$, proving Claim 2.

Claim 3: $j < 3$.
By Claim 1, we need only show that $j \neq 3$. Suppose that $j = 3$. By Lemma 8.1, u_1 and u_2 each have some neighbor, say w_1 and w_2 respectively, in $\mathrm{Int}(C_Q)$. By Theorem 8.4, there are no 1-chords across v, so w_1 is not adjacent to w_j. By the minimality of Q, w_1 and a are not adjacent, and since G has girth 5, w_1 is not adjacent to v, u_0 or u_j. Similarly, w_2 is not adjacent to any of w_0, a, v, u_0 or u_j. Since C_Q dominates its interior, w_1 is adjacent to w_0 and w_2 is adjacent to w_3. By Claim 2, this is not possible, proving Claim 3.

There remain two cases to consider.

Case 1: Exactly one of w_0 or w_j has a neighbor in $\mathrm{Int}(C_Q)$.
Assume without loss of generality that w_0 has some neighbor, call it x, in $\mathrm{Int}(C_Q)$. The vertex v has $d(v) \geq 8$, and by Claim 3, at most one neighbor of v is contained in $\mathrm{Int}(C_Q)$. Thus v has at least five neighbors in the exterior of C_Q. The face f_{j+2} is bounded by a 5-cycle $v, u_{j+2}, s, t, u_{j+3}$, where s and t are vertices of $N_2(v)$. Both s and t are within distance 3 of x. It is possible that x is adjacent to u_j. However, x is not adjacent to any other vertex of $V(C_Q) - \{w_0\}$, since G has girth 5. As there are no 1-chords across v by Theorem 8.4, there are two ways that G may exhibit a $t-x$ geodesic.

Case 1.1: The vertices t and a are adjacent.
This case yields the path t, a, w_0, x (see Figure 32). Since G has girth 5, s is not

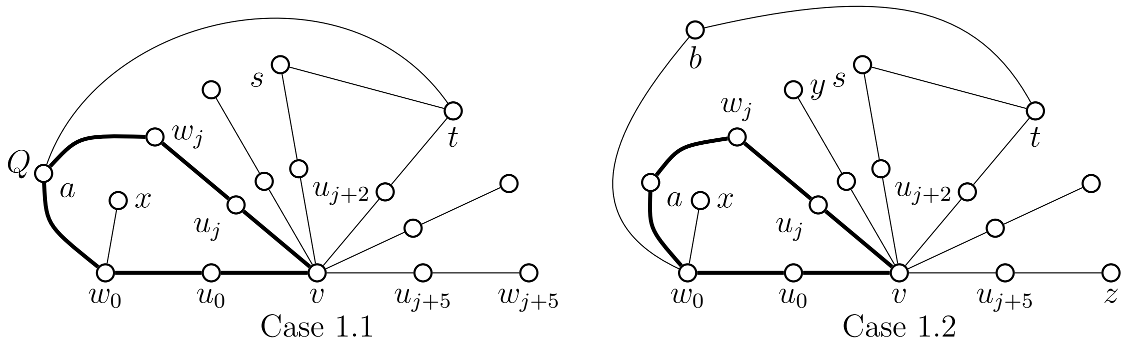


Figure 32: There are two possibilities in Case 1, either t, a, w_0, x is a $t-x$ path, as in subcase 1.1, or t, b, w_0, x is, as in subcase 1.2.

adjacent to a, b or w_j. Because there are no 1-chords across v by Theorem 8.4, t is not adjacent to w_0 (no neighbor of s is adjacent to w_0). Thus $d(s, x) > 3$.

Case 1.2: There is a vertex b in $N_3(v)$ that is adjacent to both w_0 and t.
We have the $t-x$ geodesic t, b, w_0, x (see Figure 32). There are two possibilities for an $s-x$ path of length at most 3.

(1) either s and a are adjacent, or

(2) there is some vertex c in $N_3(v)$ that is adjacent to both s and w_0.

In either case, let y and z be vertices of $N_1(u_{j+1}) \cap N_2(v)$ and $N_1(u_{j+5}) \cap N_2(v)$, respectively. Observe that $d(y, z) > 3$, completing Case 1.

Case 2: Neither w_0 nor w_j has a neighbor in $\text{Int}(C_Q)$.
We claim that both u_0 and u_j have neighbors in $\text{Int}(C_Q)$. Assume to the contrary and without loss of generality that u_0 has no neighbor in $\text{Int}(C_Q)$. Since w_0 has no neighbor in $\text{Int}(C_Q)$, the path a, w_0, u_0, v lies on the boundary of some face f in $\text{Int}(C_Q)$. Since f is bounded by a 5-cycle, there is some vertex z that is adjacent to both a and v. Thus v, z, a is a $v-a$ path of length 2, which contradicts the fact that $Q : w_0, a, w_j$ is a 2-chord (i.e., a is in $N_3(v)$). Hence there exist vertices x and y in $\text{Int}(C_Q)$ that are adjacent to u_0 and u_j respectively. Since G contains no 4-cycles, $x \neq y$, and neither x nor y is adjacent to a. The face f_{j+2} is bounded by a 5-cycle $v, u_{j+2}, s, t, u_{j+3}$, where s and t are vertices of $N_2(v)$. Because there are no 1-chords across v (by Theorem 8.4), s is not adjacent to any vertex of $N_2(v) \cap N_1(u_0)$ or $N_2(v) \cap N_1(u_j)$. As $d(s, x) \leq 3$, s is adjacent to a (and there is some vertex adjacent to both a and x). Similarly, since $d(t, x) \leq 3$, t is adjacent to a. However, we have a triangle on a, s and t, a contradiction that completes the proof. □

Theorem 8.6. *There does not exist a pentagulation with diameter* 3*, girth* 5 *and maximum degree greater than or equal to* 8*.*

Proof. Assume to the contrary that G is a pentagulation of girth 5, diameter 3 and maximum degree $\Delta \geq 8$. Let v be a vertex of G with maximum degree, and label the neighbors $u_1, u_2, \ldots, u_\Delta$ of v such that each path u_i, v, u_{i+1} lies on the boundary of a face (subscripts taken mod Δ). By Lemma 8.1, for each i in $\{1, 2, \ldots, \Delta\}$, there is a vertex w_i in $N(u_i) \cap N_2(v)$. Note that each vertex w_i is not adjacent to u_j for any $j \neq i$, and $d(w_0, w_4) \leq 3$.

We claim that any $w_0 - w_4$ geodesic Q is a 3-chord across v, i.e., the path Q is of the form w_0, a, b, w_4, where a and b are vertices of $N_3(v)$. By Theorem 8.4, there are no 1-chords across v, so w_0 and w_4 are not adjacent. Similarly, there are no 2-chords across v by Theorem 8.5, so Q is not of the form w_0, c, w_4, where c is some vertex of $N_3(v)$. The vertex v is not in Q, since Q has length at most 3 and $d(v, x_0) = d(v, x_4) = 2$. The path Q does not contain any vertex of $N(v)$: If Q contains a vertex u_j of $N(v)$, and Q had length 2, then Q is of the form $Q : w_0, u_j, w_4$, which is impossible. If Q contains u_j and has length 3, it is either of the form w_0, u_j, x, w_4 or w_0, x, u_j, w_4, where x is some vertex of $N_2(v)$. But then either xw_4 or w_0x is a 1-chord across v, which is impossible, so $V(Q) \cap N(v) = \emptyset$. To complete the proof of the claim, it suffices to show that $V(Q) \cap N_2(v) = \{w_0, w_4\}$. Assume to the contrary that there is a vertex x of Q, that is not w_0 or w_4, in $N_2(v)$. If Q has length 2, then it is of the form w_0, x, w_4. Since there are no 1-chords across v, x is adjacent to u_1 or $u_{\Delta-1}$, so xw_4 is a 1-chord across v, a contradiction. If Q has length 3, then it is either w_0, x, y, w_4 or w_0, y, x, w_4, where y is a vertex of $N_2(v)$ (y is not in $N_3(v)$, since there are no 2-chords across v). By symmetry, we may assume without loss of generality that $Q : w_0, x, y, w_4$. Since there are no 1-chords across v, x is a neighbor of u_1 or $u_{\Delta-1}$, and y is a neighbor of u_3 or u_5. In all possible cases, xy is a 1-chord across v, which proves the claim.

The cycle $C_Q : w_0, a, b, w_4, u_4, v, u_0$ under $Q : w_0, a, b, w_4$ is a separating cycle that dominates either its interior or exterior. Thus either w_2 or w_6 is adjacent to a vertex of C_Q. Suppose w_2 is adjacent to a vertex of C_Q (the proof for w_6 is identical). As G has girth 5, w_2 is not adjacent to any of u_0, v or u_4. Because G contains no 1-chords across v, w_2 is not adjacent to either w_0 or w_4. Thus w_2 is adjacent to a or b. If w_2 is adjacent to a, then w_2, a, w_0 is a 2-chord across v, and if w_2 is adjacent to b, then w_2, b, w_4 is a 2-chord. In either case we obtain a contradiction, completing the proof. □

The main result follows immediately from Corollary 3.6, Theorem 8.6 and Theorem 7.3.

Theorem 8.7. *Let G be a pentagulation of diameter 3, order n and maximum degree $\Delta \geq 8$. The order of G satisfies $n \leq 3\Delta - 1$.*

The bound in Theorem 8.7 is sharp for odd Δ. Consider the graph $\mathcal{H}$ in Figure 3. We create a graph $G(\Delta)$ of maximum degree $\Delta = 2k + 1$ from $\mathcal{H}$ as follows: replace each white-vertex path of length 3 by a collection of internally disjoint paths: k paths of length 3 and $k - 1$ paths of length 2 (so $\mathcal{H}$ itself is $G(3)$). By embedding the length 2 and length 3 paths in an alternating pattern, we see that $G(\Delta)$ can be embedded such that each face is bounded by a 5-cycle, and that it has diameter 3, maximum degree Δ and $n = 3\Delta - 1$ vertices.

9 Conclusion

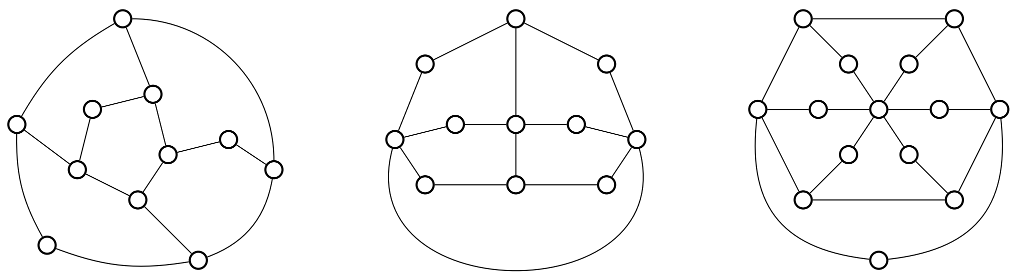
Theorem 8.7 and the sharpness example below it largely solve the degree-diameter problem for diameter 3 pentagulations. Between Theorem 8.7 and the results of [3, 8, 17], the degree-diameter problem has been solved exactly for all plane graphs of diameter 3 in which all faces are bounded by cycles of the same length. A rough summary of the upper bounds is given in Table 1.

	$\rho = 3$	$\rho = 4$	$\rho = 5$	$\rho = 6$	$\rho = 7$
$d = 2$	$\frac{3}{2}\Delta + 1^*$	$\Delta + 2$	5	—	—
$d = 3$	unknown	$3\Delta - 1^\dagger$	$3\Delta - 1^{*\dagger}$	$2\Delta + 2$	7

Table 1: Table of maximum orders $n(\Delta, d)$ among plane graphs in which each face is bounded by a cycle of length ρ. Bounds with an asterisk $*$ are sharp for Δ odd, others are always sharp. Bounds with a dagger † are sharp only for $\Delta \geq 8$.

We have not addressed diameter 3 pentagulations in which $\Delta < 8$. For $\Delta = 5$ and $\Delta = 7$, the largest diameter 3 pentagulations the author has found are $G(5)$ and $G(7)$ (constructed at the end of Section 8), with orders 14 and 20 respectively. For pentagulations with $\Delta \in \{3, 4, 6\}$, see Figure 33. It seems possible that the $n \leq 3\Delta - 1$ bound is not sharp for even values of $\Delta \geq 8$. However, improving the $n \leq 3\Delta - 1$ bound for Δ even appears extremely involved (if possible at all). This leaves two questions to consider:

- For each Δ in $\{3, 4, 5, 6, 7\}$, what is the maximum order n of a pentagulation with diameter 3 and maximum degree Δ?
- Do there exist diameter 3 pentagulations with even degree $\Delta \geq 8$ and order $n = 3\Delta - 1$? If not, what are the largest such pentagulations?



$\Delta = 3, n = 11$ $\Delta = 4, n = 11$ $\Delta = 6, n = 14$

Figure 33: Largest known pentagulations with diameter 3, and $\Delta \in \{3, 4, 6\}$.

For large diameter, getting exact bounds is both difficult and tedious. The last likely tractable exact bound still unknown is the bound for diameter 3 triangulations ($d = \rho = 3$). We end with some further problems:

- What is the maximum order of a diameter 3 triangulation?

- Let μ denote the size of the smallest face of a plane graph. What is the smallest function $\mu(d)$ such that every plane graph of diameter d and smallest face size $\mu(d)$ has order $\mathcal{O}(\Delta)$?
- Find bounds on $n(\Delta, d)$ in plane graphs where every face has the same size ρ, or where every face has at least minimum size μ.

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